

Kähler-Ricci Flow and the Poincaré-Lelong Equation

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Introduction.

In [M-S-Y], Mok-Siu-Yau studied complete Kähler manifolds with nonnegative holomorphic bisectional curvature by solving the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial}u = \text{Ric} \quad (0.1)$$

where Ric is the Ricci form of the manifold. In [M-S-Y], the authors solved (0.1) under the assumptions that the manifold is of maximal volume growth and the scalar curvature decays quadratically. On the other hand, in a series of papers of W.-X. Shi [Sh2-4], Kähler-Ricci flow

$$\frac{\partial}{\partial t}g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}} \quad (0.2)$$

has been studied extensively and important applications were given. In [N1] and [N-S-T], the Poincaré-Lelong equation has been solved under more general conditions than in [M-S-Y]. The conditions in [N-S-T] are more in line with the conditions in [Sh2-4]. Since a solution of (0.1) is a potential for the Ricci tensor, it is interesting to see if one can apply (0.1) to study solutions of (0.2).

In this work, on the one hand we shall study the Kähler-Ricci flows by using solutions of the Poincaré-Lelong equation. On the other hand, we will also refine some of the results in [Sh3, C-Z, C-T-Z] and give new applications. The hinge between the equations (0.1) and (0.2) is that by solving (0.1) one can then construct a function $u(x, t)$ which satisfies the time-dependent heat equation $(\frac{\partial}{\partial t} - \Delta)u(x, t) = 0$ and the time-dependent Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \text{Ric}_{g(t)}$ simultaneously. It then can simplify the study of (0.2) quite a bit. It also suggests some of the refined

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estimates in the second part of this paper. We should point out here that the simplification in this paper is that $|\nabla u|^2$ helps to obtain a sharp uniform curvature estimates (Cf. Theorem 1.3), which holds as an equality for the Kähler-Ricci soliton. It is different from the compact case as in [Co1], where one restricts the deformation of the metric within a fixed cohomology class and can then appeal to Yau's solution to the Monge-Ampère equation by reducing (0.2) to a single equation.

Let $(M^m, g_{\alpha\bar{\beta}}(x))$ be a complete noncompact Kähler manifold with bounded and nonnegative holomorphic bisectional curvature. Let \mathcal{R}_0 be the scalar curvature of M . In [Sh3], it was proved that (0.2) has long time solution with initial metric $g_{\alpha\bar{\beta}}(x)$ satisfying the assumption that

$$k(x, r) \leq C(1 + r)^{-\theta} \quad (0.3)$$

for some constants C and $\theta > 0$ for all x and r . Here $k(x, r)$ denotes the average of \mathcal{R}_0 on $B(x, r)$, the geodesic ball of radius r with center at x . The idea of the proof of the long time existence in [Sh3] is to use the parabolic version of the third derivative estimate for the Monge-Ampère equation together with a careful estimate of the volume element. The computation is rather tedious. In this work, we will use the solution to (0.1) constructed in [N-S-T] (more precisely the uniform curvature estimate (1.24) in Theorem 1.3) to give an alternate (and much simpler, we believe) proof for the long time existence under the assumption that

$$\int_0^\infty k(x, r) dr \leq C \quad (0.4)$$

for some C independent of x . Our proof uses a maximum principle which is a generalization of that in [K-L], and an idea similar to those in [Cw]. Our assumption here is different from but somewhat stronger than Shi's (0.3). However it has covered the interesting cases in [Sh2-3], namely the cases $k(x, r) \leq C(1 + r)^{-1-\delta}$, on which interesting geometric results could be obtained. On the other hand we also can prove a long time existence result under a more flexible condition. Namely, we show that there exists long time solution to (0.2) if

$$k(x, r) \leq \epsilon(r) \quad (0.5)$$

for *all* x (with some fixed function $\epsilon(r)$) with $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. Recently in [C-T-Z], it is proved that if the complex dimension of M is $m = 2$ and M has maximal volume growth, then (0.2) has long time solution if (0.5)

holds for *some* x and for some function $\epsilon(r)$ which tends to zero as $r \rightarrow \infty$. The proof there is an indirect blow-up argument. It also used some special features in dimension 2, such as the Gauss-Bonnet formula for the four dimensional Riemannian manifolds. In order to prove the long time existence under the assumption (0.5), we need a more precise estimate for the volume element $F(x, t) = \log [\det(g_{\alpha\bar{\beta}}(x, t)) / \det(g_{\alpha\bar{\beta}}(x, 0))]$, where $g_{\alpha\bar{\beta}}(x, t)$ is the solution of (0.2). In fact, we prove the following results, see Theorem 2.1 and Corollary 2.1:

Theorem. *Suppose (0.2) has a solution on $M \times [0, T)$. Then we have the following:*

(a) *There exists a constant $C > 0$ depending only on m such that for $0 < t < T$,*

$$-F(x_0, t) \geq C \int_0^{\sqrt{t}} sk(x_0, s) ds.$$

(b) *If in addition, $k(x, r) \leq k(r)$ for some function $k(r)$ for all x , then*

$$-\mathbf{m}(t) \leq C' \int_0^R sk(s) ds$$

where $R^2 = at(1 - \mathbf{m}(t))$, C and a are constants depending only on m . Here $\mathbf{m}(t) = \inf_{x \in M} F(x, t)$.

From the two-sidedness of the above estimates on $F(x, t)$ one can see that they are almost optimal. By comparing with the previous estimates obtained in [N-S-T] and [N2] for the Poisson equation and the linear heat equation, the refined estimates here are sharp in certain cases and fit into the theory for the linear equation. The above mentioned estimates will be proved by using, the by-now standard estimates on the heat kernels of Li-Yau in [L-Y]. There is no need to construct special exhaustion functions as in [Sh2-3, C-Z, C-T-Z]. As a consequence, a little more general gap theorem, than those in [C-Z], is obtained, see Corollary 2.3. In particular, we show that *any bounded solution to the Poisson equation $\Delta u = \mathcal{R}_0(x)$ is a constant, provided M has bounded nonnegative bisectional curvature*. In other words, if M is nonflat, $\Delta u = \mathcal{R}_0(x)$ has no bounded solution. This answers a question asked by R. Hamilton. Namely, solving Poisson for $\mathcal{R}_0(x)$ is different from arbitrary $f(x)$ since one can easily construct bounded solution to $\Delta u = f(x)$ for nonzero compact supported $f(x)$. This is also related to the gradient estimates of Chow in [Cw]. In [Y], it was proved that, on a complete Riemannian manifold with nonnegative Ricci curvature, any negative (positive) harmonic function is a constant. We prove that a similar result holds for $\Delta u = \mathcal{R}_0(x)$. Namely,

$\Delta u = \mathcal{R}_0(x)$ has no nonconstant negative solution, provided M has bounded nonnegative bisectional curvature and (0.2) has long time solution.

When $(M, g_{\alpha\bar{\beta}}(x, 0))$ has the maximum volume growth, using the estimates mentioned above the results in [C-Z] on the Steinness and the topology of M can be refined. Namely we show that if $(M, g_{\alpha\bar{\beta}}(x, 0))$ is of maximum volume growth and $\int_0^r sk(x, s) ds \leq \phi(r)$ with $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$, M is Stein and diffeomorphic to \mathbb{R}^{2m} for $m \geq 3$, homeomorphic to \mathbb{R}^4 for $m = 2$.

Another application of the estimates of F and (0.1) is that one can prove the preservation of the decay rate of \mathcal{R}_0 in a certain sense. For example, we will prove in Theorem 2.3 that if $\int_0^r sk(x, s) ds \leq C \log(1+r)$ (or $C(1+r)$), where $k(x, r)$ is the average of the scalar curvature at $t = 0$, then we still have $\int_0^r sk_t(x, s) ds \leq C' \log(1+r)$ ($C'(1+r)$, respectively), where $k_t(x, r)$ is the average of the scalar curvature at time t . Note that the constant C' is independent of t . This might be useful in analyzing the singularity models obtained by the blow-up procedure as in [H3].

From the methods of proof of the estimates of F , we can show that, under a rather weak decay condition on \mathcal{R}_0 , the volume growth is preserved in the sense that for any $t > 0$,

$$\lim_{r \rightarrow \infty} \frac{V_t(o, r)}{V_0(o, r)} = 1$$

where $V_t(o, r)$ is the volume of the geodesic ball with center at o and radius r with respect to $g_{\alpha\bar{\beta}}(x, t)$. This generalizes the results of [H3, Sh2, C-Z, C-T-Z].

