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On the holomorphicity of proper harmonic maps between unit balls with the Bergman metrics

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1 Introduction

Let M^m and N^n be two Kähler manifolds with Kähler metrics $h = h_{i\bar{j}}dz_i d\bar{z}_j$ and $g = g_{\alpha\bar{\beta}}dw^\alpha d\overline{w}^\beta$, respectively. Let $u : M \to N$ be a map from M to N. When both M and N are compact, in his proof of the celebrated strong rigidity theorem for compact Kähler manifolds, Siu [S1] proved that any harmonic map u must be holomorphic or antiholomorphic, under the assumption that N has strongly negative curvature in the sense of Siu and the rank of du at one point is greater than or equal to four (the last condition excludes the case of complex dimension one when the theorem is obviously false). The key of the proof is Siu's $\partial\overline{\partial}$-Bochner formula:

$$\begin{aligned}\partial\overline{\partial}\left(g_{\alpha\bar{\beta}}u^\alpha_{\bar{i}}\overline{u}^\beta_j d\bar{z}_i \wedge dz_j\right) &= R_{\alpha\bar{\beta}\gamma\bar{\delta}}u^\alpha_{\bar{i}}\overline{u}^\beta_j u^\gamma_k \overline{u}^\delta_{\bar{\ell}} d\bar{z}_i \wedge dz_j \wedge dz_k \wedge d\bar{z}_\ell \\ &\quad - g_{\alpha\bar{\beta}} D\overline{\partial}u^\alpha \wedge \overline{D}\partial\overline{u}^\beta. \end{aligned} \tag{1.1}$$

When M is a compact manifold, the integration of the left hand side, after wedging a (m-2) power of the Kähler form, is zero from integration by parts. It was shown in [S1], that both terms of the right hand side are non-negative when u is a harmonic map and the curvature of N is strongly negative, and therefore they are pointwise zero. This fact coupled with the the rank assumption on du shows that u must be holomorphic or antiholomorphic (cf. [S1]). A general question

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one may ask is when a harmonic map u is holomorphic or antiholomorphic if N is Kähler with strongly negative curvature. When M is a complete (noncompact) manifold, the idea of the $\partial\overline{\partial}$-Bochner formula together with the integration by parts does not work any more except in the case when M has finite volume and the above mentioned general question is largely unknown. Since unit balls in $\mathbb{C}^n$ with Bergman metric are the simplest class of Kähler manifolds with strongly negative curvature, it is natural to pose the following question.

Problem 1. *Let h, g denote the Bergman metrics on B_m and B_n, respectively; and let $u : (B_m, h) \to (B_n, g)$ be a proper harmonic map so that u can be extended to C^1 map up to the boundary ∂B_m. Is u either holomorphic or anti-holomorphic?*

A closely related problem of Problem 1 is the existence and regularity of proper harmonic maps, namely

Problem 2. *Let $\phi : \partial B_m \to \partial B_n$ be a smooth map. Does there exist a proper harmonic map u so that $u = \phi$ on ∂B_m? If there exists a harmonic map extension u what can we say about the regularity of u ?*

For the real hyperbolic space, Peter Li and L. F. Tam initiated the systematic study of the existence, uniqueness and regularity of proper harmonic maps from the unit ball D^m in $\mathbb{R}^m$ to D^n in $\mathbb{R}^n$ with respect to the hyperbolic metrics (cf. [LT 1-3]). In [LT 1, 2], among other things, they proved that if $\phi : S^{m-1} \to S^{n-1}$ is a C^1 map with energy density $e(\phi)(x) \neq 0$ for all $x \in S^{m-1}$ (here $e(\phi)$ is defined with respect to the standard metrics on S^{2m-1} and S^{2n-1}) then there is a unique proper harmonic map extension $u : D^m \to D^n$ with boundary value ϕ. Moreover, if $\phi \in C^m(S^{m-1}, S^{n-1})$ then $u \in C^{m-1,\alpha}(\overline{D}^m, \overline{D}^n)$ for any $\alpha < 1$. They also proved that if $e(\phi) \neq 0$ on S^{m-1} then the energy density $e[u]$ of the harmonic map u with respect to the hyperbolic metric satisfies

$$\lim_{x\to S^{m-1}} e[u](x) = \lim_{x\to S^{m-1}} h^{ij} g_{k\ell} \frac{\partial u^k}{\partial x_i}\frac{\partial u^\ell}{\partial x_j}(x) = m, \quad x \in S^{m-1}. \tag{1.2}$$

where $h = h_{ij}dx_i dx_j$ is the hyperbolic metric for D^m and $g = g_{ij}dy_i dy_j$ is the hyperbolic metric for D^n, and (h^{ij}) is the inverse matrix of (h_{ij}).

For the complex case, the problem was first studied by H. Donnelly [D1] where he studied the case when the domain and target manifolds are rank one symmetric space of noncompact type. He generalized the above existence and uniqueness results of Li-Tam to the setting with some necessary contact conditions on the boundary map ϕ. When $e(\phi)$ vanishes on S^{m-1}, the existence of a proper harmonic extension becomes less tractable, partial progress was made by J. Wang [W], where he proved the existence under the assumption that $e(\phi)$ has finitely many zeros on S^{m-1} and ϕ is locally rotationally symmetric around those points.

The first purpose of this paper is to show that the answer to Problem 1 is negative in general if we do not assume enough regularity on the harmonic map u. More precisely, we prove the following theorem.

Theorem 1.1 *For any $0 < \epsilon < 1$, there is a proper harmonic map $u \in C^{2-\epsilon}(\overline{B}_2)$ from B_2 to B_3 with respect to the Bergman metric, which is neither holomorphic nor anti-holomorphic.*

By Theorem 1.1, one sees that, in general, a proper harmonic map is not necessarily holomorphic or anti-holomorphic. It is natural to find out what are the necessary and sufficient conditions under which the proper harmonic map is holomorphic or anti-holomorphic. The second part of this paper serves this purpose. We find that a slightly stronger condition than harmonic map equation will be sufficient. In order to state our second result clearly, we need the notation of k-harmonic maps with respect to the origin 0. We say that a map $u : B_m \to B_n$ is k-harmonic with respect to the origin 0 if the restriction of u is harmonic on the intersection of B_m and to any k-dimensional complex linear subspace through the origin. It is clear that a map is harmonic if and only if it is m-harmonic with respect to the origin, and pluriharmonic map are k-harmonic with respect to the origin for all $1 \le k \le m$. Now we are ready to state our second theorem.

Theorem 1.2 *Let $u \in C^2(\overline{B}_m, \overline{B}_n)$ $(m > 1)$ be a proper map from B_m to B_n with respect to the Bergman metrics. Then the following statements are equivalent.*

(i) *u is either holomorphic or anti-holomorphic;*
(ii) *$(m-1)$-harmonic with respect to the origin;*
(iii) *u is harmonic; and $\mathcal{L}u$ is orthogonal to u on ∂B_m where $\mathcal{L} = (\delta_{ij} - z_i\overline{z}_j)\frac{\partial^2}{\partial z_i \partial \overline{z}_j}$;*
(iv) *u is harmonic and*

$$\lim_{r\to 1^-} e[u](rz) = m \quad \text{on} \quad \left\{z \in \partial B_m : E_b[u] = |\overline{\partial}_b \overline{u}(z)|^2 + |\overline{\partial}_b u(z)|^2 \neq 0\right\},$$

where the energy density $e[u]$ is given by

$$e[u](z) = \frac{\left(1-|z|^2\right)}{\left(1-|u(z)|^2\right)^2}\left(\delta_{ij} - z_i\overline{z}_j\right)\left(\delta_{\alpha\beta}\left(1-|u|^2\right) + \overline{u}^\alpha u^\beta\right) \cdot \left(\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta}\right).$$

Here we sum i, j from 1 to m, and sum α, β from 1 to n.

With arguments of the proof of Theorem 1.2 and Fefferman's asymptotic expansion of the Bergman kernel function of a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$ (cf. [F]), one is able to prove the following corollary.

Corollary 1.3 *Let Ω_1 and Ω_2 be smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^m$ $(m > 1)$ and $\mathbb{C}^n$, respectively. Let $u \in C^2(\overline{\Omega}_1, \overline{\Omega}_2)$ be a proper pluriharmonic map from Ω_1 to Ω_2 with respect to the Bergman metrics on Ω_1 and on Ω_2. Then u is either holomorphic or antiholomorphic.*

Combining Theorem 1.2 and a recent work of X. Huang [H] on the rigidity of proper holomorphic maps we have the following corollary.

