# PARABOLIC FREQUENCY MONOTONICITY AND A THEOREM OF HARDY-PÓLYA-SZEGÖ

LEI NI

Abstract. In this note we discuss the connections between the Li-Yau-Hamilton type estimates for heat equations on Riemannian and Kähler manifolds, the monotonicity of the frequency functional and a Hardy-Pólya-Szegö type inequality.

## 1. Introduction

The frequency functional and its monotonicity on $\mathbb{R}^n$ was introduced by Almgren [A] and used in the study of the local regularity of (multiple valued) harmonic functions and minimal surfaces. The result states that *if $u$ is a harmonic function on $\mathbb{R}^n$ (or in a region containing the point $x$). Then*

$$I_2^e(x,r) \doteqdot \frac{r\int_{B(x,r)} |\nabla u|^2 \, d\mu}{\int_{\partial B(x,r)} u^2 \, dA},$$

*where $dA$ is the induced $n-1$-dimensional Haussdorff measure on $\partial B(x,r)$, is monotone non-decreasing in $r$.* When $n=2$, the result is in fact first proved by Hardy (cf. Exercise 7 on page 138 of [Co] as well as Theorem 1 on page 148 of [H]). Later on it has been used by Garofalo-Lin [GL1, GL2] and Lin [Li] to study the unique continuation properties and to estimate the size of nodal sets. Here in $I_2^e$, $e$ stands for 'elliptic' (versus the parabolic analogue which we shall introduce in the next) and 2 is for the $L^2$-norms involved. Notice that if one denotes

$$Z_2^e(x,r) = \int_{\partial B(x,r)} u^2 \, dA$$

then $2I_2^e(x,r) = r\frac{\partial}{\partial r}\left(\log Z_2^e(x,r)\right)$. One can refer to [HL, Z] for surveys and more applications of the frequency monotonicity. It is a great pleasure to contribute this note on the occasion of the 60th birthday of Professor Phong.

## 2. Parabolic frequency and its monotonicity

Let $(M,g)$ be a complete Rimemannian manifold. Let $u(x)$ be a harmonic function on $M$ satisfying certain growth conditions so that all the integration by parts can be justified. Let $H(x,y,t)$ be the fundamental solution to the heat equation $(\frac{\partial}{\partial t}-\Delta)v(x,t)=0$. Define the following quantities:

$$Z_2(x,t) \doteqdot \int_M H(x,y,t)u^2(y)\,d\mu(y), \tag{2.1}$$

$$D_2(x,t) \doteqdot \int_M H(x,y,t)|\nabla_y u|^2(y), \tag{2.2}$$

$$I_2(x,t) \doteqdot \frac{tD_2(x,t)}{Z_2(x,t)}. \tag{2.3}$$

When the reference point $x$ is not important we simply denote them by $Z_2(t), D_2(t)$ and $I_2(t)$. Direct calculation shows that

$$\begin{aligned} \frac{d}{dt}Z_2(t) &= 2D_2(t) \geq 0 \\ \frac{d}{dt}D_2(t) &= \int_M \Delta_y H(x,y,t)|\nabla_y u|^2\,d\mu(y) \\ &= -2\int_M \nabla_i H \nabla_i\nabla_j u \nabla_j u\,d\mu(y) \\ &= 2\int_M \nabla_i\nabla_j H \nabla_i u \nabla_j u\,d\mu(y). \end{aligned}$$

Recall the matrix estimate of Hamilton which asserts that on a Riemannian manifold with nonnegative sectional curvature and parallel Ricci curvature,

$$\nabla_i\nabla_j H - \frac{\nabla_i H \nabla_j H}{H} + \frac{H}{2t}g_{ij} \geq 0. \tag{2.4}$$

Using this estimate one can prove the monotonicity of $I_2(t)$.

**Theorem 2.1.** *Assume that $M$ is a Riemannian manifold with nonnegative sectional curvature and parallel Ricci. Let $u$ be a harmonic function of polynomial growth (or satisfies some mild growth conditions so that integration by parts can be carried out). Then $\frac{d\,I_2}{dt} \geq 0$, which implies that $\log Z_2(t)$ is an increasing function and convex in $\log t$.*

*Proof.* The direct computations show that

$$\begin{aligned} \frac{d}{dt}I_2(t) &= t\frac{D_2'(t)}{Z_2(t)} - 2t\left(\frac{D_2(t)}{Z_2(t)}\right)^2 + \frac{D_2(t)}{Z_2(t)} \\ &= \frac{t}{Z_2^2(t)}\Bigg[\left(2\int_M \nabla_i\nabla_j H \nabla_i u \nabla_j u\,d\mu(y)\right)Z_2(t) \\ &\quad -2\left(\int_M \langle\nabla u, \nabla H\rangle u\,d\mu\right)^2 + \frac{1}{t}D_2(t)Z_2(t)\Bigg]. \end{aligned}$$