In [Sh2], under the assumption that $\theta = 2$ in (0.3) and that M has positive holomorphic bisectional curvature, Shi proved that the rescaled metric $\hat{g}_{\alpha\bar{\beta}}(x, t) = g_{\alpha\bar{\beta}}(x, t)/g_{v\bar{v}}(x_0, t)$ subconverges to a flat Kähler metric on M , where x_0 is a fixed point and v is a fixed nonzero $(1,0)$ vector at x_0 . If M has maximal volume growth and if the limit metric is complete, then one can conclude that M is biholomorphic to \mathbb{C}^m . It is pointed out in [C-Z] that from [Sh2] it is unclear why the property of completeness is true. In Proposition 3.1, we will prove that if the scalar curvature \mathcal{R}_0 has pointwise quadratic decay, then the largest eigenvalue of the limit metric with respect to the initial metric grows at least like $r_0^a(x)$ for some $a > 0$, where $r_0(x)$ is the distance function to a fixed point with respect to the initial metric. This is a consequence of the result that volume elements of the rescaled metrics converge to the solution of the Poincaré-Lelong equation constructed in [N-S-T], see Theorem 3.1. We believe that this new piece of information will be helpful in studying the completeness of the limiting metric.

Here is how we organize this paper. In §1, we will give an alternate proof of long time existence for (0.2). In §2, we will give more refined estimates for $F(x, t)$ together with some applications. In §3, we will study the asymptotic behavior of $F(x, t)$.

We shall use the differential inequalities for Kähler-Ricci flow of Cao [Co2-3] from time to time, which is also called Harnack inequality for the Ricci flow (Cf. [H4]) since it implies a Harnack type estimate. Since this and similar results originate from the fundamental work of Li-Yau [L-Y] and Hamilton [H4], it seems to be more appropriate to call them Li-Yau-Hamilton type inequalities. We shall adopt this terminology in this work.

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1. Long time existence via Poincaré-Lelong equation.

Let $(M^m, g_{\alpha\bar{\beta}}(x))$ be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Consider the Kähler-Ricci flow:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x, t) = -R_{\alpha\bar{\beta}}(x, t) \quad (1.1)$$

such that $g_{\alpha\bar{\beta}}(x, 0) = g_{\alpha\bar{\beta}}(x)$.

In [Sh1-3], short time existence of (1.1) was established, and the long time existence was also proved under the assumption that

$$\int_{B(x,r)} \mathcal{R}_0 dV \leq Cr^{-\theta} \quad (1.2)$$

for some constants C and $\theta > 0$ for all x and r . Here \mathcal{R}_0 is the scalar curvature of the initial metric and $\int_{B(x,r)} \mathcal{R}_0 dV$ is the average of \mathcal{R}_0 on the geodesic ball $B(x, r)$ with center at x and radius r . The proof of the long time existence in [Sh2, Sh3] is rather complicated. In this section, with the help of solutions of the Poincaré-Lelong equation we shall give a simple proof of the long time existence by using a maximum principle. Our assumption on \mathcal{R}_0 is a little bit different from (1.2).

Let us recall the result on short time existence of Shi [Sh3].

Theorem 1.1. *Let $(M^m, g_{\alpha\bar{\beta}}(x))$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that the scalar curvature \mathcal{R}_0 is bounded by C_0 . Then (1.1) has a solution on $M \times [0, T)$ for some $T > 0$ depending only m and C_0 such that the following are true.*

- (i) $(M, g_{\alpha\bar{\beta}}(x, t))$ is a Kähler metric with nonnegative holomorphic bisectional curvature for $0 \leq t < T$.
- (ii) There exists $C > 0$ such that

$$C^{-1}g_{\alpha\bar{\beta}}(x, 0) \leq g_{\alpha\bar{\beta}}(x, t) \leq g_{\alpha\bar{\beta}}(x, 0), \quad (1.3)$$

and

$$0 \leq \mathcal{R}(x, t) \leq C \quad (1.4)$$

for all $(x, t) \in M \times [0, T)$.

Before we give our proof on the long time existence, let us fix the notations. For any smooth function f , let $\Delta f = g^{\alpha\bar{\beta}}(x, t) \frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta}$, $|\nabla f|^2 = g^{\alpha\bar{\beta}}(x, t) f_\alpha f_{\bar{\beta}}$. Summation convention is understood. We also use $\tilde{\Delta}$ and $\tilde{\nabla}$ to denote the Laplacian and the gradient with respect to a fixed metric $g_{\alpha\bar{\beta}}(x)$ or the initial metric $g_{\alpha\bar{\beta}}(x, 0)$ of the solution of (1.1). $B_t(x, r)$ is the geodesic ball of radius r with respect to the metric $g_{\alpha\bar{\beta}}(x, t)$ and $V_t(x, r)$ be the volume of $B_t(x, r)$ with respect to $g_{\alpha\bar{\beta}}(x, t)$. We may also use the ones without t to denote the balls and volumes for a fixed metric. The same convention applies to the distance function $r_t(x, y)$ between two points $x, y \in M$ as well as the volume element dV_t . As in [Sh2], throughout this work, let

$$F(x, t) = \log \left(\frac{\det(g_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))} \right).$$

Then for the solution of (1.1)

$$dV_t = e^F dV, \quad (1.5)$$

$$F(x, t) = - \int_0^t \mathcal{R}(x, \tau) d\tau \quad (1.6)$$

where $\mathcal{R}(x, t)$ is the scalar curvature of the metric $g_{\alpha\bar{\beta}}(x, t)$. For the solution of (1.1), we have the following maximum principle, which is of independent interest. The proof follows the idea in [K-L] (see also Li's lecture notes [Li]).

Let $g_{ij}(x, t)$ be a smooth family of complete Riemannian metrics defined on M with $0 \leq t \leq T_1$ for some $T_1 > 0$ satisfying the following properties: There exists a constant $C_1 > 0$ such that for any $T_1 \geq t_2 \geq t_1 \geq 0$

$$C_1 g_{ij}(x, t_1) \leq g_{ij}(x, t_2) \leq g_{ij}(x, t_1) \quad (1.7)$$

for all $x \in M$.

Theorem 1.2. *With the above assumptions and notations, let $f(x, t)$ be a smooth function such that $(\Delta - \frac{\partial}{\partial t})f(x, t) \geq 0$ whenever $f(x, t) \geq 0$. Assume that*

$$\int_0^{T_1} \int_M \exp(-ar_0^2(x)) f_+^2(x, s) dV_0 ds < \infty \quad (1.8)$$

for some $a > 0$, where $r_0(x)$ is the distance function to a fixed point $o \in M$ with respect to $g_{ij}(x, 0)$. Suppose $f(x, 0) \leq 0$ for all $x \in M$. Then $f(x, t) \leq 0$ for all $(x, t) \in M \times [0, T_1]$.

Proof. Let $F(x, t)$ be such that $dV_t = e^F(x, t)dV_0$. By (1.7), we have

$$\frac{\partial}{\partial t} F \leq 0. \quad (1.9)$$

Let $0 < T \leq T_1$ which will be specified later and let

$$g(x, t) = \frac{-r_T^2(x)}{4(2T - t)}, \quad \text{on } M \times [0, T].$$

Here $r_T(x)$ is the distance function to $o \in M$ with respect to $g_{\alpha\bar{\beta}}(x, T)$. It is easy to check that

$$|\nabla_T g|^2 + \frac{\partial g}{\partial t} = 0$$

Here ∇_T is the gradient with respect to $g_{\alpha\bar{\beta}}(x, T)$. By (1.7), g_{ij} is nonincreasing in t , hence we have

$$|\nabla g|^2 + \frac{\partial g}{\partial t} \leq |\nabla_T g|^2 + \frac{\partial g}{\partial t} = 0, \quad (1.10)$$

for $t \in [0, T]$. Let $\varphi(x)$ be a cut-off function which we will specify later. We have

$$\begin{aligned} 0 &\leq \int_0^T \int_M \varphi^2 e^g f_+ \left(\Delta - \frac{\partial}{\partial t} \right) f dV_s ds \\ &= \int_0^T \int_M \varphi^2 e^g f_+ (\Delta f) dV_s ds - \frac{1}{2} \int_0^T \int_M \varphi^2 e^g \frac{\partial}{\partial t} (f_+^2) dV_s ds. \end{aligned} \quad (1.11)$$

Here $f_+ := \max\{0, f\}$. Now we calculate the last two terms in the above

inequality.

$$\begin{aligned} \int_M \varphi^2 e^g f_+(\Delta f) dV_s &= - \int_M \varphi^2 e^g |\nabla f_+|^2 dV_s - 2 \int_M \varphi e^g \langle \nabla \varphi, \nabla f_+ \rangle f_+ dV_s \\ &\quad - \int_M \varphi^2 e^g f_+ \langle \nabla g, \nabla f_+ \rangle dV_s \\ &\leq 2 \int_M e^g f_+^2 |\nabla \varphi|^2 dV_s + \frac{1}{2} \int_M \varphi^2 e^g f_+^2 |\nabla g|^2 dV_s. \end{aligned} \quad (1.12)$$

