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Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds

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1. Introduction

In [L-Y], Peter Li and S.-T. Yau developed the fundamental gradient estimate, which is now widely called the Li-Yau estimate, for any positive solution $u(x,t)$ of the heat equation on a Riemannian manifold $M^n (n = dim_{\mathbb{R}} M)$ and showed how the classical Harnack inequality can be derived from their gradient estimate. When M^n is complete and of nonnegative Ricci curvature, the Li-Yau estimate is sharp. Later in [H2], Richard Hamilton extended the Li-Yau estimate to the full matrix version of the Hessian estimate of u under the stronger assumptions that M is Ricci parallel and of nonnegative sectional curvature.

In this paper, we consider positive solutions to the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0 \tag{1.1}$$

on a complete Kähler manifold M^m of complex dimension $m (n = 2m)$ with Kähler metric $g = (g_{\alpha\bar{\beta}})$. Our main result is the following complex analogue of Hamilton's Hessian estimate for any positive solution u to (1.1):

Theorem 1.1. *Let M^m be a complete Kähler manifold of complex dimension m with nonnegative holomorphic bisectional curvature, and $u(x,t)$ be a positive solution of* (1.1)*. Then, for any vector field $V = (V^\alpha)$ of type* $(1,0)$ *on M and $t > 0$, we have*

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$$u_{\alpha\bar{\beta}} + u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}} V_\alpha + u V_\alpha V_{\bar{\beta}} + \frac{u}{t} g_{\alpha\bar{\beta}} \geq 0. \tag{1.2}$$

A similar result with error terms can be formulated for complete Kähler manifolds with curvature bounded from below. Note that we have two advantages of being in the Kähler category here. Namely, not only we can replace the assumption of nonnegativity of the sectional curvature in Hamilton's result by that of the holomorphic bisectional curvature, but also we can remove the assumption of the Ricci tensor being parallel which is a rather restrictive condition and is definitely needed in Hamilton's work [H2]. Therefore, our Theorem 1.1 should be more applicable. We should mention that in a forth coming paper [N], a new matrix Li-Yau-Hamilton inequality for Kähler-Ricci flow has been proved. It is related to the result in Theorem 1.1 by an interpolation consideration. Some applications of Theorem 1.1 as well as the new matrix Li-Yau-Hamilton inequality for Kähler-Ricci flow, in particular to heat kernel comparisons, Nash-like entropy and Perelman-like reduced volume monotonicity, to name a few, are also shown there.

If we choose the optimal $V = -\nabla u/u$ and take the trace in (1.2), we obtain the gradient estimate of Li-Yau in the Kähler case:

$$u_t - \frac{|\nabla u|^2}{u} + \frac{m}{t} u \geq 0. \tag{1.3}$$

We remark that Li-Yau [L-Y] proved the Riemannian version of the trace estimate (1.3) under the assumption of nonnegative Ricci curvature only. Moreover, we will need to use Li-Yau's result in the proof of Hessian estimate (1.2) when the manifold is noncompact (see Remark 3.1 for more details). However, the conclusion in our result is stronger and therefore the result is expected to have more applications.

An immediate application of Theorem 1.1 is the following complex Hessian comparison theorem for the distance function on a complete Kähler manifolds of nonnegative holomorphic bisectional curvature:

Corollary 1.1. *Let M be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Let $r(x)$ be the distance function to a fixed point $o \in M$. Then in the sense of currents, we have*

$$(r^2)_{\alpha\bar{\beta}}(x) \leq g_{\alpha\bar{\beta}}(x). \tag{1.4}$$

In particular, when M is noncompact, Busemann functions with respect to geodesics are plurisubharmonic.

Proof of Corollary 1.1. Applying Theorem 1.1 to the heat kernel $H(x, y, t)$ with $V = -\nabla H/H$, we have

$$(\log H)_{\alpha\bar{\beta}} + \frac{1}{t} g_{\alpha\bar{\beta}} \geq 0.$$

Now it is well known that H is positive and $-t \log H \to r^2(x, y)$ as $t \to 0$, therefore $-(t \log H)_{\alpha\bar{\beta}} \to (r^2)_{\alpha\bar{\beta}}$ in the sense of distribution/barriers. The result then follows. □

Such a Hessian comparison theorem seems to be elusive from the literature even though Greene-Wu (cf [G-W]) have proved the plurisubharmonicity of the Busemann function on such manifolds. We should mention that Corollary 1.1 was also proved in [L-W] using a different and more direct method. The argument there also works for the negative curved manifolds. It is interesting to see if one can have the parabolic version of their results similar to Theorem 1.1.

