A NOTE ON PERELMAN'S LYH TYPE INEQUALITY

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Abstract

We give a proof to the Li-Yau-Hamilton type inequality claimed by Perelman on the fundamental solution to the conjugate heat equation. The rest of the paper is devoted to improving the known differential inequalities of Li-Yau-Hamilton type via monotonicity formulae.

1. Introduction

In [**P**], Perelman *proved* a Li-Yau-Hamilton type (also called differential Harnack) inequality for the fundamental solution of the conjugate heat equation, on a manifold evolving by the Ricci flow. More precisely, let $(M, g_{ij}(t))$ be a solution to Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} \tag{1.1}$$

on $M \times [0,T]$ and let $H(x,y,\tau) = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ (where $\tau = T-t$) be the fundamental solution to the conjugate heat equation $u_\tau - \Delta u + Ru = 0$. (More precisely we should write the fundamental solution as $H(y,t;x,T)$, which satisfies $\left(-\frac{\partial}{\partial t} + \Delta_y + R(y,t)\right) H(y,t;x,T) = 0$ for any (y,t) with $t < T$ and $\lim_{t\to T} \int_M H(y,t;x,T) f(y,t)\, d\mu_t(y) = f(x,T)$.) Define

$$v_H = \left[\tau\left(2\Delta f - |\nabla f|^2 + R\right) + f - n\right] H.$$

Here all the differentiations are taken with respect to y, and $n = \dim_{\mathbb{R}}(M)$. Then $v_H \le 0$ on $M \times [0,T]$. This result is a differential inequality of Li-Yau type [**LY**], which has important applications in the later part of [**P**]. For example it is essential in proving the pseudo-locality theorem in Section 10 of [**P**]. It is also crucial in localizing the entropy formula [**N3**].

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In Section 9 of [**P**], the following important differential equation

$$\left(\frac{\partial}{\partial\tau} - \Delta + R\right) v_u = -2\tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 u \tag{1.2}$$

is stated for any positive solution u to the conjugate heat equation, whose integration on M gives the celebrated entropy formula for the Ricci flow. One can consult various sources (e.g. [**N1**]) for the detailed computations of this equation, which can also be done through a straightforward calculation, after knowing the result. [**P**] then proceeds with the proof of the claim $v_H \le 0$ in a clever way by checking that for any τ_* with $T \ge \tau_* > 0$, $\int_M v_H(y)h(y)\,d\mu_{\tau_*}(y) \le 0$, for any smooth function $h(y) \ge 0$ with compact support. In order to achieve this, in [**P**] the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) h(y,t) = 0$ with the 'initial data' $h(y, T-\tau_*) = h(y)$ (more precisely $t = T - \tau_*$), the given compactly supported nonnegative function, is solved. Applying (1.2) to $u(y,\tau) = H(x,y,\tau)$, one can easily derive as in [**P**], via integration by parts, that

$$\frac{d}{d\tau}\int_M v_H h\,d\mu_\tau = -2\int_M \tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2 Hh\,d\mu_\tau \le 0. \tag{1.3}$$

The Li-Yau type inequality $v_H \le 0$ then follows from the above monotonicity, provided the claim that

$$\lim_{\tau\to 0}\int_M v_H h\,d\mu_\tau \le 0. \tag{1.4}$$

The main purpose of this note is to prove (1.4), hence provide a complete proof of the claim $v_H \le 0$. This will be done in Section 3 after some preparations in Section 2. It was written in [**P**] that '*it is easy to see*' that $\lim_{\tau\to 0}\int_M v_H h\,d\mu_\tau = 0$. It turns out that the proof found here needs to use some gradient estimates for positive solutions, quite precise estimate on the 'reduced distance', a tool also introduced by Perelman in [**P**], and the monotonicity formula (1.3). (We shall focus on the proof of (1.4) for the case when M is compact and leave the more technical details of generalizing it to the noncompact setting to the later refinements.) Indeed the claim that $\lim_{\tau\to 0}\int_M v_H h\,d\mu_\tau = 0$ follows from a blow-up argument of [**P**], after we have established (1.4). Since our argument is a bit involved, this may not be *the proof*.

In Section 4 we derive several monotonicity formulae, which improve various Li-Yau-Hamilton inequalities for linear heat equation (systems) as well as for Ricci flow, including the original Li-Yau's inequality. In Section 5 we illustrate the localization of them by applying a general scheme of [**EKNT**].

2. Estimates and results needed

We shall collect some known results and derive some estimates needed for proving (1.4) in this section. We need the asymptotic behavior of the fundamental solution to the conjugate heat equation for small τ. Let $d_\tau(x,y)$ be the distance function with respect to the metric $g(\tau)$. Let $B_\tau(x,r)$ (Vol_τ) be the ball of radius r centered at x (the volume) with respect to the metric $g(\tau)$.