Here we have used the identity

$$\int_M |\nabla u|^2(y)H(x,y,t)\,d\mu = -\int_M \langle\nabla u, \nabla H\rangle u\,d\mu.$$

Using Hamilton's matrix inequality (2.4) we have that

$$\begin{aligned}\frac{d}{dt} I_2(t) &\geq \frac{t}{Z_2^2(t)}\left[\left(2\int_M \frac{\langle \nabla u, \nabla H\rangle^2}{H}\, d\mu - \frac{D_2(t)}{t}\right) Z_2(t)\right.\\ &\quad \left. -2\left(\int_M \langle \nabla u, \nabla H\rangle u\, d\mu\right)^2 + \frac{1}{t} D_2(t) Z_2(t)\right]\\ &\geq 0.\end{aligned}$$

The last inequality above follows from the Hölder's inequality. □

The result can be generalized to the eigenfunctions, namely those $u$ with $\Delta u = -\lambda u$. If we keep the definition of $Z_2(t)$, $D_2(t)$ and $I_2(t)$, then the following identities hold:

$$\frac{d}{dt} Z_2(t) = 2D_2(t) - 2\lambda Z_2(t) \tag{2.5}$$

$$D_2(t) = -\int_M \langle \nabla H, \nabla u\rangle u\, d\mu + \lambda Z_2(t). \tag{2.6}$$

Making use of them, together with the matrix differential estimate and Hölder's inequalty we have that

$$\begin{aligned}\frac{d}{dt} D_2(t) &= \int_M \Delta H |\nabla u|^2\\ &= 2\int_M \left(\nabla_i \nabla_j H \nabla_i u \nabla_j u - \lambda \langle \nabla H, \nabla u\rangle u\right) d\mu\\ &\geq 2\int_M \frac{\langle \nabla H, \nabla u\rangle^2}{H}\, d\mu - \frac{D_2(t)}{t} + 2\lambda (D_2(t) - \lambda Z_2(t))\\ &\geq 2\frac{(D_2(t) - \lambda Z_2(t))^2}{Z_2(t)} - \frac{D_2(t)}{t} + 2\lambda(D_2(t) - \lambda Z_2(t)).\end{aligned}$$

A direct consequence is the sharp dimension count on the space of the harmonic functions of polynomial growth. For any $d > 0$, letting $r(x)$ be the distance function to some fixed point $o \in M$, define

$$\mathcal{H}_d(M) \doteqdot \{f \mid \Delta f = 0, |f|(x) \leq C(1 + r(x))^d\}.$$

**Corollary 2.2.** *Let $(M, g)$ be as in theorem. Then*

$$\dim(\mathcal{H}_d(M)) \leq \dim(\mathcal{H}_d(\mathbb{R}^n))$$

**Corollary 2.3.** *The frequency monotonicity of $I_2(t)$ holds for function $u$ satisfying $\Delta u = -\lambda u$. Moreover $\log Z_2(t)$ satisfies that $\frac{d^2}{d s^2} \log Z_2(t) \geq -2\lambda e^s$, where $s = \log t$.*

*Proof.* Using the estimates above, direct calculation shows

$$\begin{aligned}\frac{d}{d t} I_2(t) &= t\frac{\frac{d}{d t} D_2(t)}{Z_2(t)} - 2t\frac{D_2(t)(D_2(t) - \lambda Z_2(t))}{Z_2^2(t)} + \frac{D_2(t)}{Z_2(t)}\\ &\geq 2t\frac{(D_2(t) - \lambda Z_2(t))^2}{Z_2^2(t)} + 2t\lambda \frac{D_2(t) - \lambda Z_2(t)}{Z_2(t)} - 2t\frac{D_2(t)(D_2(t) - \lambda Z_2(t))}{Z_2^2(t)}\\ &= 0.\end{aligned}$$

This proves the first claim. The second claim follows from the first. □

For the solution to heat equation, a similar monotonicity holds. Now consider $u(x,t)$, a solution to the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0 \tag{2.7}$$

on $M \times (0,T)$. Now let us pick any $(x_0,t_0)$ let $\tau = t_0 - t$, let $H(x,\tau;x_0,0)$ be fundamental solution to the backward heat equation. Similarly we define

$$Z_2(t) \doteqdot \int_M H(x,\tau;x_0,0)u^2(x,t)\,d\mu(x), \tag{2.8}$$

$$D_2(t) \doteqdot \int_M H(x,\tau;x_0,0)|\nabla_x u|^2(x,t)\,d\mu(x), \tag{2.9}$$

$$I_2(t) \doteqdot \frac{\tau D_2(t)}{Z_2(t)}. \tag{2.10}$$

The following result holds.