Now we turn our attention to the trace and matrix estimates for the potential function of the Kähler-Ricci flow on a compact or complete noncompact Kähler manifold. In the study of the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -R_{\alpha\bar{\beta}}(x,t), \tag{1.5}$$

it is often useful to consider the time-dependent heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) u(x,t) = 0. \tag{1.6}$$

Here Δ_t denotes the Laplace operator with respect to the evolving metric $g_{\alpha\bar{\beta}}(x,t)$ at time t. For example, when M is compact and the first Chern class $c_1(M) = 0$, the Kähler-Ricci flow was studied by the first author in [C1]. In this case, (1.5) can be reduced to the following scalar complex Monge-Amperé flow of the (unknown) function $\varphi(x,t)$:

$$\frac{\partial}{\partial t}\varphi(x,t) = \log\frac{\det(g_{\gamma\bar{\delta}}(x,t))}{\det(g_{\gamma\bar{\delta}}(x,0))} + f(x), \tag{1.7}$$

where $g_{\alpha\bar{\beta}}(x,t) = g_{\alpha\bar{\beta}}(x,0) + \varphi_{\alpha\bar{\beta}}(x,t)$ and $f_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}(x,0)$. It is then easy to check that $u = -\varphi_t$ is a potential function of the evolving Ricci tensor $R_{\alpha\bar{\beta}}(x,t)$, i.e., the complex Hessian $u_{\alpha\bar{\beta}}(x,t) = R_{\alpha\bar{\beta}}(x,t)$, and satisfies the heat equation (1.6). It is often useful to obtain gradient estimate for positive solutions of (1.6) in general, and in particular for the potential functions of the evolving Ricci tensor. It turns out that the trace estimate of Li-Yau always holds for the positive potential functions of the evolving Ricci tensor without any assumptions on the sign of curvature:

Theorem 1.2. *Let M^m be a compact Kähler manifold with $c_1(M) = 0$. Let φ and u be given as above, and assume $u > 0$. Then we have, for $t > 0$,*

$$u_t - \frac{|\nabla u|^2}{u} + \frac{m}{t}u \geq 0. \tag{1.8}$$

A similar result also holds for the $c_1(M) > 0$ case. See the statement of Theorem 2.1 in next section.

If we assume that M is complete noncompact with nonnegative holomorphic bisectional curvature, then the matrix gradient estimate holds for the positive potential function u. (The compact case would not be of much interests since under the assumptions $c_1(M) = 0$ and M has nonnegative holomorphic bisectional curvature, M is in fact holomorphically isometric to a flat complex tori.) Namely, we have

Theorem 1.3. [1] *Let $g_{\alpha\bar{\beta}}(t)$, $0 \le t < T$, be a complete solution to the Kähler-Ricci flow* (1.5) *on a noncompact complex manifold M with nonnegative holomorphic bisectional curvature. Let $u(x,t)$ be a positive potential function of the evolving Ricci tensor. Then u satisfies the matrix estimate* (1.2).

Notice that if the initial metric $g(0)$ has bounded nonnegative holomorphic bisectional curvature, it was proved in [Sh], following the earlier work of Bando [B] and Mok [M] that $(M, g(t))$ has nonnegative bisectional curvature for $t > 0$ if the curvature is uniformly bounded. Hence we can replace the assumption on nonnegativity of $g(t)$ by that the bisectional curvature is bounded for $g(t)$ and nonnegative for $g(0)$.

In the process of proving Theorem 1.3, we in fact prove a matrix gradient estimate for any positive solution u to the heat equation (1.6) coupled with the Kähler-Ricci flow (1.5), provided u is plurisubharmonic. See Theorem 3.1 in the last section.

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2. The compact case

Throughout this section, we assume that M^m is compact so that one can apply the tensor maximum principle of Hamilton in [H1] without worrying about any growth assumption on the tensor. We shall first present the proof of Theorem 1.1 in the compact case, and then the proof of Theorem 1.2 as well as the analogous case of $c_1(M) > 0$. The proof of Theorem 1.1 in the noncompact case will be given in Section 3.