Corollary 1.4 *Let $u \in C^2(\overline{B}_m, \overline{B}_n)$ $(m > 1)$ be a proper $(m-1)$-harmonic map with respect to the origin from $B_m \to B_n$ with respect to the Bergman metric with $n \le 2m-2$. Then there are $\phi \in Aut(B_m)$ and $\psi \in Aut(B_n)$ such that either $\psi \circ u \circ \phi$ or $\psi \circ \overline{u} \circ \phi$ is a holomorphic linear map.*

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The paper is organized as follows: In Section 2, we give some preliminary results. In Section 3, we prove that if $f \in C^2(\partial B_m)$ such that for each point $z \in \partial B_m$ we have either $\overline{\partial}_b f(z) = 0$ or $\overline{\partial}_b \overline{f}(z) = 0$, then either f or $\overline{f}$ is CR on ∂B_m; Theorem 1.2 is proved in Section 4. Finally, in Section 5, under the assumption of $E_b(u) \neq 0$ on ∂B_m (cf. Theorem 1.2 for the definition of $E_b(u)$), we pose a sufficient and necessary condition for the existence of the harmonic map extension, which is simpler than the contact conditions posed in [D1], and we push the existence of the proper harmonic map extension to $C^{1,\alpha}$ boundary maps, which slightly generalizes a previous result by Donnelly. Theorem 1.1 is also proved there.

2 Preliminary

Let $M = \Omega_1$ be a smoothly bounded domain in $\mathbb{C}^m$ with Bergman metric tensor:

$$h = \sum_{i,j} h_{i\overline{j}} dz^i d\overline{z}^j = \frac{1}{m+1} \sum_{i,j=1}^{n} \frac{\partial^2 \log K_1(z,z)}{\partial z^i \partial \overline{z}^j} dz^i d\overline{z}^j. \tag{2.1}$$

and let $N = \Omega_2$ be a smoothly bounded domain in $\mathbb{C}^n$ with Bergman metric

$$g = g^{\alpha\beta} dw^\alpha d\overline{w}^\beta = \frac{1}{n+1} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \log K_2(w,w)}{\partial w^\alpha \partial \overline{w}^\beta} dw^\alpha d\overline{w}^\beta \tag{2.2}$$

where K_j are Bergman kernel functions of Ω_j for $j = 1, 2$.

Let $\Gamma^s_{t\gamma}$ be the Christoffel symbols of the Hermitian metric g on N, and let $u = (u^1, u^2, \cdots, u^n) : M \to N \subset \mathbb{C}^n$ be a map. First we introduce the following definitions:

(a) We say that u is *harmonic* if the tension field

$$\tau^s[u] = \triangle_M u^s + \sum_{t,\gamma=1}^{n} \sum_{i,j=1}^{m} \Gamma^s_{t\gamma} h^{i\overline{j}} \partial_i u^t \overline{\partial}_j u^\gamma = 0, \quad \text{for } 1 \le s \le n \tag{2.3}$$

where $\triangle_M = h^{i\bar{j}}\partial^2_{i\bar{j}}$ and $(h^{i\bar{j}})$ is the inverse matrix of the matrix $(h_{i\bar{j}})$.

(b) We say that u is *pluriharmonic* if

$$\partial\overline{\partial}u^s + \sum_{t,\gamma} \Gamma^s_{t\gamma}\partial u^t \overline{\partial} u^\gamma = 0, \quad \text{for } 1 \le s \le n. \tag{2.4}$$

(c) We say that u is *holomorphic* if $\overline{\partial}u^s = 0$ for $1 \le s \le n$.

Since N is Kähler, it is well-known that any pluriharmonic maps are harmonic, and any holomorphic maps are pluriharmonic.

Let $M = B_m$ and $N = B_n$ be the unit balls in $\mathbb{C}^m$ and $\mathbb{C}^n$, with the corresponding Bergman metrics. We denote

$$R = \sum_{i=1}^m z_i \frac{\partial}{\partial z_i}, \quad \overline{R} = \sum_{i=1}^m \overline{z}_i \frac{\partial}{\partial \overline{z}_i}, \quad X_j = \sum_{i=1}^m (\delta_{ij} - \overline{z}_j z_i)\frac{\partial}{\partial z_i},$$
$$\overline{X}_j = \sum_{i=1}^m (\delta_{ij} - z_j \overline{z}_i)\frac{\partial}{\partial \overline{z}_i}.$$

Here R is the complex normal vector field to ∂B_m at z, and $\{X_1, \cdots, X_n\}$ generates the complex tangent space $T^{1,0}_z(\partial B_m)$. Since the Bergman kernel function of B_m is

$$K_m(z, w) = (1 - \langle z, w\rangle)^{-(m+1)}, \quad z, w \in B_m.$$

It is easy to show that

$$h_{i\bar{j}}[z] = \left(1 - |z|^2\right)^{-2}\left[\left(1 - |z|^2\right)\delta_{ij} + \overline{z}_i z_j\right],$$

$$g_{\alpha,\overline{\beta}}[w] = \left(1 - |w|^2\right)^{-2}\left[\left(1 - |w|^2\right)\delta_{\alpha\beta} + \overline{w}^\alpha w^\beta\right],$$

$$h^{i\bar{j}}[z] = \left(1 - |z|^2\right)\left(\delta_{ij} - z_i\overline{z}_j\right),$$

$$g^{\alpha\overline{\beta}}[w] = \left(1 - |w|^2\right)\left(\delta_{\alpha\beta} - w^\alpha\overline{w}^\beta\right).$$

and

$$\Gamma^s_{t\gamma}[u] = \left(1 - |u|^2\right)^{-1}\left(\overline{u}^\gamma\delta_{ts} + \overline{u}^t\delta_{\gamma s}\right)$$

By the above formula for Christoffel symbols, the tension field now can be written as

$$\tau^s[u] = \left(1 - |z|^2\right)(\delta_{ij} - z_i\overline{z}_j)\frac{\partial^2 u^s}{\partial z^i \partial \overline{z}^j} + Q^s[u] = \left(1 - |z|^2\right)\mathcal{L}u^s + Q^s[u], \tag{2.5}$$

where

$$Q^s[u] = \frac{1-|z|^2}{1-|u|^2}\left(\delta_{ij} - z_i\bar{z}_j\right)\left(u_i^s u_{\bar{j}}^t \bar{u}^t + u_i^t u_{\bar{j}}^s \bar{u}^t\right),$$

$$\mathcal{L} = \left(\delta_{ij} - z_i\bar{z}_j\right)\frac{\partial^2}{\partial z_i \partial \bar{z}_j}. \tag{2.6}$$

Here (in the right-hand side of $Q^s[u]$) we sum i, j from 1 to m, and sum t from 1 to n.

The first version of the following lemma was given in [LT1]. We modify both the statement and the proof to fit our case.

Lemma 2.1 *Let $u^s \in C^2(B_m) \cap C^1(\overline{B}_m)$ for all $1 \le s \le n$. Then*

$$\lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m}} \int_{B(z,\epsilon(z))} \overline{u}^s \mathcal{L} u^s(w)\, dv(w) = 0. \tag{2.7}$$

Here $\epsilon(z) = (1-|z|)/2$.

Proof. Since $\mathcal{L} = \sum_j X_j \frac{\partial}{\partial \bar{z}_j}$, we have

$$\begin{aligned}
&\lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m}} \int_{B(z,\epsilon(z))} \overline{u}^s \mathcal{L} u^s dv(w) \\
&= \lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m}} \int_{B(z,\epsilon(z))} \overline{u}^s \frac{\partial}{\partial \overline{w}_j} X_j u^s + \overline{u}^s \left[X_j, \frac{\partial}{\partial \overline{w}_j}\right] u^s\, dv(w) \\
&= \lim_{z\to\partial B_m} \left[\frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m}} \int_{B(z,\epsilon(z))} \frac{\partial}{\partial \overline{w}_j}(\overline{u}^s X_j u^s) - \frac{\partial \overline{u}^s}{\partial \overline{w}_j} X_j u^s\, dv(w) + o(1)\right] \\
&= \lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m}} \int_{B(z,\epsilon(z))} \frac{\partial}{\partial \overline{w}_j}(\overline{u}^s X_j u^s)\, dv(w) \\
&= \lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m+1}} \int_{\partial B(z,\epsilon(z))} (w_j - z_j)\overline{u}^s X_j u^s\, d\sigma(w) \\
&= \lim_{z\to\partial B_m} \frac{\left(1-|z|^2\right)}{\epsilon(z)^{2m+1}} \int_{\partial B(z,\epsilon(z))} (w_j - z_j)\left(\overline{u}^s X_j u^s(w) - \overline{u}^s X_j u^s\left(\frac{z}{|z|}\right)\right) d\sigma(w) \\
&= 0
\end{aligned}$$

since $(w_j - z_j)(\overline{u}^s X_j u^s(w) - \overline{u}^s X_j u^s(z/|z|)) = o(\epsilon(z))$ uniformly for $w \in B(z,\epsilon(z))$ as $z \to \partial B_m$ and the area of $\partial B(z,\epsilon(z))$ is comparable to $\epsilon(z)^{2m-1}$. Therefore, the proof of the lemma is complete. □