On the other hand,

$$\begin{aligned} -\frac{1}{2} \int_0^T \int_M \varphi^2 e^g \frac{\partial}{\partial t} (f_+^2) dV_s ds &= \frac{1}{2} \left[- \int_M \varphi^2 e^g f_+^2 dV_s \Big|_0^T \right. \\ &\quad \left. + \int_0^T \int_M \varphi^2 e^g g_s f_+^2 dV_s ds + \int_0^T \int_M \varphi^2 e^g f_+^2 F_s(y, s) dV_s ds \right] \\ &\leq \frac{1}{2} \left[- \int_M \varphi^2 e^g f_+^2 dV_s \Big|_0^T + \int_0^T \int_M \varphi^2 e^g g_s f_+^2 dV_s ds \right] \end{aligned} \quad (1.13)$$

where we have used (1.8). Combining (1.10)–(1.13), we have that

$$\int_M \varphi^2(x) e^{g(x,T)} f_+^2(x, T) dV_T \leq 4 \int_0^T \int_M e^g f_+^2 |\nabla \varphi|^2 dV_s ds.$$

Now using (1.7) we have

$$\int_M \varphi^2(x) e^{g(x,T)} f_+^2(x, T) dV_T \leq C_3 \int_0^T \int_M e^g f_+^2 |\tilde{\nabla} \varphi|^2 dV_0 ds \quad (1.14)$$

for some constant C_3 depending on C_1 in (1.7). Here $\tilde{\nabla}$ is the gradient with respect the initial metric $g_{ij}(x, 0)$. For $R > 0$, let φ be the function with compact support such that

$$\begin{aligned} \varphi(x) &= 1, \quad \text{for } x \in B_0(o, R); \\ \varphi(x) &= 0, \quad \text{for } x \in M \setminus B_0(o, 2R); \\ |\tilde{\nabla} \varphi| &\leq \frac{2}{R}. \end{aligned}$$

Letting $R \rightarrow \infty$ in (1.14) we have that

$$\int_M e^{g(x,T)} f_+^2(x, T) dV_T \leq \liminf_{R \rightarrow \infty} \frac{4C_3}{R^2} \int_0^T \int_{B_0(o, 2R) \setminus B_0(o, R)} e^{-\frac{r_0^2(x)}{C_4 T}} f_+^2 dV_0 ds$$

for some constant $C_4 > 0$ depending only on C_1 in (1.7). Now if $T < \frac{1}{aC_4}$, by (1.9), we will have

$$\int_M e^{g(x,T)} f_+^2(x,T) dV_T \leq 0.$$

This implies that $f(x,T) \leq 0$. Since C_4 depends only on C_1 , iterating this procedure we complete the proof of the theorem. \square

Let $g_{\alpha\bar{\beta}}(x,t)$ be a solution of (1.1) on $M \times [0, T)$, which is Kähler for all t . We have the following easy lemma.

Lemma 1.1. *Suppose there is a function $u_0(x)$ such that*

$$\sqrt{-1}\partial\bar{\partial}u_0 = \text{Ric}(g(\cdot, 0)) \quad (1.15)$$

where $\text{Ric}(g(0))$ is the Ricci form of the initial metric $g(0)$. Let F be the ratio of the volume element as in (1.5) and let $u(x,t) = u_0(x) - F(x,t)$. Then

$$\sqrt{-1}\partial\bar{\partial}u = \text{Ric}(g(t)), \quad (1.16)$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) u(x,t) = 0, \quad (1.17)$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) |\nabla u|^2 = \|u_{\alpha\beta}\|^2 + \|u_{\alpha\bar{\beta}}\|^2, \quad (1.18)$$

$$\left(\Delta - \frac{\partial}{\partial t}\right) (|\nabla u|^2 + 1)^{\frac{1}{2}} \geq 0, \quad (1.19)$$

and

$$\left(\Delta - \frac{\partial}{\partial t}\right) \mathcal{R} = \left(\Delta - \frac{\partial}{\partial t}\right) u_t = -\|u_{\alpha\bar{\beta}}\|^2. \quad (1.20)$$

Here $\|u_{\alpha\bar{\beta}}\|^2(x,t) = g^{\alpha\bar{\beta}}(x,t)g^{\gamma\bar{\delta}}(x,t)u_{\alpha\bar{\delta}}(x,t)u_{\gamma\bar{\beta}}(x,t)$, $\|u_{\alpha\beta}\|^2(x,t) = g^{\alpha\bar{\beta}}(x,t)g^{\gamma\bar{\delta}}(x,t)u_{\alpha\gamma}(x,t)u_{\bar{\beta}\bar{\delta}}(x,t)$.

Proof. (1.16) and (1.17) follow from the fact that $g_{\alpha\bar{\beta}}(x,t)$ is a solution of (1.1) which is Kähler, and the definition of F and u_0 .

To prove (1.18), after choosing a normal coordinates with respect to $g_{\alpha\bar{\beta}}(x,t)$ near any fixed point

$$\begin{aligned} \Delta |\nabla u|^2 &= g^{\gamma\bar{\delta}} \left(u_{\alpha} u_{\bar{\beta}} g^{\alpha\bar{\beta}} \right)_{\gamma\bar{\delta}} \\ &= u_{\alpha\gamma} u_{\bar{\alpha}\bar{\gamma}} + u_{\alpha\bar{\gamma}} u_{\bar{\alpha}\gamma} + (\Delta u)_{\alpha} u_{\bar{\alpha}} + u_{\alpha} (\Delta u)_{\bar{\alpha}} + u_{\alpha\bar{\beta}} u_{\alpha} u_{\bar{\beta}}, \end{aligned}$$

where we have used (1.1) and (1.16). Using (1.1), we have

$$\frac{\partial}{\partial t} |\nabla u|^2 = (u_t)_\alpha u_{\bar{\alpha}} + u_\alpha (u_t)_{\bar{\alpha}} + u_{\alpha\bar{\beta}} u_{\alpha\bar{\beta}}.$$

Combining this with (1.17), we have (1.18). (1.19) follows from (1.18) by direct computations.

To prove (1.20), differentiate (1.17) with respect to t . Using (1.16) we have

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t} \right) \mathcal{R} &= \left(\Delta - \frac{\partial}{\partial t} \right) u_t \\ &= -g_t^{\alpha\bar{\beta}} u_{\alpha\bar{\beta}} \\ &= g^{\xi\bar{\beta}} g^{\alpha\bar{\gamma}} g_{\xi\bar{\gamma},t} u_{\alpha\bar{\beta}} \\ &= -g^{\xi\bar{\beta}} g^{\alpha\bar{\gamma}} R_{\xi\bar{\gamma}} u_{\alpha\bar{\beta}} \\ &= -g^{\xi\bar{\beta}} g^{\alpha\bar{\gamma}} u_{\xi\bar{\gamma}} u_{\alpha\bar{\beta}} \\ &= -\|u_{\alpha\bar{\beta}}\|^2. \end{aligned}$$

This completes the proof of the lemma. \square

We are ready to prove the long time existence.

Theorem 1.3. *Let $(M^m, g_{\alpha\bar{\beta}}(x, t))$ be a complete noncompact Kähler manifold with nonnegative holomorphic bisectional curvature such that its scalar curvature \mathcal{R}_0 is bounded and satisfies*

$$\int_0^\infty k(x, s) ds \leq C_1 \tag{1.21}$$

for some constant C_1 for all x and r , where

$$k(x, s) = \int_{B(x,s)} \mathcal{R}_0 dV.$$

Then (1.1) has long time existence. Moreover, there is a function $u(x, t)$ such that

$$\sqrt{-1} \partial \bar{\partial} u(\cdot, t) = Ric(g(t)), \tag{1.22}$$

$$|\nabla u| \leq C(m)C_1, \tag{1.23}$$

and

$$\mathcal{R}(x, t) + |\nabla u|^2(x, t) \leq \sup_{x \in M} \left(\mathcal{R}_0(x) + |\tilde{\nabla} u_0|^2(x) \right) \leq \sup_{x \in M} \mathcal{R}_0(x) + (C(m)C_1)^2 \tag{1.24}$$

for some constant positive $C(m)$ depending only on m and for all (x, t) . Moreover, the equality holds for some (x_0, t_0) , with $t_0 > 0$ if and only if $g_{\alpha\bar{\beta}}(x, t)$ is a Kähler-Ricci soliton.