Proof of Theorem 1.1 (The compact case). As in [H2], it suffices to prove that for $t > 0$, the Hermitian symmetric (1,1) tensor

$$N_{\alpha\bar{\beta}} := u_{\alpha\bar{\beta}} + \frac{u}{t} g_{\alpha\bar{\beta}} - \frac{u_\alpha u_{\bar{\beta}}}{u} \ge 0. \tag{2.1}$$

[1] In [N], the author proved a new matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow. The estimate is shown to be related to Perelman's entropy formula in [P] also by an interpolation.

As always, we first apply the heat operator to the tensor $N_{\alpha\bar{\beta}}$. From direct calculations (cf. Lemma 2.1 in [N-T1]), we have

$$\left(\frac{\partial}{\partial t}-\Delta\right)u_{\alpha\bar{\beta}}=R_{\alpha\bar{\beta}\gamma\bar{\delta}}u_{\delta\bar{\gamma}}-\frac{1}{2}R_{\alpha\bar{s}}u_{s\bar{\beta}}-\frac{1}{2}R_{s\bar{\beta}}u_{\alpha\bar{s}}. \tag{2.2}$$

Using the fact that $\Delta=\frac{1}{2}\left(\nabla_s\nabla_{\bar{s}}+\nabla_{\bar{s}}\nabla_s\right)$, we also obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\left(-\frac{u_\alpha u_{\bar{\beta}}}{u}\right)=&\frac{1}{u}\left(u_{\alpha s}u_{\bar{s}\bar{\beta}}+u_{\alpha\bar{s}}u_{s\bar{\beta}}\right)+\frac{2}{u^3}u_\alpha u_{\bar{\beta}}|u_s|^2\\&+\frac{1}{2u}(R_{\alpha\bar{s}}u_s u_{\bar{\beta}}+R_{s\bar{\beta}}u_\alpha u_{\bar{s}})\\&-\frac{1}{u^2}\left(u_{\alpha s}u_{\bar{s}}u_{\bar{\beta}}+u_{\bar{\beta}s}u_\alpha u_{\bar{s}}\right.\\&\left.+u_{\alpha\bar{s}}u_{\bar{\beta}}u_s+u_{\bar{\beta}\bar{s}}u_\alpha u_s\right),\end{aligned} \tag{2.3}$$

and

$$\left(\frac{\partial}{\partial t}-\Delta\right)\left(\frac{u}{t}g_{\alpha\bar{\beta}}\right)=-\frac{u}{t^2}g_{\alpha\bar{\beta}}. \tag{2.4}$$

Combining (2.2)-(2.4), we have

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)N_{\alpha\bar{\beta}}=&R_{\alpha\bar{\beta}\gamma\bar{\delta}}N_{\delta\bar{\gamma}}-\frac{1}{2}(R_{\alpha\bar{s}}N_{s\bar{\beta}}+N_{\alpha\bar{s}}R_{s\bar{\beta}})+\frac{1}{u}N_{\alpha\bar{s}}N_{s\bar{\beta}}-\frac{2}{t}N_{\alpha\bar{\beta}}\\&+\frac{1}{u}\left(u_{\alpha s}-\frac{u_\alpha u_s}{u}\right)\left(u_{\bar{s}\bar{\beta}}-\frac{u_{\bar{s}}u_{\bar{\beta}}}{u}\right)+\frac{1}{u}R_{\alpha\bar{\beta}\gamma\bar{\delta}}u_\delta u_{\bar{\gamma}}.\end{aligned} \tag{2.5}$$

Now according to the tensor maximum principle of Hamilton in [H1], to prove $N_{\alpha\bar{\beta}}\geq 0$ it suffices to show that the right hand side of (2.5) is non-negative when applied to any null vector of $N_{\alpha\bar{\beta}}$. However, it is easy to check that in fact each term on the right hand side of (2.5) is nonnegative when evaluated at any null vector of $N_{\alpha\bar{\beta}}$. Thus the proof of Theorem 1.1 is proved in the case of M being compact. □