First let us do some computations on the notations. It is easy to see that

$$\sum_{j=1}^{m} X_j z_j = \sum_j \left(1 - |z_j|^2\right) = m - |z|^2, \tag{2.8}$$

$$\sum_j z_j X_j = R - |z|^2 R = \left(1 - |z|^2\right) R \tag{2.9}$$

and

$$X_j |z|^2 = \bar{z}_j \left(1 - |z|^2\right), \quad \overline{X}_j |z|^2 = z_j \left(1 - |z|^2\right).$$

Moreover,

$$\begin{aligned} \mathcal{L} &= \sum_j X_j \overline{X}_j + \left(m - |z|^2\right) \overline{R} + \left(1 - |z|^2\right) R\overline{R} \\ &= \sum_j \overline{X}_j X_j + \left(m - |z|^2\right) R + \left(1 - |z|^2\right) \overline{R} R. \end{aligned} \tag{2.10}$$

In fact,

$$\begin{aligned} \mathcal{L} &= \sum_{ij} \left(\delta_{ij} - z_i \bar{z}_j\right) \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \\ &= \sum_j X_j \frac{\partial}{\partial \bar{z}_j} \\ &= \sum_j X_j \overline{X}_j + \sum_j X_j z_j \overline{R} \\ &= \sum_j X_j \overline{X}_j + \left(m - |z|^2\right) \overline{R} + \sum_j z_j X_j \overline{R} \\ &= \sum_j X_j \overline{X}_j + \left(m - |z|^2\right) \overline{R} + \left(1 - |z|^2\right) R\overline{R}, \end{aligned}$$

and the other equality follows similarly.

Let $u \in C^1(\overline{B}_m, \overline{B}_n)$, and we denote

$$a[u](z) = \left(1 - |u(z)|^2\right)\left(1 - |z|^2\right)^{-1}, \tag{2.11}$$

and

$$|\partial_b u|^2 = \sum_{j=1}^{m} \sum_{s=1}^{n} |X_j u^s|^2, \quad |\overline{\partial}_b u|^2 = \sum_{j=1}^{m} \sum_{s=1}^{n} |\overline{X}_j u^s|^2;$$

$$E_b(u) = |\partial_b u|^2 + |\overline{\partial}_b u|^2. \tag{2.12}$$

We first prove the following proposition.

Proposition 2.2 *Let $u(z) \in C^1(\overline{B}_m, \overline{B}_n) \cap C^2(B_m, B_n)$ be a map from B_m to B_n. Then the followings hold*

(i) For any $z \in B_m$, we have

$$\tau^s[u] = \left(1-|z|^2\right)\left[\mathcal{L}u^s + \frac{Ru^s\left(\overline{u}^t\overline{R}u^t\right) + \overline{R}u^s\left(\overline{u}^t Ru^t\right)}{a[u](z)}\right] + \frac{\sum_i\left[(X_iu^s)\overline{u}^t\overline{X}_iu^t + (\overline{X}_iu^s)\overline{u}^t X_iu^t\right]}{a[u](z)}; \tag{2.13}$$

(ii) If u is harmonic map then

$$\sum_s \overline{u}^s X_j u^s = \sum_s \overline{u}^s\overline{X}_j u^s = 0, \quad on\ \ \partial B_m \tag{2.14}$$

and if furthermore we assume that $u \in C^2(\overline{B}_m)$ then

$$\sum_s \overline{u}^s\mathcal{L}u^s = (m-1)\sum_s \overline{u}^s Ru^s - |\partial_b u|^2 = (m-1)\sum_s \overline{u}^s\overline{R}u^s - |\overline{\partial}_b u|^2 \ \ on\ \partial B_m;$$

(iii) If $u \in C^{1,\alpha}(\overline{B}_m)$ (with $\alpha > 1/2$) is a harmonic map, then

$$a[u](z)\overline{u}^s\mathcal{L}u^s + 2(\overline{u}^s Ru^s)(\overline{u}^t\overline{R}u^t) = 0, \quad on \quad \partial B_m.$$

Remark 2.1 When $\alpha \geq 1/2$ and $u \in C^{1,\alpha}(\overline{B}_m)$ is harmonic, we have that the limit $\{\overline{u}^s\mathcal{L}u^s(z_i)\}$ exists as $z_i \to p$. We shall write $\overline{u}^s\mathcal{L}u^s(p)$ for the boundary point p without mentioning passing to the limit at each occurrence.

Proof. Since $u : B_m \to B_n$, we have $a[u](z) > 0$ on B_m. Let us prove (i) first . Since

$$\begin{aligned}
\frac{1-|u|^2}{1-|z|^2}Q^s[u] &= \left(\delta_{ij} - z_i\overline{z}_j\right)\left(\overline{u}^\gamma\partial_i u^s\partial_{\overline{j}}u^\gamma + \overline{u}^t\partial_i u^t\overline{\partial}_j u^s\right) \\
&= \left(X_ju^s\right)\left(\overline{u}^\gamma\partial_{\overline{j}}u^\gamma\right) + \left(\partial_{\overline{j}}u^s\right)\left(\overline{u}^t X_ju^t\right) \\
&= \left(X_ju^s\right)\left(\overline{u}^\gamma\overline{X}_ju^\gamma\right) + \left(z_jX_ju^s\right)\left(\overline{u}^\gamma\overline{R}u^\gamma\right) \\
&\quad + \left(\overline{X}_ju^s\right)\left(\overline{u}^t X_ju^t\right) + \left(z_j\overline{R}u^s\right)\left(\overline{u}^\gamma X_ju^\gamma\right) \\
&= \left(X_ju^s\right)\left(\overline{u}^\gamma\overline{X}_ju^\gamma\right) + \left(\overline{X}_ju^s\right)\left(\overline{u}^t X_ju^t\right) \\
&\quad + \left(1-|z|^2\right)\left[\left(Ru^s\right)\left(\overline{u}^\gamma\overline{R}u^\gamma\right) + \left(\overline{R}u^s\right)(\overline{u}^\gamma Ru^\gamma)\right]
\end{aligned}$$

it follows that

$$\begin{aligned}
\tau^s[u] &= \left(1-|z|^2\right)\mathcal{L}u^s + Q^s[u] \\
&= \left(1-|z|^2\right)\left[\mathcal{L}u^s + \frac{(Ru^s)\left(\overline{u}^t\overline{R}u^t\right) + \left(\overline{R}u^s\right)\left(\overline{u}^t Ru^t\right)}{a[u](z)}\right] \\
&\quad + \frac{\sum_j\left[\left(X_ju^s\right)\left(\overline{u}^t\overline{X}_ju^t\right) + \left(\overline{X}_ju^s\right)\left(\overline{u}^t X_ju^t\right)\right]}{a[u](z)}
\end{aligned}$$

and (i) is proved. Now we prove (ii). By (i) and the assumption that u is harmonic map, it is easy to see that

$$0 = \left(1 - |z|^2\right)\left[\overline{u}^s \mathcal{L} u^s + 2(a[u](z))^{-1}(\overline{u}^s R u^s)(\overline{u}^t \overline{R} u^t)\right] \\ +2(a[u](z))^{-1} \sum_j \left(\overline{u}^s X_j u^s\right)\left(\overline{u}^t \overline{X}_j u^t\right).$$

Thus, since $u \in C^1(\overline{B}_m)$, applying Lemma 2.1, it follows

$$\sum_j \left(\overline{u}^s X_j u^s\right)\left(\overline{u}^t \overline{X}_j u^t\right) = 0, \quad \text{on} \quad \partial B_m. \tag{2.15}$$

Since

$$|u(z)|^2 = 1, \quad \text{on } \partial B_m,$$

we have

$$0 = X_j |u(z)|^2 = \overline{u}^s X_j u^s + u^s X_j \overline{u}^s = \overline{u}^s X_j u^s + \overline{\overline{u}^s \overline{X}_j u^s}, \quad \text{on} \ \partial B_m,$$

for all $1 \leq j \leq m$. Combining this with (2.15), we have

$$\sum_s \overline{u}^s X_j u^s = \sum_s \overline{u}^s \overline{X}_j u^s = 0, \quad \text{on} \ \partial B_m, \quad \text{for all} \ 1 \leq j \leq m.$$