**Theorem 2.4.** *Assume that $M$ is a Riemannian manifold with nonnegative sectional curvature and parallel Ricci. Let $u(x,t)$ be a solution to the heat equation (2.7). Then $\frac{d\,I_2}{dt} \le 0$.*

*Proof.* First, the direct calculation as before yields:

$$I_2'(t) = \left(\frac{\tau D_2(t)}{Z_2(t)}\right)' = \frac{\tau}{Z_2^2(t)}\left(D_2'(t)\,Z_2(t) - Z_2'(t)\,D_2(t) - \frac{1}{\tau}D_2(t)Z_2(t)\right).$$

Here $(\cdot)'$ means $\frac{d}{dt}(\cdot)$. Since $\left(\frac{\partial}{\partial\tau} - \Delta\right)H(x,\tau;x_0,0) = 0$, using the matrix estimate (2.4) we have that

$$\begin{aligned}
Z_2'(t) &= -2\int_M H|\nabla u|^2\,d\mu \\
&= 2\int_M \left(\langle\nabla H,\nabla u\rangle u + Hu_t u\right)\,d\mu; \\
D_2'(t) &= \int_M (-\Delta H)|\nabla u|^2 + H(|\nabla u|^2)_t\,d\mu \\
&= 2\int_M \left(\nabla_i H\nabla_i\nabla_j u\nabla_j u + H\nabla_i u\nabla_i u_t\right)\,d\mu \\
&= -2\int_M \left(\nabla_i\nabla_j H\nabla_i u\nabla_j u + \langle\nabla H,\nabla u\rangle\Delta u + \langle\nabla H,\nabla u\rangle u_t + H(u_t)^2\right)\,d\mu \\
&\le -2\int_M \left(\frac{\nabla_i H\nabla_j H}{H}\nabla_i u\nabla_j u + 2\langle\nabla H,\nabla u\rangle u_t + H(u_t)^2\right)\,d\mu \\
&\quad + \frac{1}{\tau}\int_M H|\nabla u|^2\,d\mu \\
&= -2\int_M H\left(\langle\nabla\log H,\nabla u\rangle + u_t\right)^2 + \frac{1}{\tau}\int_M H|\nabla u|^2\,d\mu.
\end{aligned}$$

Combining the above inequalities we have that

$$\begin{aligned} I_2'(t) &\leq -\frac{2\tau}{Z_2^2(t)}\left(\left(\int_M H\left(\langle \nabla \log H, \nabla u\rangle + u_t\right)^2\, d\mu\right)\left(\int_M Hu^2\, d\mu\right)\right. \\ &\quad \left. - \left(\int_M \left(\langle \nabla H, \nabla u\rangle u + Hu_t u\right)\, d\mu\right)^2\right) \\ &\leq 0 \end{aligned}$$

by Hölder's inequality again. □

It was pointed to us by B. Kotschwar that the above result on the solution to the heat equation was essentially proved by Poon in [P].

## 3. A theorem of Hardy-Pólya-Szegö

A theorem of Pólya-Szegö (cf. [H], page 150, Theorem 1) asserts that *if $f(z)$ is a holomorphic function on $\mathbb{C}$ (or a region containing $0$). Then $Z^e_p(0,r) \doteqdot \frac{1}{2\pi}\int_0^{2\pi} |f(re^{\sqrt{-1}\theta})|^p\, d\theta$ is an increasing function of $r$ and $\log Z^e_p(0,r)$ is a convex function of $\log r$.* When $p = 2$, the result is due to Hardy. Hence the result of Pólya-Szegö is a generalization of Hardy's result. Note that for $p = 2$, the convexity of $Z^e_2(0,r)$ is the same as the monotonicity of the frequency functional. In this section we shall establish a result which serves as a parabolic version of the result of Pólya-Szegö but on Kähler manifolds with nonnegative bisectional curvature of any complex dimension. Moreover the result is related to a monotonicity proved in [N1], in the way that the result of [N1] is the limit of the result here as $p \to 0$.