Proof of Theorem 1.2. As in [L-Y], let $v=\log u$. Define $G=t\left(|\nabla v|^2-v_t\right)$. It suffices to show that $G\leq m$. Direct calculations show that

$$\Delta_t v-v_t=-|\nabla v|^2, \tag{2.6}$$

$$\Delta_t|\nabla v|^2=|v_{\alpha\gamma}|^2+|v_{\alpha\bar{\gamma}}|^2+(\Delta_t v)_\alpha v_{\bar{\alpha}}+v_\alpha(\Delta_t v)_{\bar{\alpha}}+R_{\alpha\bar{\beta}}v_\alpha v_{\bar{\beta}}, \tag{2.7}$$

and

$$\frac{\partial}{\partial t}|\nabla v|^2=R_{\alpha\bar{\beta}}v_\alpha v_{\bar{\beta}}+(v_t)_\alpha v_{\bar{\alpha}}+v_\alpha(v_t)_{\bar{\alpha}}. \tag{2.8}$$

Here the first term on the right hand side of (2.8) is due to the fact that we have a time-dependent metric evolving by the Kähler-Ricci flow (1.5).

From (2.6) we also have

$$v_{tt} - \Delta_t(v_t) = R_{\alpha\bar\beta}(v_{\beta\bar\alpha} + v_\beta v_{\bar\alpha}) + (v_t)_\alpha v_{\bar\alpha} + v_\alpha (v_t)_{\bar\alpha}. \tag{2.9}$$

From (2.7)–(2.9) it follows

$$\begin{aligned}\left(\Delta_t - \frac{\partial}{\partial t}\right)(|\nabla v|^2 - v_t) &= |v_{\alpha\gamma}|^2 + |v_{\alpha\bar\gamma}|^2 + R_{\alpha\bar\beta}(v_{\beta\bar\alpha} + v_\beta v_{\bar\alpha}) \\ &\quad -(|\nabla v|^2 - v_t)_\alpha v_{\bar\alpha} - (|\nabla v|^2 - v_t)_{\bar\alpha} v_\alpha \\ &\geq |v_{\alpha\bar\gamma}|^2 - (|\nabla v|^2 - v_t)_\alpha v_{\bar\alpha} \\ &\quad -(|\nabla v|^2 - v_t)_{\bar\alpha} v_\alpha. \end{aligned} \tag{2.10}$$

Here we have used the fact that $R_{\alpha\bar\beta}(v_{\beta\bar\alpha} + v_\beta v_{\bar\alpha}) = R_{\alpha\bar\beta} u_{\beta\bar\alpha}/u = |R_{\alpha\bar\beta}|^2/u \geq 0$. From (2.10), we obtain

$$\begin{aligned}\left(\Delta_t - \frac{\partial}{\partial t}\right) G &\geq t|v_{\alpha\bar\gamma}|^2 - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t} \\ &\geq \frac{t}{m}(\Delta_t v)^2 - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t} \\ &= \frac{G^2}{tm} - 2\langle \nabla G, \nabla v\rangle - \frac{G}{t}. \end{aligned} \tag{2.11}$$

Applying the maximum principle argument to the above inequality, it then follows that $G \leq m$, which completes the proof of the theorem. □

In the case of compact M with first Chern class $c_1(M) > 0$, we can obtain a similar result. In this case, consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{\alpha\bar\beta}(x,t) = -R_{\alpha\bar\beta}(x,t) + g_{\alpha\bar\beta}(x,t) \tag{2.12}$$

with the initial metric $g(x,0)$ and its Kähler form ω such that $c_1(M) = \pi[\omega]$. Similar to (1.5), (2.12) can also be reduced to a complex Monge-Ampere flow of the form

$$\frac{\partial}{\partial t}\varphi(x,t) = \log \frac{\det(g_{\gamma\bar\delta}(x,t))}{\det(g_{\gamma\bar\delta}(x,0))} + \varphi(x,t) + f(x). \tag{2.13}$$

Here again $g_{\alpha\bar\beta}(x,t) = g_{\alpha\bar\beta}(x,0) + \varphi_{\alpha\bar\beta}(x,t)$ and $f_{\alpha\bar\beta} = g_{\alpha\bar\beta}(x,0) - R_{\alpha\bar\beta}(x,0)$. Furthermore, it was shown by the first author [C1,C2] that the solution to (2.13), hence also (2.12), exists for all time.