So (2.14) is proved. To complete the proof of the rest of (ii), we use the equation

$$\mathcal{L} u^s(z) = \sum_j X_j \overline{X}_j u^s + \left(m - |z|^2\right) \overline{R} u^s + \left(1 - |z|^2\right) R \overline{R} u^s.$$

By (2.14), on ∂B_m, we have

$$0 = \sum_s \overline{X}_j \left(\overline{u^s} X_j u^s\right) = \sum_s \left(\overline{X_j u^s}\right)\left(X_j u^s\right) + \sum_s \overline{u^s} \overline{X}_j X_j u^s$$

Thus

$$\sum_s \overline{u^s} \overline{X}_j X_j u^s = -\sum_s \left(\overline{X_j u^s}\right)\left(X_j u^s\right) = -|\partial_b u|^2$$

and similarly

$$\sum_s \overline{u^s} X_j \overline{X}_j u^s = -\sum_s \left(\overline{\overline{X}_j u^s}\right)\left(\overline{X}_j u^s\right) = -|\overline{\partial}_b u|^2.$$

Therefore, on ∂B_m,

$$\sum_s \overline{u}^s \mathcal{L} u^s(z) = (m-1) \sum_s \overline{u}^s \overline{R} u^s + \sum_s \overline{u^s} \sum_j X_j \overline{X}_j u^s$$

$$= (m-1) \sum_s \overline{u}^s \overline{R} u^s - |\overline{\partial}_b u|^2. \tag{2.16}$$

Similarly,

$$\sum_s \overline{u}^s \mathcal{L} u^s(z) = (m-1)\sum_s \overline{u}^s R u^s - |\partial_b u|^2 \quad \text{on } \partial B_m. \tag{2.17}$$

Therefore, (ii) is proved. Next we prove (iii). First, we have from (i) that

$$a[u](z)\,\overline{u}^s \tau^s[u] = \left(1-|z|^2\right)\left[a[u](z)\overline{u}^s\mathcal{L}u^s + 2\overline{u}^s R u^s \overline{u}^t \overline{R} u^t\right] \\ +2(\overline{u}^s X_j u^s)(\overline{u}^t \overline{X}_j u^t).$$

Then, since $u \in C^{1,\alpha}(\overline{B}_m) \cap C^2(B_m)$ and $\sum_s \overline{u}^s X_j u^s = \sum_s \overline{u}^s \overline{X}_j u^s = 0$ on ∂B_m, for $1 \le j \le m$, we have the inequality:

$$|\overline{u}^s \overline{X}_j u^s| + |\overline{u}^s X_j u^s| \le C\|u\|_{C^{1,\alpha}(\overline{B}_m)}\left(1-|z|^2\right)^{\alpha},$$

for all $z \in B_m$ and $1 \le j \le m$. Thus

$$\left(1-|z|^2\right)^{-1} |\sum_i (\overline{u}^s X_i u^s)(\overline{u}^t \overline{X}_i u^t)| \le C\left(1-|z|^2\right)^{2\alpha-1}.$$

Using the fact that $\tau^s[u] = 0$ $(1 \le s \le n)$, we have

$$\left|a[u](z)\overline{u}^s\mathcal{L}u^s + 2\overline{u}^s R u^s\, \overline{u}^t \overline{R} u^t\right| \le C\|u\|_{C^{1,\alpha}(\overline{B}_m)}\left(1-|z|^2\right)^{2\alpha-1}, \quad z \in B_m.$$

In particular, for any $z^0 \in \partial B_m$, we have

$$\lim_{z\to z^0}\left[a[u](z)\,\overline{u}^s\mathcal{L}u^s + 2\overline{u}^s R u^s\, \overline{u}^t \overline{R} u^t\right] = 0,$$

because of the assumption that $\alpha > 1/2$. This completes the proof of (iii), and therefore the proof of the proposition is complete. □

We shall prove the following proposition.

Proposition 2.3 *Let $u, v \in C^2(\overline{B}_m)$ be a proper harmonic map from $B_m \to B_n$. Then*

(i) $\overline{u}^t R u^t$ and $\overline{u}^t \overline{R} u^t$ are non-negative on ∂B_m. Furthermore,

$$a[u](z) = \overline{u}^t R u^t + \overline{u}^s \overline{R} u^s \quad \text{on} \quad \partial B_m.$$

(ii) On $\{z \in \partial B_m : a[u](z) > 0\}$ we have

$$\overline{u}^s \mathcal{L} u^s(z) = \frac{-(m+1)E_b(u)(z) + D_m[u]}{4m} \quad \text{and}$$

$$a[u](z) = \frac{E_b(u)(z)}{2m} + \frac{D_m[u]}{2m(m-1)}.$$

where $D_m[u] = \sqrt{(m-1)^2 E_b(u)(z)^2 + 4m(|\partial_b u|^2 - |\overline{\partial}_b u(z)|^2)^2}$;

(iii)For $z \in \partial B_m$ we have $E_b(u) > 0$ if and only if $a[u](z) > 0$.

(iv) If $u(z) = v(z)$ and $E_b(u)(z) \ne 0$ for any $z \in \partial B_m$, then $u \equiv v$ on B_m.

Proof. For simplicity, on ∂B_m, we let $A = \overline{u}^s \mathcal{L} u^s$ and denote

$$T_1[u] = \overline{u}^t R u^t \quad \text{and} \quad T_2[u] = \overline{u}^t \overline{R} u^t .$$

By Proposition 2.2, we have

$$A = (m-1)T_1[u] - |\partial_b u|^2 = (m-1)T_2[u] - |\overline{\partial}_b u|^2, \tag{2.18}$$

and

$$a[u]A + 2T_1[u]T_2[u] = 0. \tag{2.19}$$

Direct algebraic manipulation gives the following equation for A:

$$A^2 + \left(E_b(u) + \frac{a[u](m-1)^2}{2}\right) A + |\partial_b u|^2 |\overline{\partial}_b u|^2 = 0. \tag{2.20}$$

From this we can see easily that A is a nonpositive real number. Therefore by (2.18) $T_1[u]$ and $T_2[u]$ are real.

Since $|u|^2 = 1$ on ∂B_m we have $(R - \overline{R})|u|^2 = 0$ on ∂B_m. Hence

$$a[u](z) = \frac{1}{2}\left[\overline{u}^s (R + \overline{R}) u^s + u^s (R + \overline{R}) \overline{u}^s\right] = T_1[u](z) + \overline{T_2[u](z)}.$$

Since $T_i[u]$ are real we have

$$a[u] = T_1[u] + T_2[u]. \tag{2.21}$$

Combining with (2.19) we know that $T_i[u]$ are the two roots of the following equation:

$$y^2 - a[u]y - \frac{a[u]A}{2} = 0.$$

Noticing $a[u] \geq 0$ and $A \leq 0$ we can conclude that $T_i[u]$ are nonnegative. Thus we complete the proof of (i).

Next we prove (ii). Combining (2.18) and (2.21) we can rewrite $a[u]$ as

$$a[u] = \frac{2A + E_b(u)}{m-1}.$$

Plugging this into (2.20) we have

$$mA^2 + \left(\frac{m+1}{2}\right) E_b(u) A + |\partial_b u|^2 |\overline{\partial}_b u|^2 = 0 .$$

Since $T_i[u] \geq 0$, by (2.18) we know that if $a[u] > 0$ or $|\partial_b u| \neq |\overline{\partial}_b u|$ then

$$\overline{u}^s \mathcal{L} u^s = A = \frac{-(m+1)E_b(u) + D_m[u]}{4m}, \tag{2.22}$$

where

$$D_m[u]^2 = (m+1)^2 E_b(u)^2 - 16m|\partial_b u|^2|\overline{\partial}_b u|^2$$
$$= (m-1)^2 E_b[u]^2 + 4m\left(|\overline{\partial}_b u|^2 - |\partial_b u|^2\right)^2. \tag{2.22'}$$

At the same time we have

$$a[u](z) = \frac{1}{m-1}\left[2\overline{u}^s \mathcal{L}u^s + E_b(u)\right]$$
$$= \frac{1}{m-1}\left[\frac{-(m+1)E_b(u) + D_m[u]}{2m} + E_b(u)\right]$$
$$= \frac{E_b(u)}{2m} + \frac{D_m[u]}{2m(m-1)} \tag{2.23}$$

and (ii) is proved.