Theorem 2.1. *Let $H(x,y,\tau)$ be the fundamental solution to the (backward in t) conjugate heat equation. Then as $\tau \to 0$ we have that*

$$H(x,y,\tau) \sim \frac{\exp\left(-\frac{d_0^2(x,y)}{4\tau}\right)}{(4\pi\tau)^{\frac{n}{2}}} \sum_{j=0}^{\infty} \tau^j u_j(x,y,\tau). \tag{2.1}$$

By (2.1) we mean that exists $T > 0$ and sequence $u_j \in C^\infty(M \times M \times [0,T])$ such that

$$H(x,y,\tau) - \frac{\exp\left(-\frac{d_0^2(x,y)}{4\tau}\right)}{(4\pi\tau)^{\frac{n}{2}}} \sum_{j=0}^{k} \tau^j u_j(x,y,\tau) = w_k(x,y,\tau)$$

with

$$w_k(x,y,\tau) = O\left(\tau^{k+1-\frac{n}{2}}\right)$$

as $\tau \to 0$, uniformly for all x, $y \in M$. The function $u_0(x,y,\tau)$ can be chosen so that $u_0(x,x,0) = 1$.

This result was proved in detail, for example in [**GL**], when there is no zero order term $R(y,\tau)u(y,\tau)$ in the equation $\frac{\partial}{\partial\tau}u - \Delta u + Ru = 0$ and replacing $d_0(x,y)$ by $d_\tau(x,y)$. However, one can check that the argument carries over to this case if one assumes that the metric $g(\tau)$ is C^∞ near $\tau = 0$. One can consult [**SY**, **CLN**] for intrinsic presentations.

Let

$$\mathcal{W}_h(g,H,\tau) = \int_M v_H h \, d\mu_\tau$$

where h is the previously described solution to the heat equation. It is clear that for any τ with $T \geq \tau > 0$, $\mathcal{W}_h(g,H,\tau)$ is a well-defined quantity. A priori it may blow up as $\tau \to 0$. It turns out that in our course of proving that $\lim_{\tau\to 0}\mathcal{W}_h(g,H,\tau) \leq 0$ we need to show first that exists $C > 0$, which may depends on the geometry of the Ricci flow solution $(M, g(\tau))$ defined on $M \times [0,T]$, but independent of τ (as $\tau \to 0$) so that $\mathcal{W}_h(g,H,\tau) \leq C$ for all $T \geq \tau > 0$. The following lemma (see also [**EKNT**] for a localized version of it) supplies the key estimates for this purpose.

Lemma 2.2. *Let $(M, g(t))$ be a smooth solution to the Ricci flow on $M \times [0,T]$. Assume that there exist $k_1 \geq 0$ and $k_2 \geq 0$, such that the*

Ricci curvature $R_{ij}(g(\tau)) \geq -k_1 g_{ij}(\tau)$ *and* $\max(R(y,\tau), |\nabla R|^2(y,\tau)) \leq k_2$, *on* $M \times [0,t]$.

(i) If $u \leq A$ *is a positive solution to the conjugate heat equation on* $M \times [0,T]$, *then there exists* C_1 *and* C_2 *depending on* k_1, k_2 *and* n *such that for* $0 < \tau \leq \min(1, T, \frac{1}{2k_2})$,

$$\tau \frac{|\nabla u|^2}{u^2} \leq (1 + C_1\tau)\left(\log\left(\frac{A}{u}\right) + C_2\tau\right) \tag{2.2}$$

(ii) If u *is a positive solution to the conjugate heat equation on* $M \times [0,T]$, *then there exists* B, *depending on* $(M, g(\tau))$ *so that for* $0 \leq \tau \leq \min(T, \frac{1}{2k_2}, 1)$,

$$\tau \frac{|\nabla u|^2}{u^2} \leq (2 + C_1\tau)\left(\log\left(\frac{B}{u\tau^{\frac{n}{2}}}\int_M u\, d\mu_\tau\right) + C_2\tau\right). \tag{2.3}$$

Remark 2.3. Here and thereafter we use the same C_i (B) at different lines if they differ only by a constant depending on n. Notice that $\int_M u\, d\mu_\tau$ is independent of τ and equal to 1 if u is the fundamental solution. The proof of the lemma given below is a modification of some arguments in [**H**].