Set $w = -\varphi_t$. Then w is a potential function satisfying

$$w_{\alpha\bar\beta}(x,t) = R_{\alpha\bar\beta}(x,t) - g_{\alpha\bar\beta}(x,t) \tag{2.14}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) w = w. \tag{2.15}$$

Similar to Theorem 1.2, we have

Theorem 2.1. *Let M^m be a compact Kähler manifold with $c_1(M) > 0$. Let φ and w be defined as above, and assume $w > 0$. Then we have, for $t > 0$,*

$$w_t - \frac{|\nabla w|^2}{w} + \frac{m}{t} w \geq w > 0. \tag{2.16}$$

Proof of Theorem 2.1. Let $u = e^{-t} w$, then u is a positive solution to the heat equation (1.6) coupled with (2.12). As in the proof of Theorem 1.2, we let $v = \log u$ and define $G = t\left(|\nabla v|^2 - v_t\right)$. It follows from similar calculations there that

$$\begin{aligned} \left(\Delta_t - \frac{\partial}{\partial t}\right)(|\nabla v|^2 - v_t) &= |v_{\alpha\gamma}|^2 + |v_{\alpha\bar{\gamma}}|^2 + \left(R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}}\right)(v_{\beta\bar{\alpha}} + v_\beta v_{\bar{\alpha}}) \\ &\quad -(|\nabla v|^2 - v_t)_\alpha v_{\bar{\alpha}} - (|\nabla v|^2 - v_t)_{\bar{\alpha}} v_\alpha + |\nabla v|^2 \\ &\geq |v_{\alpha\bar{\gamma}}|^2 - (|\nabla v|^2 - v_t)_\alpha v_{\bar{\alpha}} \\ &\quad -(|\nabla v|^2 - v_t)_{\bar{\alpha}} v_\alpha. \end{aligned} \tag{2.17}$$

Here we have used the fact that

$$(R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}})(v_{\beta\bar{\alpha}} + v_\beta v_{\bar{\alpha}}) = (R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}}) u_{\beta\bar{\alpha}}/u = |R_{\alpha\bar{\beta}} - g_{\alpha\bar{\beta}}|^2/w \geq 0.$$

Hence G satisfies the same differential inequality (2.11), and we can conclude the same way that $G \leq m$. Therefore, the function $u = e^{-t} w$ satisfies the estimate (1.8). Expressing this in terms of w, we obtain the desired estimate (2.16) and thus proves Theorem 2.1. □

Remark 2.1. Due to the equations (2.13) and (2.15), one can ensure $w(x, t) > 0$ by choosing $\varphi(x, 0)$ such that $-w(x, 0) = \varphi(x, 0) + f(x) < 0$.

Remark 2.2. We also should point out that Perelman also derived certain gradient estimate for the Ricci potential function w, which is anchored in a clever way through his entropy. The gradient estimate there plays a important role in obtaining geometric estimates for the Kähler-Ricci flow for $c_1(M) > 0$ case. At this moment, it is still unclear to us at this moment what is the relation between his estimate and ours.

3. The complete noncompact case

Now we consider the case when M is a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature. Due to the fact that uniqueness of the solution to the scalar heat equation fails to be true in general on a noncompact manifold, one normally needs to impose some kind of growth conditions on the function u as well as its first and second order derivatives in order to be able to apply Hamilton's tensor maximum principle (or its argument) to the tensor $N_{\alpha\bar{\beta}}$ defined in (2.1). However, in our case of proving Theorem 1.1, we shall see that we can get away without imposing any growth assumptions on u and its derivatives. The key here is that we are working with a positive solution of the heat equation, thus we can make use of the available estimate of Li-Yau to obtain the required growth estimates at any positive time. First let us collect some basic facts.

Lemma 3.1. *Let* $u(x, t)$ *be a positive solution to* (1.1). *Then we have*

$$\left(\frac{\partial}{\partial t} - \Delta\right) |\nabla u|^2 \leq -|u_{\alpha\bar{\beta}}|^2 - |u_{\alpha\beta}|^2 \tag{3.1}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) |u_{\alpha\bar{\beta}}|^2 \leq 0. \tag{3.2}$$

Proof. Both (3.1) and (3.2) can be verified by direct calculations. Here the nonnegativity of the Ricci curvature is used in (3.1) and the nonnegativity of the holomorphic bisectional curvature is used in (3.2). The elliptic version of the computation for (3.2) goes back to Bishop-Goldberg [B-G] (see also [M-S-Y, Proposition on page 185]). For more details, see for example Lemma 1.1 in [N-T1] and Lemma 1.5 in [N-T2] (or Lemma 2.1 of [N-T4] for a slightly more general, but published version). □