Now we prove (iii). By (ii), we have that if $a[u](z) > 0$ then $E_b(u)(z) > 0$. Conversely, we need only to show that if $z_0 \in \partial B_m$ such that $a[u](z_0) = 0$ then $E_b(u)(z_0) = 0$. If $z_0 \in Z(a[u])$, the zero set of $a[u]$ on ∂B_m is not interior point of $Z(a[u])$, then $E_b(u)(z_0) = 0$ by (2.23) and passing limit. Now we may assume that $z_0 \in Z(a[u])$ is an interior point of $Z(a[u])$. Since

$$|u|^2 = 1 + a[u](z)\rho(z)$$

we have

$$\overline{u}^s Ru^s + \overline{\overline{u}^s \overline{R} u^s} = Ra[u]\rho(z) + a[u]|z|^2,$$

where $\rho(z) = |z|^2 - 1$. Therefore

$$2\mathrm{Re}\,(\overline{u}^s Ru^s\, \overline{u}^t \overline{R}u^t)\, a[u]^{-1} = |Ra[u]\rho(z) + a[u]|z|^2|^2 a[u]^{-1} - \left[|\overline{u}^s Ru^s|^2 + |\overline{u}^t \overline{R}u^t|^2\right] a[u]^{-1}$$
$$= |Ra[u]|^2\rho(z)^2 a[u]^{-1} + 2|z|^2\rho(z)(R+\overline{R})a[u] + a[u]|z|^4 - \left[|\overline{u}^s Ru^s|^2 + |\overline{u}^t \overline{R}u^t|^2\right] a[u]^{-1}$$
$$\le 4\left(R\, a[u]^{1/2}\right)\left(\overline{R}\, a[u]^{1/2}\right)\rho(z)^2 + 2|z|^2\rho(z)(R+\overline{R})a[u] + a[u].$$

Since

$$\overline{u}^s X_i u^s + \overline{\overline{u}^s \overline{X}_i u^s} = X_i a[u]\rho(z) - \overline{z}_i a[u]\rho(z)$$

we have

$$2\mathrm{Re}\,\left(\overline{u}^s X_i u^s\, \overline{u}^t \overline{X}_i u^t\right) a[u]^{-1}|\rho(z)|^{-1}$$
$$= |X_i a[u]\rho(z) - \overline{z}_i a[u]\rho(z)|^2 a[u]^{-1}|\rho(z)|^{-1} - \left[|\overline{u}^s X_i u^s|^2 + |\overline{u}^s \overline{X}_i u^s|^2\right] a[u]^{-1}|\rho(z)|^{-1}$$
$$\le |X_i a[u]|^2 a[u]^{-1}|\rho(z)| - 2\mathrm{Re}\,(z_i X_i a[u])|\rho(z)| + a[u]|z|^2|\rho(z)|$$
$$= 4\left(X_i a[u]^{1/2}\right)\left(\overline{X}_i a[u]^{1/2}\right)|\rho(z)| + 2\mathrm{Re}\,(Ra[u])\rho(z)^2 + a[u]|z|^2|\rho(z)|.$$

Since $a[u] \in C^1(\overline{B}_m)$ and $a[u](z) = 0$ in $\partial B_m \cap B(z_0, \delta)$ for some $\delta > 0$, it is clear $a[u]^{1/2} \in C^{1/2}(\overline{B}_m)$. Thus

$$\lim_{r\to 1^-} \left[\left(R\, a[u]^{1/2}\right)\left(\overline{R}\, a[u]^{1/2}\right)\ \rho^2(z)\right](rz_0) = 0.$$

At the mean time, since X_i and $\overline{X}_i$ are tangential vector fields to ∂B_m, we have

$$\lim_{r\to 1^-} \left[\left(X_i\, a[u]^{1/2}\right)\left(\overline{X}_i\, a[u]^{1/2}\right)\ \rho(z)\right](rz_0) = 0.$$

Combining the above estimates, we have

$$\begin{aligned}
0 &= \limsup_{r\to 1^-} \Bigg[\mathrm{Re}\,(\overline{u}^s \mathcal{L} u^s(rz_0)) + 2\mathrm{Re}\left(\frac{\overline{u}^s R u^s \overline{u}^t \overline{R} u^t}{a[u]}(rz_0)\right) \\
&\quad +2\mathrm{Re}\left(\frac{\overline{u}^s X_i u^s \overline{u}^t \overline{X}_i u^t}{a[u]|\rho(z)|}(rz_0)\right)\Bigg] \\
&\le \limsup_{r\to 1^-} \mathrm{Re}\,\left(\overline{u}^s \mathcal{L} u^s(rz_0)\right) \\
&= \overline{u}^s \mathcal{L} u^s(z_0)
\end{aligned}$$

Therefore, $\overline{u}^s \mathcal{L} u^s(z_0) \ge 0$. By (2.16), (2.17) and $a[u](z_0) = 0$, we have $0 \le 2\overline{u}^s \mathcal{L} u^s(z_0) = -E_b(u)(z_0) \le 0$. Thus $E_b(u)(z_0) = 0$, and (iii) is proved.

Finally, we prove (iv). The statement of (iv) was proved in [D1] as well as in [LT1] for real and complex hyperbolic spaces. For the sake of convenience we provide a proof using our notation. Let $\phi(z) = u(z)$ on ∂B_n. Then we have

$$u(z) = \phi(z/|z|) + b_1(z)\rho(z), \quad v(z) = \phi(z/|z|) + b_2(z)\rho(z),$$

for z near ∂B_m. Here $\rho(z)$ is as above, $b_i(z)$ are vector valued functions defined by the above equations. Direct calculation shows that $|u(z)|^2 = 1 + 2 < \phi, b_1 > \rho + |b_1|^2\rho^2$. Using the defining expression of $a[u]$ we can write

$$a[u] = 2\langle \phi, b_1\rangle \qquad \text{on } \partial B_m.$$

Similarly we have

$$a[v] = 2\langle \phi, b_2\rangle \qquad \text{on } \partial B_m.$$

By (iii), since $E_b(\phi)(z) \ne 0$ for all $z \in \partial B_m$, we have $a[u] = a[v]$ on ∂B_m, which is also given by (2.23). Thus

$$\begin{aligned}
1 - \langle u(z), v(z)\rangle &= [\langle \phi, b_2\rangle + \langle b_1(z), \phi\rangle]\rho(z) + \langle b_1, b_2(z)\rangle \rho(z)^2 \\
&= \frac{a[u] + a[v]}{2}\rho(z) + O(|\rho|^{3/2}).
\end{aligned}$$

Then

$$\lim_{z\to p} \frac{1 - \langle u(z), v(z)\rangle}{\rho(z)} = \frac{1}{2}\left[a[u](p) + a[v](p)\right] = a[u](p)$$

and

$$\lim_{z\to p}\frac{|1-\langle u(z),v(z)\rangle|^2-(|u(z)|^2-1)(|v(z)|^2-1)}{|\rho(z)|^2}$$
$$=a[u](p)^2-a[u](p)a[v](p)=0.$$

Therefore, for any $p\in\partial B_m$, by (iii), we have $a[u](p)>0$ and

$$\lim_{z\to p}d_B(u(z),v(z))=c_n\lim_{z\to p}\log$$
$$\left(\frac{|1-\langle u(z),v(z)\rangle|+\sqrt{|1-\langle u(z),v(z)\rangle|^2-(1-|u(z)|^2)(1-|v(z)|^2)}}{|1-\langle u(z),v(z)\rangle|-\sqrt{|1-\langle u(z),v(z)\rangle|^2-(1-|u(z)|^2)(1-|v(z)|^2)}}\right)$$
$$=c_n\log\left(\frac{a[u](p)}{a[v](p)}\right)=0.$$

Since $d_B(u(z),v(z))^2$ is subharmonic for any two harmonic maps in B_m (see [SY]). The maximum principle shows that $u=v$. The proof of (iv) is complete.

Therefore, the proof of the proposition is complete. □

3 Cauchy-Riemann functions

In this section we study the following question:
Question: *Given a C^2 function $g(z)$ on ∂B_m satisfying that for each $z\in\partial B_m$ either $\overline{\partial}_b g(z)=0$ or $\overline{\partial}_b\overline{g}(z)=0$ holds, can one conclude that either g is CR on ∂B_m or $\overline{g}$ is CR on ∂B_m?*

From the previous section we know that the understanding of this question is useful and closely related to the problem we posed in the introduction. The purpose of this section is to answer this question affirmatively. In particular, we prove the following theorem.