Proof. Direct computation, using a unitary frame, gives

$$\begin{aligned}
\left(\frac{\partial}{\partial\tau} - \Delta\right)\left(\frac{|\nabla u|^2}{u}\right) &= -\frac{2}{u}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 + \frac{|\nabla u|^2}{u}R \\
&\quad + \frac{-4R_{ij}u_i u_j - 2\langle\nabla(Ru), \nabla u\rangle}{u} \\
&\leq (4+n)k_1\frac{|\nabla u|^2}{u} + 2|\nabla R||\nabla u| \\
&\leq [(4+n)k_1 + 1]\frac{|\nabla u|^2}{u} + k_2 u
\end{aligned}$$

and

$$\begin{aligned}
\left(\frac{\partial}{\partial\tau} - \Delta\right)\left(u\log\left(\frac{A}{u}\right)\right) &= \frac{|\nabla u|^2}{u} + Ru - Ru\log\left(\frac{A}{u}\right) \\
&\geq \frac{|\nabla u|^2}{u} - nk_1 u - k_2 u\log\left(\frac{A}{u}\right).
\end{aligned}$$

Combining the above two equations together we have that

$$\left(\frac{\partial}{\partial\tau} - \Delta\right)\Phi \leq 0$$

where

$$\Phi = \varphi\frac{|\nabla u|^2}{u} - e^{k_2\tau}u\log\left(\frac{A}{u}\right) - 2(k_2 + nk_1 e^{k_2})\tau u$$

with $\varphi = \frac{\tau}{1+[(4+n)k_1+1]\tau}$, which satisfies

$$\frac{d}{d\tau}\varphi + [(4+n)k_1 + 1]\,\varphi < 1.$$

By the maximum principle we have that

$$\varphi\frac{|\nabla u|^2}{u} \leq e^{k_2\tau}u\log\left(\frac{A}{u}\right) + 2(k_2 + nk_1e^{k_2})\tau u.$$

From this one can derive (2.2) easily.

To prove the second part, we claim that for u, a positive solution to the conjugate heat equation, there exists a C depending on $(M, g(\tau))$ such that

$$u(y,\tau) \leq \frac{C}{\tau^{\frac{n}{2}}}\int_M u(z,\tau)\,d\mu_\tau(z). \tag{2.4}$$

This is a mean-value type inequality, which can be proved via, for example the Moser iteration. Here we follow [**H**]. We may assume that $\sup_{y\in M, 0\leq\tau\leq 1}\tau^{\frac{n}{2}}u(y,\tau)$ is finite. Otherwise we may replacing τ by $\tau_\epsilon = \tau - \epsilon$ and let $\epsilon \to 0$ after establishing the claim for τ_ϵ. Now let $(x_0,\tau_0) \in M\times[0,1]$ be such a space-time point that $\max\tau^{\frac{n}{2}}u(y,\tau) = \tau_0^{\frac{n}{2}}u(y_0,\tau_0)$. Then we have that

$$\sup_{M\times[\frac{\tau_0}{2},\tau_0]} u(y,t) \leq \left(\frac{2}{\tau_0}\right)^{\frac{n}{2}}\tau_0^{\frac{n}{2}}u(y_0,\tau_0) = 2^{\frac{n}{2}}u(y_0,t_0).$$

Noticing this upper bound, we apply (2.2) to u on $M\times[\frac{\tau_0}{2},\tau_0]$, and conclude that

$$\frac{\tau_0}{2}\left(\frac{|\nabla u|^2}{u^2}\right)(y,\tau_0) \leq (1+C_1\tau_0)\left(\log\left(\frac{2^{\frac{n}{2}}u(y_0,\tau_0)}{u(y,\tau_0)}\right) + C_2\tau_0\right).$$

Let $g = \log\left(\frac{2^{\frac{n}{2}}u(y_0,\tau_0)}{u(y,\tau_0)}\right) + C_2\tau_0$. The above can be written as

$$|\nabla\sqrt{g}| \leq \sqrt{\frac{1+C_1\tau_0}{2\tau_0}}$$

which implies that

$$\sup_{B_{\tau_0}\left(y_0,\sqrt{\frac{\tau_0}{1+C_1\tau_0}}\right)}\sqrt{g}(y,\tau_0) \leq \sqrt{g}(y_0,\tau_0) + \frac{1}{\sqrt{2}}.$$

Rewriting the above in terms of u we have that

$$u(y,\tau_0) \geq 2^{\frac{n}{2}}u(y_0,\tau_0)e^{-\left(\frac{1}{2}+\frac{2}{\sqrt{2}}\sqrt{\frac{n}{2}\log 2+C_2}\right)} = C_3u(y_0,\tau_0)$$

for all $y \in B_{\tau_0}\left(y_0, \sqrt{\frac{\tau_0}{1+C_1\tau_0}}\right)$. Here we have also used $\tau_0 \le 1$. Noticing that

$$Vol_{\tau_0}\left(B_{\tau_0}\left(y_0, \sqrt{\frac{\tau_0}{1+C_1\tau_0}}\right)\right) \ge C_4 \tau_0^{\frac{n}{2}}$$

for some C_4 depending on the geometry of $(M, g(\tau_0))$. Therefore we have that

$$\frac{C_5}{\tau_0^{\frac{n}{2}}} \int_M u(y,\tau_0)\, d\mu_{\tau_0}(y) \ge u(y_0,\tau_0)$$

for some C_5 depending on C_3 and C_4. By the way we choose (y_0, τ_0) we have that

$$\tau^{\frac{n}{2}} u(y,\tau) \le \tau_0^{\frac{n}{2}} u(y_0,\tau_0) \le C_5 \int_M u(y,\tau_0)\, d\mu_{\tau_0}(y) = C_5 \int_M u(y,\tau)\, d\mu_\tau(y).$$