We also need to use the result of Li-Yau on the Harnack inequality for positive solution to the heat equation. Let $o \in M$ be a fixed point, and let $u(x, t)$ be a positive solution of (1.1). Since our focus here is to obtain a upper bound on u for positive time we can assume, without the lose of generality, that $u(x, t)$ is defined on $M \times [0, 2]$. By the Harnack inequality in [L-Y] (Theorem 2.2(i), page 168) we have, for $0 < t < 1$

$$u(x, t) \leq \frac{C}{t^m} u(o, 2) \exp(ar^2(x)). \tag{3.3}$$

Here $a > 0$ is a constant and $r(x)$ is the distance function from the point o. In particular, for $2 - \delta \geq t \geq \delta > 0$, there exists a constant $b > 0$ (might depends on δ and $u(o, 2)$) such that

$$u(x, t) \leq \exp(b(r^2(x) + 1)). \tag{3.4}$$

In fact using (3.1) and (3.2) together with the mean value inequality of Li-Tam we can push further to obtain the similar control on the gradient and the complex Hessian of u for $t > 2\delta$. In fact, we have the following

Lemma 3.2. *For $2-\delta \geq t \geq 2\delta$, there exists $b_1 \geq b > 0$ such that*

$$|\nabla u|^2(x,t) \leq \exp(b_1(r^2(x)+1)) \tag{3.5}$$

and

$$|u_{\alpha\bar{\beta}}|^2(x,t) \leq \exp(b_1(r^2(x)+1)). \tag{3.6}$$

Proof. First we prove that for some $b_2 > 0$, for any $T \leq 2-2\delta$,

$$\int_\delta^T \int_M \exp(-b_2(r^2(x)+1))|\nabla u|^2(x)\,dx\,dt < \infty \tag{3.7}$$

To see this, we multiply φ^2, where φ is a cut-off function, on both sides of the equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) u^2 = -2|\nabla u|^2$$

and then integrate by parts. As in [N-T2] we have

$$\begin{aligned}
2\int_0^T \int_M \varphi^2 |\nabla u|^2 dxdt &= -\int_0^T \int_M \varphi^2 \left(\frac{\partial}{\partial t} - \Delta\right) u^2 \\
&\leq \int_M \varphi^2 u_0^2(x)dx + 4\int_0^T \int_M \varphi u|\nabla\varphi|\,|\nabla u|dxdt \\
&\leq \int_M \varphi^2 u_0^2(x)dx + 4\int_0^T \int_M |\nabla\varphi|^2 u^2 dxdt \\
&\quad + \int_0^T \int_M \varphi^2|\nabla u|^2 dxdt.
\end{aligned}$$

Now (3.7) follows from (3.3). Now we apply the mean value inequality of Li-Tam (Theorem 1.1 of [L-T]) and the fact that $|\nabla u|^2$ is a sub-solution to the heat equation to obtain the point-wise upper bound (3.5) from an integral one. In fact one can get the estimate (3.5) for all $t > \frac{3\delta}{2}$. Now use (3.1) and repeating the above argument in the proof of (3.7) we can have that

$$\int_{\frac{3\delta}{2}}^T \int_M \varphi^2 |u_{\alpha\bar{\beta}}|^2 dxdt \leq \int_M \varphi^2 |\nabla u|^2(x, \frac{3\delta}{2})dx + 4\int_{\frac{3\delta}{2}}^T \int_M |\nabla\varphi|^2|\nabla u|^2 dxdt.$$

Hence by (3.5) for $t = \frac{3\delta}{2}$ we have that

$$\int_{\frac{3\delta}{2}}^T \int_M \exp(-b_2(r^2(x)+1))|u_{\alpha\bar{\beta}}|^2(x)\,dx\,dt < \infty \tag{3.8}$$

Applying Li and Tam's mean value theorem again we obtain the point-wise estimate (3.6). ◻

Now we are in the position to prove Theorem 1.1 for the complete noncomapct case. We use the perturbation techniques from [N-T1].