Theorem 3.1 *Let $g\in C^2(\partial B_m)$ such that for any point $z\in\partial B_m$ we have either $\overline{\partial}_b g(z)=0$ or $\overline{\partial}_b\overline{g}(z)=0$. Then either g is CR function on ∂B_m or $\overline{g}$ is CR on ∂B_m.*

Proof. Let $A=\{z\in\partial B_m:\overline{\partial}_b\overline{g}(z)=0\}$ and $B=\{z\in\partial B_m:\overline{\partial}_b g(z)=0\}$. By the assumption, we have $\partial B_m=A\cup B$, and that A and B are closed subsets in ∂B_m. Thus $\partial B_m=A_0\cup B_0$. Let A_0 be closure of $\mathrm{Int}(A)$ and B_0 the closure of $\mathrm{Int}(B)$. Let $A_1=A_0\cap B_0$. If $\mathrm{Int}(A_0)=\emptyset$ then g is CR function on ∂B_m, and the theorem is proved. Without loss of generality, we may assume that $A_0\neq\emptyset$. We shall prove that $\overline{g}$ is a CR function on ∂B_m.

For any point $z_0\in A_1$ we have $X_j g(z_0)=\overline{X}_j g(z_0)=0$ for all $1\le j\le m$. Moreover, since $X_k\overline{X}_j g=0$ on $\mathrm{Int}(B)$, $\overline{X}_k X_j g=0$ on $\mathrm{Int}(A)$ and $g\in C^2(\partial B_m)$

we have that $X_k\overline{X}_j g = 0$ on B_0, $\overline{X}_k X_j g = 0$ on A_0 and both equal zero on A_1 for all $1 \le j, k \le m$. Thus

$$(m-1)(R-\overline{R})g(z_0) = (X_j\overline{X}_j - \overline{X}_j X_j)g(z_0) = 0.$$

Let

$$G(z) = (R-\overline{R})g(z).$$

Then $G(z) = 0$ on A_1. Let

$$\tilde{G}(z) = \begin{cases} 0, & \text{if } z \in B_0 \\ G(z), & \text{if } z \in A_0 \end{cases}$$

Then $\tilde{G} \in C(\partial B_m)$. Since

$$\begin{aligned}
\left[X_j, (R-\overline{R})\right] &= X_j(R-\overline{R}) - (R-\overline{R})X_j \\
&= X_jR - X_j\overline{R} - RX_j + \overline{R}X_j \\
&= \partial_j R - \overline{z}_j RR - \partial_j\overline{R} + \overline{z}_j R\overline{R} - R\partial_j + \overline{z}_j RR \\
&\quad +\overline{R}\partial_j - \overline{z}_j R - \overline{z}_j\overline{R}R \\
&= \partial_j R - R\partial_j - \overline{z}_j R \\
&= \partial_j + R\partial_j - R\partial_j - \overline{z}_j R \\
&= X_j.
\end{aligned}$$

Therefore

$$X_j(R-\overline{R}) = (R-\overline{R})X_j + [X_j, R-\overline{R}] = (R-\overline{R})X_j + X_j.$$

Thus, for any $z \in A_0$, since $g \in C^2(\partial B_m)$ and A_0 is closure of $\mathrm{Int}(A)$, we have

$$X_jG(z) = X_j(R-\overline{R})g = (R-\overline{R})X_jg + X_jg = 0, \quad 1 \le j \le m.$$

This implies that $X_j\tilde{G} = 0$ on ∂B_m for all $1 \le j \le m$, i.e., $\overline{\tilde{G}}$ is a CR function on ∂B_m. It then follows that either $G \equiv 0$ or $B_0 = \emptyset$. If $B_0 = \emptyset$ then $\overline{g}$ is CR on ∂B_m, and theorem is proved. Without loss of generality, we may assume that $G \equiv 0$ on ∂B_m. Since

$$\begin{aligned}
&\overline{X}_k X_j - X_j\overline{X}_k \\
&= (\overline{\partial}_k - z_k\overline{R})(\partial_j - \overline{z}_j R) - (\partial_j - \overline{z}_j R)(\overline{\partial}_k - z_k\overline{R}) \\
&= \partial_{\overline{k}j} - \delta_{jk}R - \overline{z}_j\overline{\partial}_k R - z_k\overline{R}\partial_j + z_k\overline{z}_j R + z_k\overline{z}_j\overline{R}R - \partial_{j\overline{k}} + z_k\partial_j\overline{R} \\
&\quad +\delta_{jk}\overline{R} + \overline{z}_j R\partial_{\overline{k}} - \overline{z}_j z_k\overline{R} - \overline{z}_j z_k R\overline{R} \\
&= -\delta_{jk}R - z_k\overline{R}\partial_j + z_k\overline{z}_j R + \delta_{jk}\overline{R} - \overline{z}_j z_k\overline{R} + z_k\partial_j\overline{R} \\
&= (\delta_{kj} - z_k\overline{z}_j)(\overline{R} - R)
\end{aligned}$$

for all $1 \le j, k \le m$. It follows that

$$\overline{X}_k X_j g = X_j \overline{X}_k g + (\delta_{kj} - z_k \overline{z}_j)(\overline{R} - R) g = X_j \overline{X}_k g \quad z \in \partial B_m$$

for all $1 \le j, k \le m$. This implies that $\overline{X}_k (X_j g) = 0$ for all $1 \le j, k \le m$. Thus $X_j g$ has holomorphic extension to B_m, and $X_j g = 0$ on $A_0 \ne \emptyset$. Thus $X_j g \equiv 0$ for $1 \le j \le m$. Therefore, we have $g \equiv$ constant, and the proof of the theorem is complete. □

4 The proof of Theorem 1.2

In this section, we shall prove Theorem 1.2.

Lemma 4.1 *If $u \in C^{1,\alpha}(\overline{B}_m)$ with $\alpha > 1/2$ be a harmonic map from $B_m \to B_n$ then for any $z_0 \in \partial B_m$ with $a[u](z_0) > 0$ we have*

$$\lim_{z \to z_0} e[u](z) = m + 2 \frac{\overline{u}^\alpha R u^\alpha \, \overline{u}^\beta \overline{R} u^\beta}{a[u](z_0)^2}.$$

Proof. We first compute the energy density:

$$\begin{aligned}
e[u](z) &= h^{i\overline{j}} g_{\alpha\overline{\beta}} \left(\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta} \right) \\
&= \frac{1 - |z|^2}{1 - |u(z)|^2} \left(\delta_{ij} - z_i \overline{z}_j \right) \left(\delta_{\alpha\overline{\beta}} + \frac{\overline{u}^\alpha u^\beta}{1 - |u(z)|^2} \right) \left(\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta} \right) \\
&= \frac{1}{a[u](z)} \left(\delta_{ij} - z_i \overline{z}_j \right) \Bigg[\partial_i u^\alpha \overline{\partial_j u^\alpha} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\alpha} \\
&\quad + \frac{\overline{u}^\alpha u^\beta}{1 - |u(z)|^2} \left(\partial_i u^\alpha \overline{\partial_j u^\beta} + \partial_{\overline{j}} u^\alpha \overline{\partial_{\overline{i}} u^\beta} \right) \Bigg] \\
&= \frac{1}{a[u](z)} \Bigg[(X_j u^\alpha) \overline{\partial_j u^\alpha} + (\overline{X}_i u^\alpha) \overline{\partial_{\overline{i}} u^\alpha} \\
&\quad + \frac{(\overline{u}^\alpha X_j u^\alpha) \overline{\overline{u}^\beta \partial_j u^\beta} + (\overline{u}^\alpha \overline{X}_i u^\alpha) \overline{\overline{u}^\beta \partial_{\overline{i}} u^\beta}}{1 - |u(z)|^2} \Bigg] \\
&= \frac{1}{a[u](z)} \Bigg[(X_j u^\alpha) \overline{X_j u^\alpha} + \left(1 - |z|^2\right) (R u^\alpha) \overline{R u^\alpha} \\
&\quad + (\overline{X}_i u^\alpha) \overline{\overline{X}_i u^\alpha} + \left(1 - |z|^2\right) (\overline{R} u^\alpha) \overline{\overline{R} u^\alpha} \\
&\quad + \frac{\left(\overline{u}^\alpha X_j u^\alpha\right) \overline{\overline{u}^\beta X_j u^\beta} + \left(1 - |z|^2\right) \left(\overline{u}^\alpha R u^\alpha\right) \overline{\overline{u}^\beta R u^\beta}}{1 - |u(z)|^2} \\
&\quad + \frac{(\overline{u}^\alpha \overline{X}_i u^\alpha) \overline{\overline{u}^\beta \overline{X}_i u^\beta} + \left(1 - |z|^2\right) (\overline{u}^\alpha \overline{R} u^\alpha) \overline{\overline{u}^\beta \overline{R} u^\beta}}{1 - |u(z)|^2} \Bigg]
\end{aligned}$$

$$= \frac{1}{a[u](z)} \Big[(|X_j u^\alpha|^2 + |\overline{X}_i u^\alpha|^2) + \big(1-|z|^2\big) (|Ru^\alpha|^2 + |\overline{R}u^\alpha|^2)$$
$$+ \frac{(\overline{u}^\alpha X_j u^\alpha)\overline{\overline{u}^\beta X_j u^\beta} + (\overline{u}^\alpha \overline{X}_i u^\alpha)\overline{\overline{u}^\beta \overline{X}_i u^\beta} + \big(1-|z|^2\big)[|(\overline{u}^\alpha R u^\alpha)|^2 + |\overline{u}^\alpha \overline{R} u^\alpha|^2]}{1-|u(z)|^2} \Big].$$