Proof. By Theorem 1.1, there is a maximal $\infty \geq T_{\max} > 0$ such that (1.1) has a solution $g_{\alpha\bar{\beta}}(x, t)$ which satisfies condition (i) in Theorem 1.1 for $0 \leq t < T_{\max}$, and satisfies the following condition: For any $0 < T < T_{\max}$, there is a constant $C > 0$ such that (1.3) and (1.4) are true on $M \times [0, T]$. By (1.21) and the results in [N-S-T, Theorems 1.3 and 5.1], there is a function $u_0(x)$ such that

$$\sqrt{-1}\partial\bar{\partial}u_0 = \text{Ric}(g(0))$$

and

$$|\tilde{\nabla}u_0|(x) \leq C(m)C_1 \tag{1.25}$$

for all x for some constant $C(m)$ depending only on m . Let $u(x, t) = u_0(x) - F(x, t)$ and let $0 < T < T_{\max}$ be fixed. By (1.3), (1.4), (1.6) and (1.25), it is easy to see that there is a constant C_2 such that for $(x, t) \in M \times [0, T]$

$$|u(x, t)| \leq C_2(r_0(x) + 1) \tag{1.26}$$

where $r_0(x)$ is the distance from a fixed point o with respect to $g(0)$. By Lemma 1.1 (1.16), we have $\Delta u(x, t) = \mathcal{R}(x, t)$. Combining this with (1.4) and (1.26), it is not hard to prove that

$$\int_{B_t(o, r)} |\nabla u|^2 \leq C_3 r^{2m+1} \tag{1.27}$$

for some constant C_3 for all $0 \leq t \leq T$ and for all r . Here we have used the fact that $g_{\alpha\bar{\beta}}(x, t)$ has nonnegative Ricci curvature and volume comparison. Hence using (1.3), we conclude that the function $f = (|\nabla u|^2 + 1)^{\frac{1}{2}} - (C^2(m)C_1^2 + 1)^{\frac{1}{2}}$ satisfies the condition (1.8) in Theorem 1.2 with T_1 replaced by T . Here $C(m)$ is the constant in (1.25). By (1.19) of Lemma 1.1 and Theorem 1.2, we can conclude that (1.23) is true for $x \in M$ and $0 \leq t \leq T_{\max}$, because T can be any positive number less than T_{\max} .

By (1.18) and (1.20) of Lemma 1.1, we have

$$\left(\Delta - \frac{\partial}{\partial t}\right) (|\nabla u|^2 + \mathcal{R}) = \|u_{\alpha\bar{\beta}}\|^2. \tag{1.28}$$

By (1.23) and (1.4), we conclude that $|\nabla u|^2 + \mathcal{R}$ is uniformly bounded on $M \times [0, T]$. By (1.28), we can apply Theorem 1.2 again and conclude that

(1.24) is true for all $x \in M$ and $0 \leq t \leq T_{\max}$. In particular \mathcal{R} is uniformly bounded on $M \times [0, T_{\max}]$. By Theorem 1.1, T_{\max} must be infinity. If for some (x_0, t_0) , $t_0 > 0$,

$$(\mathcal{R} + |\nabla u|^2)(x_0, t_0) = \sup_{x \in M} (\mathcal{R} + |\nabla u|^2)(x, 0)$$

we can conclude that $\mathcal{R}(x, t) + |\nabla u|^2(x, t)$ is constant, by the strong maximum principle. Thus $u_{\alpha\beta}(x, t) = 0$ by (1.28). Together with the fact $u_{\alpha\bar{\beta}}(x, t) = R_{\alpha\bar{\beta}}(x, t)$, it implies that $g_{\alpha\bar{\beta}}(x, t)$ is a Kähler-Ricci soliton. It is easy to check that for a Kähler Ricci soliton (1.24) holds with the equality (Cf. [C-H]). \square

2. Some properties preserved by the Kähler-Ricci flow.

In this section, we shall investigate the behavior of $\int_{B_t(x_0, r)} \mathcal{R} dV_t$. To do this, we shall give some generalizations of the estimates in [Sh2-3, C-Z, C-T-Z] from above and below on the volume element $F(x, t)$ defined in (1.5). More precisely, we shall obtain upper and lower estimates on $F(x, t)$ in terms of the integral

$$\int_0^r sk(x, s) ds$$

where $k(x, s)$ is the average of the scalar curvature \mathcal{R}_0 over $B_0(x, s)$ at $t = 0$. Our proofs use the well-known estimates of the heat kernels and the Green's functions for manifolds with nonnegative Ricci curvature of Li-Yau [L-Y]. Our proofs seem to be simpler than those in [Sh2-3], etc. Also we do not use the complicated construction of exhaustion functions as in the [Sh2-3, C-Z, C-T-Z]. To derive our estimates we need the following lemma, which is a direct consequence of the mean value inequality of Li-Schoen [L-S] on subharmonic functions.

Lemma 2.1 (Generalized mean value inequality). *Let M^n be a complete noncompact Riemannian manifold with nonnegative Ricci curvature with real dimension n . Let $u \geq 0$ be a smooth function such that $\tilde{\Delta} u \geq -f$ with $f \geq 0$. For any $x_0 \in M$ and $r > 0$, we have*

$$u(x_0) \leq \int_{B(x_0, r)} G_r(x_0, y) f(y) dy + C(n) \int_{B(x_0, r)} u \quad (2.1)$$

for some constant $C(n)$ depending only on n , where $G_r(x, y)$ is the positive Green's function on $B(x_0, r)$ with zero boundary value.

Proof. Let v be such that $\tilde{\Delta}v = -f$ on $B(x_0, r)$ and $v = 0$ on $\partial B(x_0, r)$. Note that $v \geq 0$ in $B(x_0, r)$. Since $w = \max\{u - v, 0\}$ is Lipschitz, subharmonic and nonnegative, by the mean value inequality of Li-Schoen [L-S], we have

$$w(x_0) \leq C \int_{B(x_0, r)} w$$

for some constant $C = C(n)$ depending only on n . If $u(x_0) - v(x_0) \leq 0$, then we have

$$u(x_0) \leq v(x_0) = \int_{B(x_0, r)} G_r(x_0, y) f(y) dy.$$

In this case, (2.1) is true. If $u(x_0) - v(x_0) > 0$ then

$$\begin{aligned} u(x_0) &= w(x_0) + v(x_0) \\ &\leq C \int_{B(x_0, r)} w + v(x_0) \\ &\leq C \int_{B(x_0, r)} u + v(x_0) \\ &\leq C \int_{B(x_0, r)} u + \int_{B(x_0, r)} G_r(x_0, y) f(y) dy. \end{aligned}$$

Therefore (2.1) is also true for this case. □

We should mention that the above lemma was also proved in a somewhat different form in [Sh2-3] with a more complicated proof (Cf. Lemma 6.10 of [Sh2] and Lemma 6.8 of [Sh3]). We also need the following estimates of Green's functions.

Lemma 2.2. *Let M^n be as in Lemma 1.1. For any function $f \geq 0$, let $k(x, r) = \int_{B(x, r)} f$. Then we have*

$$\int_{B(x, r)} G_r(x, y) f(y) dy \geq C(n) \left(r^2 k(x, \frac{r}{5}) + \int_0^{\frac{r}{5}} s k(x, s) dr \right),$$

for some constant $C(n) > 0$ depending only on n , where G_r is the Green's function on $B(x, r)$ where zero boundary value. If in addition, M supports a minimal positive Green's function $G(x, y)$ such that

$$\alpha \cdot \frac{r^2(x, y)}{V(x, r(x, y))} \leq G(x, y) \leq \frac{1}{\alpha} \cdot \frac{r^2(x, y)}{V(x, r(x, y))}.$$

for some $\alpha > 0$ for all $x, y \in M$, then

$$\int_{B(x,r)} G(x,y)f(y)dy \leq C(n,\alpha) \left(r^2 k(x,r) + \int_0^r sk(x,s)dr \right),$$

for some positive constant $C(n,\alpha)$ depending only on n and α .

Proof. See the proofs of [N-S-T, Theorems 1.1, 2.1]. \square

In the rest of this section, we assume M^m is a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature such that $g_{\alpha\bar{\beta}}$ is a solution of (1.1) on $M \times [0, T]$ with $T \leq \infty$. We also assume that conditions (i) and (ii) are satisfied by $g_{\alpha\bar{\beta}}$ on $M \times [0, T_1]$ for any $T_1 < T$. Let $\mathbf{m}(t) = \inf_M F(\cdot, t)$. Then $\mathbf{m}(t) \leq 0$.

With the notations as in §1, we also need the following result of Shi [Sh3, p. 156].

Lemma 2.3.

$$\begin{aligned} \mathcal{R}_0(x) &\geq \mathcal{R}_0(x) + e^F F_t \\ &\geq \mathcal{R}_0(x) - g^{\alpha\bar{\beta}}(x, 0) R_{\alpha\bar{\beta}}(x, t) \\ &= \tilde{\Delta} F(x, t) \\ &\geq \mathcal{R}_0(x) - \mathcal{R}(x, t) \end{aligned} \tag{2.2}$$

where $\tilde{\Delta}$ is the Laplacian of the metric $g(0)$.