Proof of Theorem 1.1 (The noncompact case). We first shift the time by 2δ. By doing so, $u(x,t)$ together with its gradient and complex Hessian satisfy (3.3)–(3.6). If we can prove the theorem for this case, then we would have (1.2) when replacing t by $t+2\delta$. By letting $\delta \to 0$ we would complete the proof of Theorem 1.1. Therefore, without loss of generality, we can assume (3.3)–(3.6) hold. By a similar argument we also can assume $u \geq \delta$ in the proof.

We first construct a function $\phi(x,t)$ such that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi = \phi$$

and

$$\phi(x,t) \geq C_1 \exp(2b_1(r^2(x)+1))$$

for some constant $C_1 > 0$. This can be done by Lemma 1.1 of [N-T1].

Let $N_{\alpha\bar{\beta}}$ be the Hermitian (1,1) tensor defined in (2.1). We consider the (1,1) tensor $Z_{\alpha\bar{\beta}} = t^2 N_{\alpha\bar{\beta}} + \epsilon\phi g_{\alpha\bar{\beta}}$, where $g_{\alpha\bar{\beta}}$ is the metric tensor. Clearly we only need to show that $Z_{\alpha\bar{\beta}} \geq 0$ for any $\epsilon > 0$. We shall prove this by contradiction. Suppose it is not true, then by the growth nature of ϕ and the fact that $N_{\alpha\bar{\beta}} > 0$ at time $t = 0$, we know that there exists a first time $t_0 > 0$, and a point $x_0 \in M$ and a unit vector $V = v^\alpha \frac{\partial}{\partial z_\alpha} \in T_{x_0}M$ such that $Z_{\alpha\bar{\beta}}(x_0,t_0)v^\alpha \bar{v}^\beta = 0$. Now we choose a normal coordinate around x_0 and extend V to be a local unit vector field near x_0 by parallel translation along the geodesics emanating from x_0. It then follows from the direct calculation that, at point x_0,

$$\Delta\left(Z_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta\right) = \left(\Delta Z_{\alpha\bar{\beta}}\right) v^\alpha \bar{v}^\beta.$$

Since $Z_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta \geq 0$ for all (x,t) with $t \leq t_0$ and x close to x_0, and $Z_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta = 0$ at (x_0,t_0) we see that at (x_0,t_0),

$$0 \geq \left(\frac{\partial}{\partial t} - \Delta\right)\left(Z_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta\right).$$

On the other hand, using (2.5) we also have, at (x_0,t_0),

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(Z_{\alpha\bar{\beta}} v^\alpha \bar{v}^\beta\right) &= \left(\left(\frac{\partial}{\partial t} - \Delta\right) Z_{\alpha\bar{\beta}}\right) v^\alpha \bar{v}^\beta \\ &\geq t^2\left(R_{\alpha\bar{\beta}\gamma\bar{\delta}} N_{\bar{\gamma}\delta} - \frac{1}{2} R_{\alpha\bar{s}} N_{s\bar{\beta}} - \frac{1}{2} R_{s\bar{\beta}} N_{\alpha\bar{s}}\right) v^\alpha \bar{v}^\beta \\ &\quad + \frac{t^2}{u} R_{\alpha\bar{\beta}\gamma\bar{\delta}} u_{\bar{\gamma}} u_\delta v^\alpha \bar{v}^\beta + \epsilon\phi |V|^2\end{aligned}$$

$$\geq R_{\alpha\bar{\beta}\gamma\bar{\delta}} Z_{\bar{\gamma}\delta} v^{\alpha} \bar{v}^{\beta} - \frac{1}{2} R_{\alpha\bar{s}} Z_{s\bar{\beta}} v^{\alpha} \bar{v}^{\beta}$$
$$-\frac{1}{2} R_{s\bar{\beta}} Z_{\alpha\bar{s}} v^{\alpha} \bar{v}^{\beta} + \epsilon\phi |V|^2 > 0.$$

We now have a contradiction. This shows that $Z_{\alpha\bar{\beta}} \geq 0$ for all $t \leq 2 - 2\delta$. Letting $\epsilon \to 0$, $\delta \to 0$ and repeating the argument to the later time (if necessary) we complete the proof Theorem 1.1 in case M is complete noncompact. □

Remark 3.1. (i) As we mentioned before, our result does not contain Li-Yau's gradient estimate even though by taking trace we obtain their estimate (1.3) since we need to use the Harnack inequality and the mean value inequality of Li-Tam in Lemma 3.2. The proof of both the Harnack inequality and Li-Tam's mean value inequality rely on Li-Yau's gradient estimate. Also we need to assume nonnegativity of the holomorphic bisectional curvature in stead of the Ricci curvature. On the other hand our result is about the full Hessian matrix of the function and therefore is stronger than Li-Yau's trace estimate.