This proves the claim (2.4). Now the estimate (2.3) follows from (2.2), applied to u on $M \times [\frac{\tau}{2}, \tau]$, and the just proved (2.4), which ensures the needed upper bound for applying the estimate (2.2). q.e.d.

If $u(y,\tau) = \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}$ is the fundamental solution $H(y,\tau;x_0,0)$ (expressed in terms of τ) to the conjugate heat equation we have that $\int_M u\, d\mu_\tau = 1$. Therefore, by (2.3), we have that

(2.5)

$$\int_M \tau |\nabla f|^2 uh\, d\mu_\tau \le (2 + C_1\tau) \int_M \left(\log B + f + \frac{n}{2}\log(4\pi) + C_2\tau\right) uh\, d\mu_\tau.$$

On the other hand, integrating by parts we can rewrite

$$\begin{aligned} \mathcal{W}_h(g,u,\tau) &= \int_M \tau|\nabla f|^2 uh\, d\mu_\tau - 2\tau \int_M \langle \nabla f, \nabla h\rangle u\, d\mu_\tau \\ &\quad + \tau \int_M Ruh\, d\mu_\tau + \int_M (f-n) uh\, d\mu_\tau \\ &= I + II + III + IV. \end{aligned}$$

The I term can be estimated by (2.5), whose right hand side contains only one 'bad' term $\int_M fuh\, d\mu_\tau$ in the sense that it could possibly blow up. The second term

$$II = 2\tau \int_M \langle \nabla u, \nabla h\rangle\, d\mu = -2\tau \int_M u\Delta h\, d\mu_\tau$$

is clearly bounded as $\tau \to 0$. In fact $II \to 0$ as $\tau \to 0$. The same conclusion obviously holds for III. Summarizing above, we reduce the question of bounding from above the quantity $\mathcal{W}_h(u,g,\tau)$ to bounding one single term

$$V = \int_M fuh\, d\mu_\tau$$

from above (as $\tau \to 0$). We shall show later that $\lim_{\tau\to 0} V \leq 0$. To do this we need to use the 'reduced distance', introduced by Perelman in [**P**] for the Ricci flow geometry.

Let x be a fixed point in M. Let $\ell(y,\tau)$ be the reduced distance in [**P**], with respect to $(x,0)$ (more precisely $\tau = 0$). We collect the relevant properties of $\ell(y,\tau)$ in the following lemma (Cf. [**Ye**, **CLN**]).

Lemma 2.4. *Let $\bar{L}(y,\tau) = 4\tau\ell(y,\tau)$.*

(i) Assume that there exists a constant k_1 such that $R_{ij}(g(\tau)) \geq -k_1 g_{ij}(\tau)$, $\bar{L}(y,\tau)$ is a local Lipschitz function on $M \times [0,T]$;

(ii) Assume that there exist constant k_1 and k_2 so that $-k_1 g_{ij}(\tau) \leq R_{ij}(g(\tau)) \leq k_2 g_{ij}(\tau)$. Then

$$\bar{L}(y,\tau) \leq e^{2k_2\tau} d_0^2(x,y) + \frac{4k_2 n}{3}\tau^2 \tag{2.6}$$

and

$$d_0^2(x,y) \leq e^{2k_1\tau}\left(\bar{L}(y,\tau) + \frac{4k_1 n}{3}\tau^2\right); \tag{2.7}$$

(iii)

$$\left(\frac{\partial}{\partial\tau} - \Delta + R\right)\left(\frac{\exp\left(-\frac{\bar{L}(y,\tau)}{4\tau}\right)}{(4\pi\tau)^{\frac{n}{2}}}\right) \leq 0. \tag{2.8}$$

Proof. The first two claims follow from the definition by straight forward checking. For (iii), it was proved in Section 7 of [**P**]. By now there are various sources where a detailed proof can be found. See for example [**Ye**] and [**CLN**]. q.e.d.