Therefore, if u is a proper harmonic map and for any $z_0 \in \partial B_m$ with $a[u](z_0) > 0$ we have

$$\begin{aligned}
&\lim_{z\to z_0} e[u](z) \\
&= \frac{1}{a[u](z_0)} \left[|\partial_b u(z_0)|^2 + |\overline{\partial}_b u(z_0)|^2 + \frac{(\overline{u}^\alpha R u^\alpha)^2 + (u^\alpha \overline{R} u^\alpha)^2}{a[u](z_0)} \right] \\
&= \frac{1}{a[u](z_0)} \Big[-2\overline{u}^\alpha \mathcal{L} u^\alpha(z_0) + (m-1)(\overline{u}^\alpha R u^\alpha + \overline{u}^\alpha \overline{R} u^\alpha) \\
&\quad + \frac{(\overline{u}^\alpha R u^\alpha)^2 + (u^\alpha \overline{R} u^\alpha)^2}{a[u](z_0)} \Big] \\
&= m - 1 + \frac{1}{a[u](z_0)} \left[4 \frac{(\overline{u} R u^\alpha)(\overline{u}^\beta \overline{R} u^\beta)(z_0)}{a[u](z_0)} + \frac{(\overline{u}^\alpha R u^\alpha)^2 + (u^\alpha \overline{R} u^\alpha)^2}{a[u](z_0)} \right] \\
&= m + 2 \frac{(\overline{u}^\alpha R u^\alpha)(\overline{u}^\beta \overline{R} u^\beta)(z_0)}{a[u](z_0)^2}
\end{aligned}$$

and the proof is complete. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First, we need the following fact pointed out to us by Peter Li: If u is a proper harmonic map then $u|_{\partial B_m}$ is not a constant map (One can consult [LW] for more general results and arguments of the proof).

Notice that $\mathcal{L}u$ is orthogonal to u on ∂B_m if and only if $\overline{u}^s \mathcal{L} u^s = 0$ on ∂B_m. By Proposition 2.2 and Lemma 4.1, we have, for any $z \in \partial B_m$ that if $a[u](z) \neq 0$ then

$$\overline{u}^s \mathcal{L} u^s(z) = 0 \iff \overline{u}^s R u^s(z) \overline{u}^t \overline{R} u^t(z) = 0 \iff \lim_{w\to z} e[u](z) = m.$$

This implies that (iii) and (iv) are equivalent.

It is obvious that (i) implies (iii). Now we prove that (iii) implies (i). Since $\overline{u}^s \mathcal{L} u^s = 0$ on ∂B_m, by Proposition 2.3, we have

$$|\partial_b u|^2 = (m-1) \sum_s \overline{u}^s R u^s, \quad |\overline{\partial}_b u|^2 = (m-1) \sum_s \overline{u}^s \overline{R} u^s.$$

Combining this with Proposition 2.2, we have $|\partial_b u||\overline{\partial}_b u| = 0$. Applying Theorem 3.1, we have that $u|_{\partial B_m}$ has an either holomorphic or anti-holomorphic map extension v. On the other hand, since u is proper harmonic, we have $u|_{\partial B_m}$ is non-constant. Thus, by Hopf's lemma we have $a[v] > 0$ on ∂B_m. By Proposition 2.2, we have $a[u] = a[v]$, and Part (iv) of Proposition 2.3 gives that $u = v$. Thus (i) holds.

It is clear that (i) implies (ii). Next we prove (ii) implies (i). It is sufficient to prove that for any $z_0 \in \partial B_m$ with $a[u](z_0) > 0$, we have

$$\overline{u}^s \mathcal{L} u^s(z_0) = 0.$$

By rotation, without loss of generality, we may assume that $z_0 = (0, 0, \cdots, 0, 1)$. Since u is $(m-1)$-harmonic with respect to the origin, we have $u(z_1, \cdots, z_{j-1}, 0, z_{j+1}, \cdots, z_n)$ is proper harmonic on B_{m-1} for all $1 \le j \le m-1$ as a function of $\hat{z}_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_m)$. If we denote

$$v(j)(z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_m) = u(z_1, \cdots, z_{j-1}, 0, z_{j+1}, \cdots, z_m)$$

then

$$\overline{v}^s(j) R v^s(j)(0, \cdots, 0, 1) = \overline{u}^s R u^s(z_0).$$

Applying Proposition 2.2 to $v(j)$, we have

$$\sum_{k=1, k\neq j}^{m} \overline{u}^s(z_0) u^s_{k\overline{k}}(z_0) + 2\frac{\overline{u}^s R u^s\, \overline{u}^t \overline{R} u^t(z_0)}{a[u](z_0)} = 0.$$

In other words,

$$\overline{u}^s \mathcal{L} u^s(z_0) - \overline{u}^s(z_0) u^s_{j\overline{j}}(z_0) + 2\frac{\overline{u}^s R u^s \overline{u}^t \overline{R} u^t(z_0)}{a[u](z_0)} = 0.$$

Since u is proper $(m-1)$-harmonic map with respect to the origin, u is proper harmonic. Therefore, by Proposition 2.2, we have

$$\overline{u}^s \mathcal{L} u^s(z_0) + 2\frac{\overline{u}^s R u^s \overline{u}^t \overline{R} u^t(z_0)}{a[u](z_0)} = 0.$$

Combining the above two equalities, we have

$$\overline{u}^s(z_0) u^s_{j\overline{j}}(z_0) = 0, \quad 1 \le j \le m-1.$$

This implies

$$\overline{u}^s(z_0) \mathcal{L} u^s(z_0) = \sum_{j=1}^{m-1} \overline{u}^s u^s_{j\overline{j}}(z_0) = 0.$$

We complete our proof of Theorem 1.2. □

Corollary 4.2 *There is no non-constant proper harmonic map u from $B_m \to B_1$ with $m > 1$ so that $u \in C^1(\overline{B}_m)$.*

Proof. Let $u \in C^1(\overline{B}_m)$ be a proper harmonic map from $B_m \to B_1$. Then $X_j u = \overline{X}_j u = 0$ on ∂B_m. Therefore, $u|_{\partial B_m}$ has a holomorphic and antiholomorphic extension on B_m. By uniqueness, we have that u is both holomorphic and anti-holomorphic. Therefore, u must be a constant. □

5 Proof of Theorem 1.1

In this section, we shall prove Theorem 1.1 by constructing a counterexample. That is, we show that there is a harmonic map which is neither holomorphic nor anti-holomorphic. First we shall prove the following existence theorem, which was proved in [D1] for $C^{2,\alpha}$ boundary maps with nonvanishing energy density. Here using our previous calculation we can push it to $C^{1,\alpha}$ boundary maps.

Theorem 5.1 *Let $k \geq 1$ and $0 < \alpha \leq 1$, and let $\phi : \partial B_m \to \partial B_n$ so that $\phi \in C^{k,\alpha}(\partial B_m)$ satisfy that $E[\phi](z) \neq 0$ on ∂B_m and the necessary condition*

$$\sum_s \overline{\phi^s}(z) X_j \phi^s(z) = 0, \quad z \in \partial B_m, \ \ 1 \leq j \leq m. \tag{5.1}$$

Then there is a unique proper harmonic map $u \in C^{l,\beta}(\overline{B}_m)$ such that $u = \phi$ on ∂B_m for all $0 < l + \beta < \min\{m, k + \alpha\}$.