(ii) Due to the recently established tensor maximum principle [N-T4, Theorem 2.1] we can further simplify the proof a little bit since one can replace the pointwise control (3.5) and (3.6) by the corresponding weaker integral control (3.7) and (3.8) in order to apply the maximum principle in [N-T4]. The main advantage of the tensor maximum principle in [N-T4] is to allow one apply to tensors which are only nonnegative at the initial time in the weak sense, the situation the current argument above does not work anymore.

Finally, we consider the matrix gradient estimate for any positive solution u to the heat equation (1.6) coupled with the Kähler-Ricci flow (1.5). In this case, (2.2) and (2.3) remain the same but (2.4) and (2.5) become, respectively,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{u}{t} g_{\alpha\bar{\beta}}\right) = -\frac{u}{t^2} g_{\alpha\bar{\beta}} - \frac{u}{t} R_{\alpha\bar{\beta}}. \tag{2.4'}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta\right) N_{\alpha\bar{\beta}} = -\frac{1}{2} R_{\alpha\bar{s}} N_{s\bar{\beta}} - \frac{1}{2} N_{\alpha\bar{s}} R_{s\bar{\beta}} + \frac{1}{u} N_{\alpha\bar{s}} N_{s\bar{\beta}} - \frac{2}{t} N_{\alpha\bar{\beta}}$$
$$+\frac{1}{u}\left(u_{\alpha s} - \frac{u_\alpha u_s}{u}\right)\left(u_{\bar{s}\bar{\beta}} - \frac{u_{\bar{s}} u_{\bar{\beta}}}{u}\right) + R_{\alpha\bar{\beta}\gamma\bar{\delta}} u_{\bar{\gamma}\delta}. \tag{2.5'}$$

Notice that under the extra assumption that $u(x, t)$ is plurisubharmonic, the last term is nonnegative definite. Therefore we can prove a similar result as in Theorem 1.1 for the coupled case:

Theorem 3.1. *Let $g_{\alpha\bar{\beta}}(x,t)$ be complete Kähler metrics evolving by the Kähler-Ricci flow* (1.5) *on M^m, and $u(x,t)$ be a positive solution to the time-dependent heat equation* (1.6). *Assume that the holomorphic bisectional curvature of $g_{\alpha\bar{\beta}}(x,t)$ is nonnegative and $u(x,t)$ is plurisubharmonic. Then*

$$u_{\alpha\bar{\beta}} + u_\alpha V_{\bar{\beta}} + u_{\bar{\beta}} V_\alpha + u V_\alpha V_{\bar{\beta}} + \frac{u}{t} g_{\alpha\bar{\beta}} \geq 0.$$

Proof of Theorem 1.3. Apply Theorem 3.1 to the potential function u, which is plurisubharmonic since its complex Hessian is equal to the Ricci tensor.

As pointed out in the introduction one can replace the nonnegativity of the bisectional curvature for $g(t)$ by assuming that the bisectional curvature is bounded for $g(t)$ and nonnegative for $g(0)$, using a result of [Sh,B,M].

Notice in [N-T3], the authors proved that under some average curvature decay assumption one indeed can obtain the potential function $u(x,t)$ for the Ricci tensor by solving the Poincaré-Lelong equation and utilizing the volume element. But it is hard to get positive function in this case since the potential function obtained can not be bounded.

Remark 3.2. (i) Taking the trace in Theorem 3.1, we can obtain the gradient estimate for u obtained before in [N-T1]. Notice again that the gradient estimate in [N-T1] is not entirely a corollary of Theorem 3.1 since there they only need to assume that the Ricci curvature is nonnegative while in Theorem 3.1 we need to assume that the holomorphic bisectional curvature is nonnegative.

(ii) In [N-T1,N-T4], the authors also studied the question under what conditions the plurisubharmonicity of $u(x,t)$ will be preserved by the heat flow in the time-dependent or independent case. A optimal result was proved in [N-T4]. This is a separate but technically harder issue which one has to deal with in order to make use of the Li-Yau-Hamilton inequality.

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