As a consequence of (2.6) and (2.7)

$$\lim_{\tau\to 0} \frac{\exp\left(-\frac{\bar{L}(y,\tau)}{4\tau}\right)}{(4\pi\tau)^{\frac{n}{2}}} = \delta_x(y),$$

which together with (2.8) implies that H, the fundamental solution to the conjugate heat equation, is bounded from below as

$$H(x,y,\tau) \geq \frac{\exp\left(-\frac{\bar{L}(y,\tau)}{4\tau}\right)}{(4\pi\tau)^{\frac{n}{2}}},$$

by the heat kernel comparison principle (cf. Proposition 1 of [**CLY**], noticing the duality between the fundamental solution of the heat equation and the fundamental solution of the conjugate heat equation). Hence

$$f(y,\tau) \leq \frac{\bar{L}(y,\tau)}{4\tau}. \tag{2.9}$$

This was proved in [**P**] making use of the inequality $v_H \leq 0$. Since we are in the middle of proving $v_H \leq 0$, we provide the above alternative of obtaining (2.9).

3. Synthesis

Now we assemble the results in the previous section to prove (1.4). As the first step we show that $\mathcal{W}_h(g, H, \tau)$ is bounded (thanks to the monotonicity (1.3), it is sufficient to bound it from above) as $\tau \to 0$, where $H(x, y, \tau)$ is the fundamental solution to the conjugate heat equation with $H(x, y, 0) = \delta_x(y)$. By the reduction done in the previous section we only need to show that

$$V = \int_M f H h \, d\mu_\tau$$

is bounded from above as $\tau \to 0$. By (2.9) we have that

$$\begin{aligned}
\limsup_{\tau\to 0} \int_M f H h \, d\mu_\tau &\leq \limsup_{\tau\to 0} \int_M \frac{\bar{L}(y,\tau)}{4\tau} H(x, y, \tau) h(y, \tau) \, d\mu_\tau(y) \\
&\leq \limsup_{\tau\to 0} \int_M \frac{d_0^2(x, y)}{4\tau} H(x, y, \tau) h(y, \tau) \, d\mu_\tau(y) \\
&+ \lim_{\tau\to 0} \int_M \left(\frac{e^{k_2\tau} - 1}{4\tau} d_0^2(x, y) + \frac{k_2 n}{3}\tau \right) H(x, y, \tau) h(y, \tau) \, d\mu_\tau(y).
\end{aligned}$$

Here we have used (2.6) in the last inequality. By Theorem 2.1, some elementary computations give that

$$\lim_{\tau\to 0} \int_M \frac{d_0^2(x, y)}{4\tau} H(x, y, \tau) h(y, \tau) \, d\mu_\tau(y) = \frac{n}{2} h(x, 0).$$

Since $\frac{e^{k_2\tau}-1}{4\tau} d_0^2(x, y) + \frac{k_2 n}{3}\tau$ is a bounded continuous function even at $\tau = 0$, we have that

$$\lim_{\tau\to 0} \int_M \left(\frac{e^{k_2\tau} - 1}{4\tau} d_0^2(x, y) + \frac{k_2 n}{3}\tau \right) H(x, y, \tau) h(y, \tau) \, d\mu_\tau(y) = 0.$$

This completes our proof of the finiteness of $\limsup_{\tau\to 0} \int_M f H h \, d\mu_\tau$. In fact we have proved that

$$\limsup_{\tau\to 0} \int_M (f - \frac{n}{2}) H h \, d\mu_\tau \leq 0. \tag{3.1}$$

By the just proved finiteness of $\mathcal{W}_h(g, H, \tau)$ as $\tau \to 0$, and the (entropy) monotonicity (1.3), we know that the limit $\lim_{\tau\to 0} \mathcal{W}_h(g, H, \tau)$ exists. Let

$$\lim_{\tau\to 0} \mathcal{W}_h(g, H, \tau) = \lim_{\tau\to 0} \int_M v_H h \, d\mu_\tau = \alpha$$

for some finite α. Hence $\lim_{\tau\to 0}\left(\mathcal{W}_h(g,H,\tau)-\mathcal{W}_h(g,H,\frac{\tau}{2})\right)=0$. By (1.3) and the mean-value theorem we can find $\tau_k\to 0$ such that

$$\lim_{\tau_k\to 0}\tau_k^2\int_M |R_{ij}+\nabla_i\nabla_j f-\frac{1}{2\tau_k}g_{ij}|^2 Hh\,d\mu_{\tau_k}=0.$$

By the Cauchy-Schwartz inequality and the Hölder inequality we have that

$$\lim_{\tau_k\to 0}\tau_k\int_M\left(R+\Delta f-\frac{n}{2\tau_k}\right)Hh\,d\mu_{\tau_k}=0.$$

This implies that

$$\lim_{\tau\to 0}\mathcal{W}_h(g,H,\tau)=\lim_{\tau_k\to 0}\int_M\left(\tau_k(\Delta f-|\nabla f|^2)+f-\frac{n}{2}\right)Hh\,d\mu_{\tau_k}.$$

Again integration by parts shows that

$$\begin{aligned}\int_M\tau_k(\Delta f-|\nabla f|^2)Hh\,d\mu_{\tau_k} &= \int_M\tau_k\langle\nabla H,\nabla h\rangle\,d\mu_{\tau_k}\\ &= -\tau_k\int_M H\Delta h\,d\mu_{\tau_k}\to 0.\end{aligned}$$

Hence by (3.1)

$$\lim_{\tau\to 0}\mathcal{W}_h(g,H,\tau)=\lim_{\tau_k\to 0}\int_M(f-\frac{n}{2})Hh\,d\mu_{\tau_k}\le 0.$$

This proves $\alpha\le 0$, namely (1.4).