Theorem 2.1. *With the above assumptions and notations, the following estimates are true. Namely there exists $C_1 > 0$ depending only on m such that for all $(x_0, t) \in M \times [0, T]$*

$$-F(x_0, t) \geq C_1^{-1} \int_0^{\sqrt{t}} sk(x_0, s)ds \tag{2.3}$$

and

$$-F(x_0, t) \leq C_1 \left[\left(1 + \frac{t(1-\mathbf{m}(t))}{R^2} \right) \int_0^R sk(x_0, s)ds - \frac{t\mathbf{m}(t)(1-\mathbf{m}(t))}{R^2} \right], \tag{2.4}$$

where $k(x_0, t) = \int_{B_0(x_0, r)} \mathcal{R}_0 dV_0$.

Proof. To prove (2.3), by Lemma 2.3 we have

$$\tilde{\Delta}F \geq \mathcal{R}_0 - \mathcal{R} = \mathcal{R}_0 + F_t$$

and so

$$\left(\tilde{\Delta} - \frac{\partial}{\partial t}\right)(-F) \leq -\mathcal{R}_0. \quad (2.5)$$

Let $H(x, y, t)$ be the heat kernel of M with respect to the metric $g(0)$, and let

$$v(x, t) = \int_0^t \int_M H(x, y, t) \mathcal{R}_0(y) dV_0(y).$$

Then $\tilde{\Delta}v - v_t = -\mathcal{R}_0$ and $v = 0$ at $t = 0$. By (2.5) and the fact that $F(\cdot, 0) \equiv 0$, by the maximum principle and the estimate of the heat kernel [L-Y], we have for $(x, t) \in M \times [0, T]$

$$\begin{aligned} -F(x, t) &\geq v(x, t) \\ &= \int_0^t \int_M H(x, y, \tau) \mathcal{R}_0(y) dV_0 d\tau \\ &\geq C_2 \int_0^t \int_0^\infty \frac{1}{V_0(x, \sqrt{\tau})} e^{-\frac{r^2}{5\tau}} \int_{\partial B_0(x, r)} \mathcal{R}_0(y) dA_0 dr d\tau \\ &\geq C_2 \int_0^t \int_0^{\sqrt{\tau}} \frac{1}{V_0(x, \sqrt{\tau})} e^{-\frac{r^2}{5\tau}} \int_{\partial B_0(x, r)} \mathcal{R}_0(y) dA_0 dr d\tau \\ &= C_3 \int_0^t k(x, \sqrt{\tau}) d\tau \\ &= 2C_3 \int_0^{\sqrt{t}} \tau k(x, \tau) d\tau. \end{aligned}$$

for some positive constants $C_2 - C_3$ depending only on m . Hence (2.3) is true.

To prove (2.4), by Lemma 2.3, $\tilde{\Delta}F \leq \mathcal{R}_0 + e^F F_t$. Hence for any $(x_0, t) \in M \times [0, T]$ for any $R > 0$, integrating the above inequality over $B_0(x_0, R) \times [0, t]$, we have

$$\begin{aligned} &\int_0^t \int_{B_0(x_0, R)} G_R(x_0, y) \tilde{\Delta}F(y, s) dV_0 ds \\ &\leq t \int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 + \int_{B_0(x_0, R)} G_R(x_0, y) (e^{F(y, t)} - 1) dV_0, \end{aligned}$$

and

$$\begin{aligned} & \int_{B_0(x_0, R)} G_R(x_0, y)(1 - e^{F(y, t)}) dV_0 \quad (2.6) \\ & \leq t \int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 + \int_0^t \int_{B_0(x_0, R)} G_R(x_0, y) \tilde{\Delta}(-F(y, s)) dV_0. \end{aligned}$$

By the Green's formula, for each $0 \leq s \leq t$

$$\begin{aligned} \int_{B_0(x_0, R)} G_R(x_0, y) \tilde{\Delta}(-F(y, s)) dV_0 &= F(x_0, s) + \int_{\partial B_0(x_0, R)} F(y, s) \frac{\partial G_R(x_0, y)}{\partial \nu} \\ &\leq -\mathbf{m}(t), \end{aligned}$$

where we have used the fact that $\mathbf{m}(t)$ is nonincreasing, $F \leq 0$, $\frac{\partial}{\partial \nu} G_R(x_0, y) \leq 0$ and $\int_{\partial B_0(x_0, R)} \frac{\partial}{\partial \nu} G_R(x_0, y) = -1$. Combining this with (2.6), we have

$$\int_{B_0(x_0, R)} G_R(x_0, y)(1 - e^{F(y, t)}) dV_0 \leq t \left(\int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 - \mathbf{m}(t) \right).$$

Using the first inequality in Lemma 2.2, this implies

$$R^2 \int_{B_0(x_0, \frac{1}{5}R)} (1 - e^{F(y, t)}) dV_0 \leq C_4 t \left(\int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 - \mathbf{m}(t) \right) \quad (2.7)$$

for some constant C_4 depending only on m . Since if $0 \leq x \leq 1$, $1 - e^{-x} \geq \frac{1}{3}x$, we have $(1 - e^F)(1 - \mathbf{m}(t)) \geq -CF$ for some absolute positive constant C . Hence (2.7) implies that

$$\begin{aligned} & R^2 \int_{B_0(x_0, \frac{1}{5}R)} (-F(y, t)) dV_0 \\ & \leq C_5 t (1 - \mathbf{m}(t)) \left(\int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 - \mathbf{m}(t) \right) \quad (2.8) \end{aligned}$$

for some constant C_5 depending only on m . By Lemma 2.3, $\tilde{\Delta}(-F) \geq -\mathcal{R}_0$. By Lemma 2.1 and (2.8), there is a constant C_6 depending only on m such

that

$$\begin{aligned}
-F(x_0, t) &\leq \int_{B_0(x_0, \frac{1}{5}R)} G_{\frac{1}{5}R}(x_0, y) \mathcal{R}_0(y) dV_0 + C(n) \int_{B_0(x_0, \frac{1}{5}R)} (-F(y, t)) dV_0 \\
&\leq \int_{B_0(x_0, \frac{1}{5}R)} G_{\frac{1}{5}R}(x_0, y) \mathcal{R}_0(y) dV_0 \\
&\quad + \frac{C_6 t (1 - \mathbf{m}(t))}{R^2} \left(\int_{B_0(x_0, R)} G_R(x_0, y) \mathcal{R}_0(y) dV_0 - \mathbf{m}(t) \right), \quad (2.9)
\end{aligned}$$

where $G_{\frac{1}{5}R}$ is the Green's function on $B_0(x_0, \frac{1}{5}R)$. As in [Sh3], by considering $M \times \mathbb{C}^2$, we may assume that M has positive Green's function which satisfies the condition in Lemma 2.2. Applying Lemma 2.2, we can conclude from (2.9) that

$$-F(x_0, t) \leq C_7 \left[\left(1 + \frac{t(1 - \mathbf{m}(t))}{R^2} \right) \int_0^{2R} sk(x_0, s) ds - \frac{t\mathbf{m}(t)(1 - \mathbf{m}(t))}{R^2} \right],$$

for some constant C_7 depending only on m . This completes the proof of the theorem. \square

Corollary 2.1. *Same assumptions and notations as in Theorem 2.1. Suppose $k(x, r) \leq k(r)$ for some function $k(r)$ for all $x \in M$. Then there exist positive constants C , a depending only on m such that for $0 \leq t < T$*

$$-\mathbf{m}(t) \leq C \int_0^{R^2} sk(s) ds \quad (2.10)$$

where $R^2 = at(1 - \mathbf{m}(t))$.

Proof. By (2.4), we have for any $R > 0$

$$-\mathbf{m}(t) \leq C_1 \left[\left(1 + \frac{t(1 - \mathbf{m}(t))}{R^2} \right) \int_0^R sk(s) ds - \frac{t\mathbf{m}(t)(1 - \mathbf{m}(t))}{R^2} \right]$$

where C_1 is a constant depending only on m . Let $R^2 = 2C_1 t(1 - \mathbf{m}(t))$, we have

$$-\mathbf{m}(t) \leq 2C_1 \left(1 + \frac{1}{2C_1} \right) \int_0^R sk(s) ds.$$

From this the result follows. \square

Corollary 2.2. *With the same assumptions as in Corollary 2.1. Suppose*

$$\int_0^r sk(s)ds \leq r^2\phi(r)$$

for all r , where $\phi(r)$ is a nonincreasing function of r such that $\lim_{r \rightarrow \infty} \phi(r) = 0$. For $0 < \tau \leq \sup \phi$, let

$$\psi(\tau) = \sup\{r \mid \phi(r) \geq \tau\}.$$

Then for $0 \leq t < T$,

$$-\mathbf{m}(t) \leq \max\left\{1, \frac{C'}{t}\psi^2\left(\frac{C''}{t}\right)\right\}$$

for some positive constants C' and C'' depending only on m . In particular, the Kähler-Ricci flow has long time existence.