Proof. We first consider $\phi \in C^{k,\alpha}(\partial B_m)$ with $k \geq 2$ and $\alpha \geq 0$. Let $\phi(z)$ denote the radial extension of ϕ from ∂B_m to $\overline{B}_m$. We try to apply Li-Tam's general existence theorem of [LT3]. So as the first step we construct an approximating harmonic map. We consider an extension $v(z)$ given by

$$v(z) = \phi(z) + \rho(z) b(z),$$

where $\rho(z)$ is defined as in last section, $b(z)$ is a vector valued function which will be given later. Since

$$\mathcal{L}v(z) = \mathcal{L}\phi(z) + \mathcal{L}\rho(z) b(z) + \rho(z)\mathcal{L}b + X_i \rho\, \partial_{\bar{i}} b + \overline{X}_j \rho\, \partial_j b(z),$$

we have

$$\mathcal{L}v(z) = \frac{1}{2}(\overline{X}_j X_j + X_j \overline{X}_j)\phi(z) + b(z)(m-1), \quad z \in \partial B_m.$$

Let

$$b(z) = b_0(z)\phi(z), \quad z \in \overline{B}_m,$$

where b_0 is a non-negative function, which will be determined later. Since $|\phi| = 1$, as in the proof of Proposition 2.3, we have

$$a[v](z) = 2\overline{v}^s b^s = 2b_0(z) \quad \text{on} \quad \partial B_m.$$

Since $|\phi|^2 = 1$, and ϕ satisfies (5.1), we have

$$\begin{aligned}(m-1)\left(\sum_s \overline{\phi}^s (R - \overline{R})\phi^s\right) &= \overline{\phi}^s (X_j \overline{X}_j - \overline{X}_j X_j)\phi^s \\ &= -|\overline{\partial}_b \phi|^2 + |\partial_b \phi|^2, \quad z \in \partial B_m.\end{aligned}$$

Thus, on ∂B_m, we have

$$\begin{aligned}&a[v]\overline{v}^s \mathcal{L} v^s + 2(\overline{v}^s R v^s)(\overline{v}^t \overline{R} v^t) \\ &= 2b_0 \overline{v}^s \mathcal{L} v^s + \frac{1}{2}\left[a[v]^2 - \left(\sum_s \overline{v}^s (R - \overline{R})\phi^s\right)^2\right] \\ &= b_0 \overline{\phi}^s \left(\overline{X}_j X_j + X_j \overline{X}_j\right)\phi^s + b_0(m-1)a[v] + 2b_0^2 - \frac{1}{2}\left[|\partial_b \phi|^2 - |\overline{\partial}_b \phi|^2\right]^2 \\ &= -b_0 E_b(\phi) + 2mb_0^2 - \frac{1}{2}\left[-|\partial_b \phi|^2 + |\overline{\partial}_b \phi|^2\right].\end{aligned}$$

Now let

$$\begin{aligned}4mb_0(z) &= E_b(\phi) + \sqrt{[E_b(\phi) + \frac{4m}{(m-1)^2}[|\overline{\partial}_h \phi|^2 - |\partial_b \phi|^2]^2} \\ &= E_b(\phi) + \frac{1}{m-1}\sqrt{(m+1)^2 E_b(\phi)^2 - 16|\overline{\partial}_b \phi|^2 |\partial_b \phi|^2} \\ &\geq 2E_b(\phi).\end{aligned}$$

Then $b_0(z) > 0$ since $E(\phi) > 0$. Thus $a[v] = 2b_0 \geq E_b(\phi)/(2m) > 0$ on ∂B_m and

$$a[v]\overline{v}^s \mathcal{L} v^2 + (\overline{v}^s R v^s)(\overline{v}^t \overline{R} v^t) = 0, \quad z \in \partial B_m.$$

The fact $b_0 > 0$ also implies that $v(z)$ map B_m to B_n. Since

$$\begin{aligned}|\tau[v]|_g^2 &= g_{\alpha\overline{\beta}} \tau^\alpha[v] \overline{\tau^\beta[v]} \\ &= \frac{\sum_\alpha |\tau^\alpha[v]|^2}{\left(1 - |v|^2\right)} + \frac{\overline{v}^\alpha \tau^\alpha \overline{\overline{v}^\beta \tau^\beta[v]}}{\left(1 - |v|^2\right)^2} \\ &= \frac{\sum_\alpha |\tau^\alpha[v]|^2}{\left(1 - |v|^2\right)} \\ &\quad + \frac{\left[\left(1 - |z|^2\right)\left(a[v]\overline{v}^\alpha \mathcal{L} v^s + 2\overline{v}^s R v^s \overline{v}^s \overline{R} v^s\right)\right) + \overline{v}^s X_i v^s \overline{v}^t \overline{X}_i v^t\right]^2}{a[v]^2 \left(1 - |v|^2\right)^2} \\ &= O\left(1 - |z|^2\right),\end{aligned}$$

we have $|\tau[u]|_g \in L^{2p}(B_m, d\lambda_m)$ for $p > m$ where $d\lambda_m(z) = K_{B_m}(z,z)dv(z)$ and $K_{B_m}(z,z) = c_m\left(1-|z|^2\right)^{-m-1} dv(z)$ is the Bergman kernel function of B_m.

When $\phi \in C^{k,\alpha}(\partial B_m)$ with $k \geq 1$ and $0 < \alpha \leq 1$, the above construction shows that

$$|\tau[v]|_g^2 = O\left(\left(1-|z|^2\right)^{\alpha}\right).$$

Thus $|\tau[v]|_g \in L^{2p}(B_m, d\lambda_m)$ when $p > m/\alpha$. Applying the existence theorem of [LT3] and the argument of proving regularity in [LT1], we have completed the proof of Theorem 5.1. □

Proof of Theorem 1.1.

Let

$$\phi(z) = ((|z|^2 - |w|^2), \sqrt{2}zw, \sqrt{2}\overline{zw}) = (\phi^1(z,w), \phi^2(z,w), \phi^3(z,w)).$$

Then it is easy to verify that $\phi : \partial B_2 \to \partial B_3$. To prove Theorem 1.1, it suffices to check that ϕ satisfies the conditions in Theorem 5.1. Let

$$X = \overline{z}\frac{\partial}{\partial w} - \overline{w}\frac{\partial}{\partial z}$$

Then X spans the complex tangent space $T^{(1,0)}(\partial B_2)$. Since

$$X\phi^1(z,w) = -\overline{zw} - \overline{wz} = -2\overline{zw},$$

$$X\phi^2(z,w) = \sqrt{2}(|z|^2 - |w|^2)$$

and

$$X\phi^3(z,w) = 0.$$

We have

$$\begin{aligned}\overline{\phi}^s X\phi^s(z,w) &= -2\overline{zw}(|z|^2-|w|^2) + \sqrt{2}\overline{zw}\sqrt{2}(|z|^2-|w|^2)\\ &= 0 \quad \text{for} \quad (z,w) \in \partial B_2,\end{aligned}$$

which is the first assumption of Theorem 5.1.

On the other hand, for any $(z,w) \in \partial B_2$, we have

$$|X\phi^1|^2 + |X\phi^2|^2 + |X\phi^3|^2 = 4|z|^2|w|^2 + 2(|z|^2-|w|^2)^2 = 2(|z|^4+|w|^4) \geq 1.$$

By Theorem 5.1, there is a proper harmonic map $u \in C^{1,\alpha}(\overline{D}_2)$ with $u = \phi$ on ∂B_2 for all $0 < \alpha < 1$. It is obvious that $\overline{X}\phi^1(z,w) = -2zw \not\equiv 0$ and $X\phi^1(z,w) = -2\overline{zw} \not\equiv 0$ on ∂B_2. Thus ϕ has neither a proper holomorphic nor an anti-holomorphic extension to B_2, and the proof of Theorem 1.1 is complete. □

Finally, we give the following remark.

Remark 5.1 It was proved by C. R. Graham and J. M. Lee [GL] that the Dirichlet problem

$$\Delta_{B_m} f = 0 \quad \text{in } B_m, \quad f = \phi \quad \text{on } \partial B_m$$

has a unique solution $f \in C^{m-1,\alpha}(\overline{B}_m)$ if $\phi \in C^k(\partial B_m)$ when $k \geq m$, no matter how big k is. Similar regularity result for Einstein-Kähler metric was given by J. Lee and R. Melrose [LM]. It was also proved in [GL] that if $f \in C^m(\overline{B}_m)$ then f must be pluriharmoynic. A natural question can be asked is: If $u \in C^m(\overline{B}_m)$ is a proper harmonic map from B_m to B_n in the Bergman metric, is u pluriharmonic? If it is true, then Theorem 1.2 will show it is holomorphic or anti-holomorphic.

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