The claim that $\alpha=\lim_{\tau\to 0}\mathcal{W}_h(g,H,\tau)=0$ can now be proved by the blow-up argument as in Section 4 of [**P**]. Assume that $\alpha<0$. One can easily check that this would imply that $\lim_{\tau\to 0}\mu(g,\tau)<0$. Here $\mu(g,\tau)$ is the invariant defined in Section 4 of [**P**]. In fact, noticing that $h(y,\tau)>0$ for all $\tau\le\tau_*$ (where the τ_* is the one we fixed in the introduction). Therefore by multiple $\frac{1}{h(x,0)}$ (more precisely $\frac{1}{h(x,\cdot)}$ at $\tau=0$) to the original $h(y,\tau)$, we may assume that $h(x,0)=\int_M H(x,y,\tau)h(y,\tau)\,d\mu_\tau=1$. Let $\tilde{u}(y,\tau)=H(x,y,\tau)h(y,\tau)$ and $\tilde{f}=-\log\tilde{u}-\frac{n}{2}\log(4\pi)$. Now direct computation yields that

$$\mathcal{W}_h(g,H,\tau)=\mathcal{W}(g,\tilde{u},\tau)+\int_M\left(\tau\left(\frac{|\nabla h|^2}{h}\right)-h\log h\right)H\,d\mu_\tau.$$

Noticing that the second integration goes to 0 as $\tau\to 0$, we can deduce that $\mathcal{W}(g,\tilde{u},\tau)<0$ for sufficient small τ if $\alpha<0$. This, together with the fact $\int_M\tilde{u}\,d\mu=1$, implies that $\mu(g,\tau)<0$ for sufficiently small τ. Now Perelman's blow-up argument in the Section 4 of [**P**] gives a contradiction with the sharp logarithmic Sobolev inequality on the Euclidean space [**G**]. (One can consult, for example [**N1**, **STW**], for more details of this part.)

REMARK 3.1. The method of proof here follows a similar idea used in [**N1**], where the asymptotic limit of the entropy as $\tau \to \infty$ was computed. Note that we have to use properties of the reduced distance, introduced in Section 7 of [**P**], in our proof, while the similar, but slightly easier, claim that $\lim_{\tau\to 0} \mathcal{W}(g, H, \tau) = 0$ appears much earlier in Section 4 of [**P**].

REMARK 3.2. R. Hamilton asked whether or not the LYH-type estimate $v_u \leq 0$ still holds for more general positive solution u to the conjugate heat equation, other than the fundamental solution. The proof presented here can be adapted to show that it still holds for finite sum of fundamental solutions. Namely let $u(y,\tau) = \sum_{i=1}^{k} H(y,t;x_i,T)$. Then the estimate (1.4), hence $v_u \leq 0$, still holds for such u.

The proof can be easily modified to give the asymptotic behavior of the entropy defined in [**N1**] for the fundamental solution to the linear heat equation, with respect to a fixed Riemannian metric. Indeed if we restrict to the class of complete Riemannian manifolds with non-negative Ricci curvature we have the following estimates.

Proposition 3.3. *For any $\delta > 0$, there exists $C(\delta)$ such that*

$$\frac{|\nabla H|^2}{H}(x,y,\tau) \leq 2\frac{H(x,y,\tau)}{\tau}\left(C(\delta) + \frac{d^2(x,y)}{(4-\delta)\tau}\right) \tag{3.2}$$

and

$$\Delta H(x,y,\tau) + \frac{|\nabla H|^2}{H}(x,y,\tau) \leq 2\frac{H(x,y,\tau)}{\tau}\left(C(\delta) + 4\frac{d^2(x,y)}{(4-\delta)\tau}\right). \tag{3.3}$$

The previous argument for the Ricci flow case can be transplanted to show that

$$\tau(2\Delta f - |\nabla f|^2) + f - n \leq 0$$

where $H(y,\tau;x,0) = \frac{1}{(4\pi\tau)^{n/2}}e^{-f}$ is the fundamental solution to the heat operator $\frac{\partial}{\partial\tau} - \Delta$. This gives a rigorous argument for the inequality (1.5) (Theorem 1.2) of [**N1**], for both the compact manifolds and complete manifolds with non-negative Ricci (or Ricci curvature bounded from below). For the full detailed account please see [**CLN**].