Proof. Note the $\psi(\tau)$ is finite and nonincreasing for $0 < \tau \leq \sup \phi$ because $\phi(r) \rightarrow 0$ as $r \rightarrow \infty$. By Corollary 2.1, there exist constants a and C_1 depending only on m such that

$$-\mathbf{m}(t) \leq C_1 \int_0^{R^2} sk(s)ds \leq C_1 at(1 - \mathbf{m}(t))\phi\left(\sqrt{at(1 - \mathbf{m}(t))}\right)$$

where $R^2 = at(1 - \mathbf{m}(t))$. Suppose $-\mathbf{m}(t) \geq 1$, then the above inequality implies that

$$\phi\left(\sqrt{at(1 - \mathbf{m}(t))}\right) \geq \frac{1}{2C_1 at}.$$

In particular, $\frac{1}{2C_1 at} \leq \sup \phi$. Hence

$$\sqrt{at(1 - \mathbf{m}(t))} \leq \psi\left(\frac{1}{2C_1 at}\right).$$

Hence

$$-\mathbf{m}(t) \leq \max\left\{1, \frac{C'}{t}\psi^2\left(\frac{C''}{t}\right)\right\}$$

for some positive constants C' and C'' depending only on m .

The last statement follows from the method in [Sh3, §7]. Here we cannot use the method in Theorem 1.3 because we do not have a good solution for the Poincaré-Lelong equation. \square

Remark 2.1. *The condition for long time existence in the corollary is weaker than that in [Sh3]. In [C-T-Z], the long time existence is proved for the case of surfaces under the assumptions that the surface has maximal volume growth and that $\int_0^r sk(x_0, s) = o(r^2)$. The last assumption is a little bit weaker than ours.*

Remark 2.2. *By the corollary, we may have the estimates in [Sh2-3]. For example, if $k(r) = C(1+r)^{-2}$, then it is easy to see that $-\mathfrak{m}(t) \leq C \log(t+1)$. If $k(r) = C(1+r)^{-\theta}$ for $0 < \theta < 2$, then $-\mathfrak{m}(t) \leq C(t+1)^{(2-\theta)/\theta}$. In addition to these results in [Sh2-3], we may have the following estimate. Namely, if $\int_0^\infty k(r)dr < \infty$, then $-\mathfrak{m}(t) = o(t)$ and if $\int_0^r sk(s)ds \leq Cr^2 / \log(2+r)$, then we have $-\mathfrak{m}(t) \leq e^{Ct}$ for some $C > 0$.*

Another application of the corollary is a slight generalization of a gap theorem of Chen-Zhu [C-Z]. In [C-Z], it is proved that if M is a complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature such that

$$k(x_0, r) = \int_{B_0(x,r)} \mathcal{R}_0 dV_0 \leq \epsilon(r)r^{-2}$$

for all x and r , where $\epsilon(r) \rightarrow 0$ as $r \rightarrow \infty$. Then M must be flat. Note that under this condition, the Kähler-Ricci flow has long time solution such that $R(x, t)$ is uniformly bounded on $M \times [0, \infty)$ by Theorem 1.3 and so $-\mathfrak{m}(t) \leq Ct$. Moreover

$$\int_0^r sk(x, s)ds = o(\log r)$$

uniformly.

Using Corollary 2.1, we have:

Corollary 2.3. *Let (M^m, g) be complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature such that the Kähler-Ricci flow (1.1) has long time solution.*

- (a) *Suppose M is nonflat and $-\mathfrak{m}(t) \leq Ct^k$ for some constant C and $k > 0$. Then*

$$\liminf_{r \rightarrow \infty} \frac{\int_0^r sk(x, s)ds}{\log r} > 0, \tag{2.11}$$

$$\liminf_{t \rightarrow \infty} \frac{-F(x, t)}{\log t} > 0, \tag{2.12}$$

and

$$\liminf_{t \rightarrow \infty} t\mathcal{R}(x, t) > 0, \quad (2.13)$$

for all x , where $k(x, s) = \int_{B_0(x, r)} \mathcal{R}_0 dV_0$.

- (b) *If the Poisson equation $\tilde{\Delta}u = \mathcal{R}_0$ has a solution u which is bounded from above, then M is flat. In particular, any bounded from above solution is a constant.*

Proof. Note that if (2.11) is true for some x , it is true for all x . Suppose M is nonflat, then there exists x_0 such that $\mathcal{R}_0(x_0) > 0$. If (2.11) is not true, then there exists $R_i \rightarrow \infty$ such that

$$\int_0^{R_i} sk(x_0, s) ds \leq \frac{1}{i} \log R_i. \quad (2.14)$$

Let $t_i \rightarrow \infty$ be such that $t_i(1 - (\mathfrak{m}(t_i))^2) = R_i^2$. By (2.4), we have

$$\begin{aligned} -F(x_0, t_i) &\leq C_1 \left(\int_0^{R_i} sk(x_0, s) ds + 1 \right) \\ &\leq C_1 \left(\frac{1}{i} \log R_i + 1 \right) \\ &\leq C_2 \left(\frac{1}{i} \log t_i + 1 \right) \end{aligned} \quad (2.15)$$

for some constants $C_1 - C_2$ independent of i . Here we have used the assumption that $-\mathfrak{m}(t) \leq Ct^k$. We can then proceed as in [C-Z]. For any $T > 0$, by the Li-Yau-Hamilton type inequality [Co2-3] for $t > T$,

$$\frac{T}{t} \mathcal{R}(x_0, T) \leq \mathcal{R}(x_0, t).$$

Integrating from T to t_i , we have

$$T \log \frac{t_i}{T} \mathcal{R}(x_0, T) \leq -F(x_0, t_i) \leq C_2 \left(\frac{1}{i} \log t_i + 1 \right).$$

Dividing both sides by $\log t_i$ and let $t_i \rightarrow \infty$, we have $\mathcal{R}(x_0, T) = 0$. Since T is arbitrary, we conclude that $\mathcal{R}(x_0) = 0$. This is a contradiction. Hence (2.11) is true.

If (2.12) is not true for some x , then by (2.3) in Theorem 2.1, (2.11) is not true for this x . Hence M must be flat by the previous result.

By (2.12), for any $x \in M$ there exists $C_3 > 0$ and $t_0 > 0$ such that

$$-F(x, t) \geq C_3 \log t, \tag{2.16}$$

for all $t \geq t_0$. By the Li-Yau-Hamilton type inequality in [Co2-3], for all $t > t_0$ and $s \leq t$,

$$\frac{t}{s} \mathcal{R}(x, t) \geq \mathcal{R}(x, s).$$

Integrating over s from 1 to t and using (2.16) we have

$$\begin{aligned} (t \log t) \mathcal{R}(x, t) &\geq \int_1^t \mathcal{R}(x, s) ds \\ &= -F(x, t) - \int_0^1 \mathcal{R}(x, s) ds \\ &\geq C_3 \log t - \int_0^1 \mathcal{R}(x, s) ds. \end{aligned}$$

From this (2.13) follows.

The proof of (b) follows from the proof of (a) and Theorem 2.1 of [N-S-T].

□

Remark 2.3. *The argument above in fact also shows that any bounded solution to $\tilde{\Delta}u = \mathcal{R}_0(x)$ is a constant since if $\tilde{\Delta}u = \mathcal{R}_0(x)$ has a bounded solution, we then have long time solution to (0.2) by Theorem 1.3 and Theorem 2.1 of [N-S-T]. In [Cw], a gradient estimate is obtained for the Kähler-Ricci flow under the assumption that there is a bounded potential function for the Ricci tensor. If we assume the manifold has nonnegative holomorphic bisectional curvature, then this is only possible for flat manifolds.*

Corollary 2.4. *Same assumptions and notations as in Corollary 2.1. If we assume that*

$$\lim_{r \rightarrow \infty} \frac{\int_0^r sk(s) ds}{r} = 0$$

we have long time existence for the Kähler-Ricci flow with

$$\lim_{t \rightarrow \infty} \frac{-\mathbf{m}(t)}{t} = 0$$

and

$$\lim_{t \rightarrow \infty} R(x, t) = 0$$

uniformly for $x \in M$. If in addition, we assume that $(M, g(0))$ has maximum volume growth, M is diffeomorphic to \mathbb{R}^{2m} , in case $m \geq 3$ and homeomorphic to \mathbb{R}^4 , in case $m = 2$. Moreover, M is a Stein manifold.