Let $M$ be a Kähler manifold with nonnegative bisectional curvature. Assume that $f$ is a holomorphic function. Let $H(x,y,t)$ be the heat kernel. Let

$$\begin{aligned} Z_p(x,t) &= \int_M H(x,y,t)|f|^p(y)\, d\mu, \\ D_p(x,t) &= \frac{p}{4}\int_M H(x,y,t)|\nabla f|^2|f|^{p-2}\, d\mu, \\ I_p(x,t) &= \frac{tD_p(x,t)}{Z_p(x,t)}. \end{aligned}$$

As before we sometimes just write as $Z_p(t), D_p(t), I_p(t)$ by omitting the reference to $x$. These integrals are finite if we assume that $f$ is of finite order in the sense of Hadamard [N1]. Recall that $H(x,y,t)$ satisfies the estimate [CN, N2]:

$$(\log H)_{i\bar{j}} + \frac{1}{t}g_{i\bar{j}} \geq 0. \tag{3.1}$$

Here $g_{i\bar{j}}$ is the Kähler metric. For Kähler manifolds we use the convention that $\Delta = \frac{1}{2}(\nabla_i\nabla_{\bar{i}} + \nabla_{\bar{i}}\nabla_i)$, under a normal coordinate, for tensors. Also $\langle \nabla F, \nabla G\rangle = \frac{1}{2}(\nabla_i F \nabla_{\bar{i}} G +$

$\nabla_i G \nabla_{\bar{i}} F)$. The direct calculation shows that

$$\begin{aligned}
\int_M \Delta_y H(x,y,t)|f|^p(y)\, d\mu &= -\frac{p}{4}\int_M H_i f\bar{f}_{\bar{i}}|f|^{p-2} + H_{\bar{i}} f_i \bar{f}|f|^{p-2} \\
&= -\frac{p}{2}\int_M \langle \nabla H, \nabla |f|^2\rangle |f|^{p-2}\, d\mu; \\
-\frac{1}{2}\int_M \langle \nabla H, \nabla |f|^2\rangle |f|^{p-2}\, d\mu &= -\frac{1}{4}\int_M H_i \bar{f}_{\bar{i}} f|f|^{p-2} + H_{\bar{i}} f_i \bar{f}|f|^{p-2}\, d\mu \\
&= \frac{1}{2}\left[\int_M H|\nabla f|^2|f|^{p-2} + (\frac{p}{2}-1)H|f|^{p-2} f_i \bar{f}_{\bar{i}}\, d\mu\right] \\
&= D_p(t).
\end{aligned}$$

These imply that

$$\frac{1}{p}\frac{d}{dt}(\log Z_p(x,t)) = \frac{D_p(t)}{Z_p(t)}. \tag{3.2}$$

Further computation shows that

$$\begin{aligned}
\frac{4}{p}\frac{d}{dt}D_p(t) &= \int_M \Delta H|\nabla f|^2|f|^{p-2}\, d\mu \\
&= -\int_M H_i |f|^{p-2} f_j \bar{f}_{\bar{j}\bar{i}} - (\frac{p}{2}-1)H_i|f|^{p-4}\bar{f}_{\bar{i}} f|\nabla f|^2\, d\mu \\
&= \int_M H_{i\bar{j}} f_j \bar{f}_{\bar{i}}|f|^{p-2} + (\frac{p}{2}-1)H_i|f|^{p-4} f\bar{f}_{\bar{j}} f_j \bar{f}_{\bar{i}} - (\frac{p}{2}-1)H_i|f|^{p-4}\bar{f}_{\bar{i}} f|\nabla f|^2\, d\mu \\
&= \int_M H_{i\bar{j}} f_j \bar{f}_{\bar{i}}|f|^{p-2}\, d\mu.
\end{aligned}$$

Combining them together with the estimate (3.1) we have the following theorem.

**Theorem 3.1.** *Let $(M^m, g)$ be a complete Kähler manifold with nonnegative bisectional curvature. Assume that $f$ is a holomorphic function of finite order. Then for any $p>0$, $\frac{1}{p}\log Z_p(x,t)$ is an increasing, convex function of $\log t$.*

*Proof.* We only need to prove the convexity, which is equivalent to the monotonicity of $I_p(x,t)$. Direct calculation shows that

$$\frac{d}{dt}I_p(t) = \frac{t\frac{d}{dt}D_p(t)}{Z_p(t)} - pt\left(\frac{D_p(t)}{Z_p(t)}\right)^2 + \frac{D_p(t)}{Z_p(t)}.$$