4. Improving Li-Yau-Hamilton estimates via monotonicity formulae

The proof of (1.4) indicates a close relation between the monotonicity formulae and the differential inequalities of Li-Yau type. The hinge is simply Green's second identity. This was discussed very generally in [**EKNT**]. Moreover if we chose h in the introduction to be the fundamental solution to the time dependent heat equation $(\frac{\partial}{\partial t} - \Delta)$ centered at (x_0, t_0) we can have a better upper bound on $v_H(x_0, t_0)$ in

terms of the a weighted integral which is non-positive. In fact, this follows from the representation formula for the solutions to the non-homogenous conjugate heat equation. More precisely, since $h(y,t;x_0,t_0)$ is the fundamental solution to the heat equation (to make it very clear, v_H is defined with respect to $H = H(y,t;x,T)$, the fundamental solution to the conjugate heat equation centered at (x,T) with $T > t_0$), we have that

$$\lim_{t\to t_0}\int_M h(y,t;x_0,t_0)v_H(y,t)\,d\mu_t(y) = v_H(x_0,t_0).$$

On the other hand from (1.2) we have that (by Green's second identity)

$$\frac{d}{dt}\int_M hv_H\,d\mu_t = 2\tau\int_M |R_{ij}+f_{ij}-\frac{1}{2\tau}|^2 Hh\,d\mu_t.$$

Therefore

$$\lim_{t\to T}\int_M hv_H\,d\mu_t - v_H(x_0,t_0) = \int_{t_0}^T 2\tau\int_M |R_{ij}+f_{ij}-\frac{1}{2\tau}g_{ij}|^2 Hh\,d\mu_t\,dt.$$

Using the fact that $\lim_{t\to T} v_H = 0$ we have that

$$v_H(x_0,t_0) = -2\int_{t_0}^T (T-t)\int_M \left|R_{ij}+f_{ij}-\frac{1}{2(T-t)}\right|^2 Hh\,d\mu_t\,dt \le 0,$$

which sharpens the estimate $v_H \le 0$ by providing a non-positive upper bound. Noticing also the duality $h(y,t;x_0,t_0) = H(x_0,t_0;y,t)$ for any $t > t_0$ (cf. [**F**]) we can express everything in terms of the fundamental solution to the *(backward) conjugate heat equation*.

Below we show a few new monotonicity formulae, which expand the list of examples shown in [**EKNT**], and more importantly improve the earlier established Li-Yau-Hamilton estimates in a similar way as the above.

For the simplicity let us just consider the Kähler-Ricci flow case even though often the discussions are also valid for the Riemannian (Ricci flow) case, after replacing the assumption on the nonnegativity of the bisectional curvature by the nonnegativity of the curvature operator whenever necessary.

We first let $(M, g_{\alpha\bar\beta}(x,t))$ ($m = \dim_{\mathbb{C}} M$) be a solution to the Kähler-Ricci flow:

$$\frac{\partial}{\partial t}g_{\alpha\bar\beta} = -R_{\alpha\bar\beta}.$$

Let $\Upsilon_{\alpha\bar\beta}(x,t)$ be a Hermitian symmetric tensor defined on $M\times[0,T]$, which is deformed by the complex Lichnerowicz-Laplacian heat equation (or L-heat equation in short):

$$\left(\frac{\partial}{\partial t}-\Delta\right)\Upsilon_{\gamma\bar\delta} = R_{\beta\bar\alpha\gamma\bar\delta}\Upsilon_{\alpha\bar\beta} - \frac{1}{2}\left(R_{\gamma\bar p}k_{p\bar\delta} + R_{p\bar\delta}\Upsilon_{\gamma\bar p}\right).$$

Let $div(\Upsilon)_\alpha = g^{\gamma\bar\delta}\nabla_\gamma \Upsilon_{\alpha\bar\delta}$ and $div(\Upsilon)_{\bar\beta} = g^{\gamma\bar\delta}\nabla_{\bar\delta}\Upsilon_{\gamma\bar\beta}$. Consider the quantity

$$\begin{aligned} Z &= g^{\alpha\bar\beta}g^{\gamma\bar\delta}\left[\frac{1}{2}\left(\nabla_{\bar\beta}\nabla_\gamma + \nabla_\gamma\nabla_{\bar\beta}\right)\Upsilon_{\alpha\bar\delta} + R_{\alpha\bar\delta}\Upsilon_{\gamma\bar\beta}\right. \\ &\quad \left. + \left(\nabla_\gamma \Upsilon_{\alpha\bar\delta}V_{\bar\beta} + \nabla_{\bar\beta}\Upsilon_{\alpha\bar\delta}V_\gamma\right) + \Upsilon_{\alpha\bar\delta}V_{\bar\beta}V_\gamma\right] + \frac{K}{t} \\ &= \frac{1}{2}[g^{\alpha\bar\beta}\nabla_{\bar\beta}div(\Upsilon)_\alpha + g^{\gamma\bar\delta}\nabla_\gamma div(\Upsilon)_{\bar\delta}] \\ &\quad + g^{\alpha\bar\beta}g^{\gamma\bar\delta}[R_{\alpha\bar\delta}\Upsilon_{\gamma\bar\beta} + \nabla_\gamma\Upsilon_{\alpha\bar\delta}V_{\bar\beta} + \nabla_{\bar\beta}\Upsilon_{\alpha\bar\delta}V_\gamma + \Upsilon_{\alpha\bar\delta}V_{\bar\beta}V_\gamma] + \frac{K}{t} \end{aligned}$$