Proof. The first part just follows from Corollary 2.1 and the Li-Yau-Hamilton type inequality of Cao [Co2-3] as in the proof of Corollary 2.3. To prove that M is Stein and topologically \mathbb{R}^{2m} one just need to use the observation that the injectivity radius of M has a uniform lower bound in the case of the maximum volume growth and bounded curvature tensor. Also $|R(x, t)| \rightarrow 0$, as $t \rightarrow \infty$, means that the Kähler-Ricci flow will improves the injectivity radius to ∞ along the flow. The rest argument is same as in section 3 of [C-Z]. \square

Another corollary of the proof of Theorem 2.1 is a result on the preservation of volume growth under the Kähler-Ricci flow. In [Sh2] it was proved that the property of having maximum volume growth is preserved under the assumption that $\mathcal{R}_0(x)$ is of quadratic decay. In [C-Z, C-T-Z] it was generalized to the case of more relaxed decay conditions on $\mathcal{R}_0(x)$ using the same argument as [Sh2]. In [H3], it was proved under the Ricci flow with nonnegative Ricci curvature, and under the stronger assumption that the Riemannian curvature tensor of the initial metric goes to zero pointwisely, then the volume ratio $\lim_{r \rightarrow \infty} r^{-n} V_t(r)$ is preserved. In our case, we have the following stronger result:

Theorem 2.2. *With the same assumptions and notations as in Theorem 2.1. Suppose*

$$\int_0^r sk(x, s)ds = o(r^2) \quad \text{as } r \rightarrow \infty.$$

Let $o \in M$ be a fixed point. Then for any $0 < t < T$,

$$\lim_{r \rightarrow \infty} \frac{V_t(o, r)}{V_0(o, r)} = 1$$

where $V_t(o, r)$ is the volume of the geodesic ball $B_t(o, r)$ with respect to the metric $g(t)$ for $0 \leq t < T$.

Proof. Since $\mathcal{R}(x, t)$ is uniformly bounded on $M \times [0, t]$, by Theorem 17.2 in [H3], $B_t(o, r) \subset B_0(o, r + C_1 t)$ for some constant C_1 independent of r . Using the fact that $g(t)$ is nonincreasing in t , we have that

$$\begin{aligned} V_t(o, r) &\leq V_t(B_0(o, r + C_1 t)) \\ &\leq V_0(B_0(o, r + C_1 t)) \\ &\leq V_0(o, r) \cdot \left(\frac{r + C_1 t}{r} \right)^{2m}. \end{aligned}$$

This implies that

$$\limsup_{r \rightarrow \infty} \frac{V_t(o, r)}{V_0(o, r)} \leq 1.$$

Using the fact that $g(t)$ is nonincreasing in t again, we have

$$\begin{aligned} V_t(o, r) &\geq \int_{B_0(o, r)} dV_t \\ &= \int_{B_0(o, r)} e^{F(y, t)} dV_0 \\ &= V_0(o, r) + \int_{B_0(o, r)} (e^{F(y, t)} - 1) dV_0. \end{aligned} \tag{2.17}$$

On the other hand, using (2.7) in the proof of Theorem 2.1 and using Lemma 2.2 as in the proof of (2.4), we have

$$\int_{B_0(o, r)} (1 - e^{F(y, t)}) dV_0 \leq C_2 r^{-2} t \left(\int_0^{10r} sk(o, s) ds - \mathfrak{m}(t) \right)$$

for some constant C_2 independent on r . Combining this with (2.17), we have

$$\frac{V_t(o, r)}{V_0(o, r)} \geq 1 - C_2 r^{-2} t \left(\int_0^{10r} sk(o, s) ds - \mathfrak{m}(t) \right).$$

Since $\int_0^R sk(s) ds = o(R^2)$, we have

$$\liminf_{r \rightarrow \infty} \frac{V_t(o, r)}{V_0(o, r)} \geq 1.$$

The theorem then follows. □

It was proved in [H3] that the condition $|Rm| \rightarrow 0$ as $x \rightarrow \infty$ is preserved under the Ricci flow. Applying Theorem 2.1, we can prove that the decay rate of the scalar curvature in the average sense is preserved under the Kähler-Ricci flow in a certain sense.

Theorem 2.3. *Let M^m be a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. Suppose (1.1) has long time existence, such that for any $T > 0$ the conditions (i) and (ii) in Theorem 1.1 are satisfied. Then the following are true:*

- (a) *Suppose $\int_0^r sk(x, s) ds \leq C(1 + r)^{1-\epsilon}$ for some constants $C > 0$ and $\epsilon > 0$ for all x and r . Then $\int_0^r sk_t(x, s) ds \leq C'(1 + r)^\delta$ where $\delta = \min\{1, 2(1 - \epsilon)/(1 + \epsilon)\}$ for some constant C' independent of x, t, r .*

(b) Suppose $\int_0^r sk(x, s)ds \leq C \log(r + 2)$ for some constants $C > 0$ for all x and r . Then $\int_0^r sk_t(x, s)ds \leq C' \log(r + 2)$ for some constant C' independent of x, t, r .

Here $k(x, r) = \int_{B_0(x, r)} \mathcal{R}_0 dV_0$ and $k_t(x, r) = \int_{B_t(x, r)} \mathcal{R}(y, t) dV_t$.

Proof. We prove (b) first. For $T \geq 0$, let

$$F(x, t; T) = \log \left[\frac{\det(g_{\alpha\bar{\beta}}(x, t + T))}{\det(g_{\alpha\bar{\beta}}(x, T))} \right].$$

Considering the flow $g_{\alpha\bar{\beta}}(x, t + T)$ with initial data $g_{\alpha\bar{\beta}}(x, T)$ and using (2.3) in Theorem 2.1, we have for any $t > 0$

$$-F(x, t; T) \geq C_1 \int_0^{\sqrt{t}} sk_T(x, s)ds. \quad (2.18)$$

for some constant $C_1 > 0$ depending only on m . On the other hand, by the Li-Yau-Hamilton inequality [Co2-3]

$$T\mathcal{R}(x, T) \leq t\mathcal{R}(x, t)$$

for all $t \geq T$. We have

$$\int_T^t \frac{T}{s} \mathcal{R}(x, T) ds \leq \int_T^t \mathcal{R}(x, s) ds \leq -F(x, t; 0) \leq C_2 \log(t + 2)$$

for some constant C_2 independent of x and t , where we have used Corollary 2.1 and the assumption on $k(x, r)$. Dividing both sides by $\log t$ and let $t \rightarrow \infty$, using the fact that \mathcal{R} is uniformly bounded on $M \times [0, \infty)$ by Theorem 1.3, we have

$$\mathcal{R}(x, T) \leq \frac{C_3}{T + 1} \quad (2.19)$$

for some constant C_3 independent of x and t . Since the metric is nonincreasing along the Ricci flow $\det(g_{\alpha\bar{\beta}}(x, T)) \leq \det(g_{\alpha\bar{\beta}}(x, 0))$, by (2.18) and Theorem 2.1, for all $t > 0$

$$\begin{aligned} \log(t + T + 2) &\geq -C_4 F(x, t + T; 0) \\ &\geq -C_4 F(x, t; T) \\ &\geq C_5 \int_0^{\sqrt{t}} sk_T(x, s)ds \end{aligned} \quad (2.20)$$

for some positive constants $C_4 - C_5$ independent of x, t and T . Suppose $r^2 \geq T$, then we take $t = r^2$ in (2.20), we have

$$\int_0^r sk_T(x, s)ds \leq C_6 \log(r + 2) \tag{2.21}$$

for some constant C_6 independent of x, t, T . Suppose $r^2 \leq T$, then by (2.19), we have

$$\int_0^r sk_T(x, s)ds \leq C_3 \frac{r^2}{T + 1} \leq C_7 \log(r + 2) \tag{2.22}$$

where C_7 is a constant independent of x, t, T . (b) follows from (2.21) and (2.22).