Putting the above computation together and applying (3.1) one has that

$$\begin{aligned}
\frac{d}{dt}I_p(t) &= \frac{t}{Z_p(t)}\left(\frac{p}{4}\int_M H_{i\bar{j}} f_j \bar{f}_{\bar{i}}|f|^{p-2}\, d\mu - \frac{p}{4Z_p(t)}\left(\int_M \langle \nabla H, \nabla |f|^2\rangle |f|^{p-2}\, d\mu\right)^2 + \frac{D_p(t)}{t}\right) \\
&\geq \frac{pt}{4Z_p(t)}\left(\int_M 4\left|\langle \nabla H, \nabla \bar{f}\rangle\right|^2 \frac{|f|^{p-2}}{H}\, d\mu - \frac{1}{Z_p(t)}\left(\int_M \langle \nabla H, \nabla |f|^2\rangle |f|^{p-2}\, d\mu\right)^2\right).
\end{aligned}$$

Applying Hölder inequality we have that $\frac{d}{dt}I_p(t) \geq 0$. $\square$

In [N1], Corollary 2.1 (see also Theorem 3.3 of [N2]) we proved that

$$t\int_M H(x,y,t)\Delta \log |f|^2\, d\mu$$

is an increasing function of $t$. This fact plays the key role in the resolution of one of Yau's conjecture on the sharp dimension comparison on the space of holomorphic functions of polynomial growth. It is easy to see that the monotonicity is equivalent to the following consequence of Theorem 3.1.

**Corollary 3.2.** *Let $(M^m, g)$ and $f$ be as in Theorem 3.1. Then*

$$\log Z_0(x,t) \doteqdot \int_M H(x,y,t)\log|f|(y)\, d\mu_y$$

*is an increasing convex function of* $\log t$.

*Proof.* Letting $p \to 0$, we have that

$$\begin{aligned} \frac{1}{p}\log Z_p(x,t) &= \frac{1}{p}\log\left(\int_M He^{p\log|f|}\right) d\mu \\ &= \frac{1}{p}\log\left(\int_M H\left(1+p\log|f| + \mathrm{O}(p^2)\right)\right) d\mu \\ &= \frac{1}{p}\log\left(1+p\int_M H\log|f|\, d\mu + \mathrm{O}(p^2)\right) \\ &\to \int_M H(x,y,t)\log|f|(y)\, d\mu. \end{aligned}$$

Hence the claimed result follows from Theorem 3.1. □

Since the frequency monotonicity can be viewed as certain *entropy* property in *a statistical ensemble* (cf. page 568 of [Z]), the earlier result in [N1] can be viewed similarly.

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## References

[A] F. J. Almgren, *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents.* Minimal submanifolds and geodesics (M. Obata, ed.) 1–6, North Holland, Amsterdam, 1979.

[CN] H.-D. Cao and L. Ni, *Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds.* Math. Ann. **331** (2005), no. 4, 795–807.

[Co] J. Conway, *Functions of one complex variable.* Second edition. Graduate Texts in Mathematics, 11. Springer-Verlag, New York-Berlin, 1978. xiii+317 pp.

[GL1] N. Garofalo and F.-H. Lin, *Monotonicity properties of variational integrals, $A_p$ weights and unique continuation.* Indiana Univ. Math. J. **35**(1986), no. 2, 245–268.

[GL2] N. Garofalo and F.-H. Lin, *Unique continuation for elliptic operators: a geometric-variational approach.* Comm. Pure Appl. Math. **40**(1987), no. 3, 347–366.

[HL] Q. Han and F.-H. Lin, *Nodal sets of solutions of elliptic differential equation.* Book in preparation.

[H] L.-K. Hua, *An introduction to higher mathematics.* Second Printing (in chinese). Vol 3, Higher Education Press, Beijing, 2009.

[Li] F.-H. Lin, *Nodal sets of solutions of elliptic and parabolic equations.* Comm. Pure Appl. Math. **44**(1991), no. 3, 287–308.

[N1] L. Ni, *A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature.* J. Amer. Math. Soc. **17** (2004), no. 4, 909–946 (electronic).

[N2] L. Ni, *A matrix Li-Yau-Hamilton estimate for Kähler-Ricci flow.* J. Differential Geom. **75** (2007), no. 2, 303–358.

[P] C.-C. Poon, *Unique continuation for parabolic equations.* Comm. Partial Differential Equations **21** (1996), no. 3-4, 521–539.

[Z] S. Zelditch, *Local and global analysis of eigenfunctions on Riemannian manifolds.* Handbook of geometric analysis. No. 1, 545–658, Adv. Lect. Math. (ALM), **7**, Int. Press, Somerville, MA, 2008.

ADDRESSES:

LEI NI, Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA

email: lni@math.ucsd.edu