To prove (a), if $2(1 - \epsilon)/(1 + \epsilon) < 1$, the proof is similar to the proof of (b). If $2(1 - \epsilon)/(1 + \epsilon) \geq 1$, the assumption in (a) implies that $\int_0^\infty k(x, s)ds \leq C_8$ for all r and for all x . By Theorem 1.3, for any t we can solve the Poincaré-Lelong equation $\sqrt{-1}\partial\bar{\partial}u = \text{Ric}(g(t))$ with $|\nabla u|(x, t) \leq C_9$ for some constant independent of x and t . By Theorem 2.1 in [N-S-T], the result follows. \square

3. Asymptotic behavior of the volume element.

In §2, we gave some estimates of the volume element $-F(x, t)$ under the Kähler-Ricci flow. In general, $-F(x, t)$ has no limit as $t \rightarrow \infty$ unless the original manifold is flat. In this section, we will use the Poincaré-Lelong equation and the results in [N-S-T] to obtain information on asymptotic behavior of the rescaled volume element $-F(x, t) + F(x_0, t)$. Let us assume that $(M^m, g_{\alpha\bar{\beta}}(x))$ is a complete noncompact Kähler manifold with bounded nonnegative holomorphic bisectional curvature. As before, denote

$$k(x, r) = \int_{B(x, r)} \mathcal{R}_0 dV_0$$

where \mathcal{R}_0 is the scalar curvature of $g_{\alpha\bar{\beta}}$. We also assume that

$$k(x, r) \leq k(r) \tag{3.1}$$

for all $x \in M$, with $\int_0^\infty k(r)dr < \infty$. By Theorem 1.3, (1.1) has a long time solution $g_{\alpha\bar{\beta}}(x, t)$ with $g_{\alpha\bar{\beta}}(x, 0) = g_{\alpha\bar{\beta}}(x)$. On the other hand, by the result in [N-S-T], there is a unique function u such that

$$\sqrt{-1}\partial\bar{\partial}u_0 = \text{Ric}(g(0)) \tag{3.2}$$

with $u_0(o) = 0$ and $|u_0| = o(r)$. We have the following:

Theorem 3.1. *Let $x_0 \in M$ be a fixed point. For any $t_j \rightarrow \infty$, there is a subsequence, which is also denoted by t_j , such that*

$$\lim_{j \rightarrow \infty} (F(x, t_j) - F(x_0, t_j)) = u_0(x) - u_0(x_0) - v(x)$$

where u_0 is the function in (3.2) and $v(x)$ is a pluriharmonic function of at most linear growth (with respect to the initial metric). The convergence is uniform on compact sets. If in addition, $\int_0^r sk(x, s) ds \leq C(1+r)^{1-\epsilon}$ with $\epsilon > 1/3$, then

$$\lim_{t \rightarrow \infty} (F(x, t) - F(x_0, t)) = u_0(x) - u_0(x_0)$$

and the convergence is uniform on compact sets of M .

Proof. Let $h(x, t) = (u_0(x) - F(x, t)) - (u_0(x_0) - F(x_0, t))$. By Theorem 1.3, there exists a constant C_1 such that for all $(x, t) \in M \times [0, \infty)$

$$\left| \tilde{\nabla} h(x, t) \right| \leq |\nabla h(x, t)| \leq C_1, \quad (3.3)$$

where $\tilde{\nabla} h$ is the gradient with respect to the initial metric $g(0)$, and we have used the fact that $g_{\alpha\bar{\beta}}$ is nonincreasing. Since $h(x_0, t) = 0$ for all t , it is easy to see that for any $t_j \rightarrow \infty$, there is a subsequence, which will be denoted by t_j again, such that

$$\lim_{j \rightarrow \infty} h(x, t_j) = v(x)$$

for some Lipschitz continuous function $v(x)$ on M with bounded gradient. Since

$$\tilde{\Delta} h(x, t) = g^{\alpha\bar{\beta}}(x, 0) R_{\alpha\bar{\beta}}(x, t) \quad (3.4)$$

for all x , where $\tilde{\Delta}$ is the Laplacian with respect to $g(0)$,

$$0 \leq g^{\alpha\bar{\beta}}(x, 0) R_{\alpha\bar{\beta}}(x, t) \leq g^{\alpha\bar{\beta}}(x, t) R_{\alpha\bar{\beta}}(x, t) = \mathcal{R}(x, t),$$

Since by Corollary 2.4, $\lim_{t \rightarrow \infty} \mathcal{R}(x, t) = 0$ uniformly on M , we conclude that $v(x)$ is a harmonic function of at most linear growth. Notice that $h(x, t)$ is plurisubharmonic. Thus v is also plurisubharmonic. Together with the fact that it is also harmonic, v must be pluriharmonic.

Suppose $\int_0^r sk(x, s) ds \leq C(1+r)^{1-\epsilon}$ with $\epsilon > 1/3$. Then by Theorem 2.3, we have

$$\int_0^r sk_t(x, s) ds \leq C_2(1+r)^\delta \quad (3.5)$$

for some constant $C_2 > 0$ independent of x and t . Here $k_t(x, s) = \int_{B_t(x, s)} \mathcal{R} dV_t$ and $\delta = 2(1 - \epsilon)/(1 + \epsilon) < 1$. By Theorem 1.2 in [N-S-T] and the fact that $h(x, t) = o(r_t(x, x_0))$ for fixed t , we can conclude from (3.5) that

$$h(x, t) \leq C_3(1 + r_t(x, x_0))^\delta \leq C_3(1 + r_0(x, x_0))^\delta$$

for some constant independent of t . Hence the harmonic function $v(x)$ is of sublinear growth and must be constant by [C-Y]. Since $v(x_0) = 0$, v must be identically zero. \square

In [Sh2] and later in [C-Z], it was proved that if M is a complete non-compact Kähler manifold with *positive* and bounded holomorphic bisectional curvature such that the scalar curvature satisfies $\int_{B(x, r)} \mathcal{R}_0 \leq k(r)$ for all x and r with $k(r) \leq C(1 + r)^{-1-\epsilon}$, $\epsilon > 1/2$, then the long time solution of the Kähler-Ricci flow subconverges after rescaling in the following sense. Let x_0 be a fixed point in M and let v be a fixed $(1, 0)$ vector at x_0 with unit length with respect to the initial metric. Let $\widehat{g}_{\alpha\bar{\beta}}(x, t) = g_{\alpha\bar{\beta}}(x, t)/g_{v\bar{v}}(x_0, t)$. Then for any $t_j \rightarrow \infty$, we can find a subsequence, also denoted by t_j , such that $\widehat{g}_{\alpha\bar{\beta}}(x, t_j)$ converge uniformly on compact sets of M to a flat Kähler metric. However, as pointed out in [C-Z], it is unclear whether the metric is complete. Using Theorem 3.1, we can get some preliminary estimates for the limiting metric.

Proposition 3.1. *Let $(M^m, g_{\alpha\bar{\beta}})$ be a complete noncompact Kähler manifold with positive and bounded holomorphic bisectional curvature such that the scalar curvature \mathcal{R}_0 satisfies*

$$\int_{B(x, r)} \mathcal{R}_0 dV_0 \leq k(r)$$

for all x and r , where $k(r) \leq C(1 + r)^{-1-\epsilon}$ with $\epsilon > 1/2$. Let $g_{\alpha\bar{\beta}}(x, t)$ be the long time solution of (1.1) with $g_{\alpha\bar{\beta}}(x, 0) = g_{\alpha\bar{\beta}}(x)$.

(a) *The rescaled metrics*

$$\tilde{g}_{\alpha\bar{\beta}}(x, t) = e^{-\frac{F(x_0, t)}{m}} g_{\alpha\bar{\beta}}(x, t)$$

subconverge to a flat Kähler metric $h_{\alpha\bar{\beta}}$ on M . The convergence is uniform on compact sets, where x_0 is a fixed point and

$$F(x, t) = \log \frac{\det(g_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))}.$$

(b) If, in addition, $\epsilon = 1$ and $\mathcal{R}_0(x) \leq Cr_0^{-2}(x)$, where $r_0(x)$ is the distance function from x_0 with respect to the initial metric, then

$$\frac{\det(h_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))} \geq C' r_0^a(x) - C'' \quad (3.6)$$

for some positive constants a , C' and C'' . In particular, the maximal eigenvalue $\lambda_{\max}(x)$ of $h_{\alpha\bar{\beta}}(x)$ with respect to $g_{\alpha\bar{\beta}}(x, 0)$ satisfies

$$\lambda_{\max}(x) \geq C''' r_0^{\frac{a}{m}}(x) \quad (3.7)$$

for some positive constant C''' , provided $r_0(x)$ is large enough.

Proof. Part (a) follows from the results in [Sh2, C-Z]. Since

$$\log \frac{\det(h_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))} = \lim_{t \rightarrow \infty} (F(x, t) - F(x_0, t)),$$

by Theorem 3.1 we have

$$\log \frac{\det(h_{\alpha\bar{\beta}}(x, t))}{\det(g_{\alpha\bar{\beta}}(x, 0))} = u(x) - u(x_0) \quad (3.8)$$

where $u(x)$ is the solution for the Poincaré-Lelong equation obtained in [N-S-T, Theorem 5.1]. Since M is nonflat, by Remark 2.2 and Corollary 2.3, we have

$$\liminf_{r \rightarrow \infty} \frac{\int_0^r sk(x_0, s) ds}{\log r} > 0. \quad (3.9)$$

By [N-S-T, Corollary 1.1], (3.8) and (3.9), we conclude that (3.6) is true.

(3.7) follows from (3.6) immediately. \square

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