where K is the trace of $\Upsilon_{\alpha\bar\beta}$ with respect to $g_{\alpha\bar\beta}(x,t)$. In [**NT**] the following result, which is the Kähler analogue of an earlier result in [**CH**], was showed by the maximum principle.

Theorem 4.1. *Let $\Upsilon_{\alpha\bar\beta}$ be a Hermitian symmetric tensor satisfying the L-heat equation on $M\times[0,T]$. Suppose $\Upsilon_{\alpha\bar\beta}(x,0)\geq 0$ (and satisfies some growth assumptions in the case M is noncompact). Then $Z\geq 0$ on $M\times(0,T]$ for any smooth vector field V of type $(1,0)$.*

The use of the maximum principle in the proof can be replaced by the integration argument as in the proof of (1.4). For any $T\geq t_0>0$, in order to prove that $Z\geq 0$ at t_0 it suffices to show that when $t=t_0$, $\int_M t^2 Zh\,d\mu_t \geq 0$ for any compact-supported nonnegative function h. Now we solve the *conjugate heat equation* $\left(\frac{\partial}{\partial\tau} - \Delta + R\right)h(y,\tau) = 0$ with $\tau = t_0 - t$ and $h(y,\tau=0) = h(y)$, the given compact-supported function at t_0. By the perturbation argument we may as well as assume that $\Upsilon > 0$. Let $Z_m(y,t) = \inf_V Z(y,t)$. It was shown in [**NT**] that

$$\left(\frac{\partial}{\partial t} - \Delta\right)Z_m = Y_1 + Y_2 - 2\frac{Z_m}{t}$$

where

$$\begin{aligned} Y_1 &= \Upsilon_{\bar p q}\left(\Delta R_{p\bar q} + R_{p\bar q\alpha\bar\beta}R_{\bar\alpha\beta} + \nabla_\alpha R_{p\bar q}V_{\bar\alpha} + \nabla_{\bar\alpha}R_{p\bar q}V_\alpha\right. \\ &\quad \left. + R_{p\bar q\alpha\bar\beta}V_{\bar\alpha}V_\beta + \frac{R_{p\bar q}}{t}\right) \end{aligned}$$

and

$$\begin{aligned} Y_2 &= \Upsilon_{\gamma\bar\alpha}\left[\nabla_p V_{\bar\gamma} - R_{p\bar\gamma} - \frac{1}{t}g_{p\bar\gamma}\right]\left[\nabla_{\bar p}V_\alpha - R_{\alpha\bar p} - \frac{1}{t}g_{\bar p\alpha}\right] \\ &\quad + \Upsilon_{\gamma\bar\alpha}\nabla_{\bar p}V_{\bar\gamma}\nabla_p V_\alpha \\ &\geq 0. \end{aligned}$$

Notice that in the above expressions, at every point (y,t) the vector $V(y,t)$ is the vector minimizing Z_m. This implies the monotonicity

$$\frac{d}{dt}\int_M t^2 Z_m h\,d\mu_t = t^2\int_M (Y_1+Y_2)\,h\,d\mu_t \geq 0.$$

Since $\lim_{t\to 0} t^2 Z_m = 0$, which is certainly the case if Υ is smooth at $t = 0$ and can be assumed so in general by shifting t with a $\epsilon > 0$, we have that $\int_M t^2 Z_m h \, d\mu|_{t=t_0} \geq 0$. This proof via the integration by parts implies the following monotonicity formula.

Proposition 4.2. *Let $(M, g(t))$, Υ and Z be as in Theorem* 4.1. *For any space-time point (x_0, t_0) with $0 < t_0 \leq T$, let $\ell(y, \tau)$ be the reduced distance function with respect to (x_0, t_0). Then*

(4.1)
$$\frac{d}{dt}\int_M t^2 Z_m \left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right) d\mu_t \geq t^2 \int_M (Y_1 + Y_2)\left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right) d\mu_t \geq 0.$$

In particular,

$$t_0^2 Z(x_0, t_0) \geq \int_0^{t_0} t^2 \left(\int_M (Y_1 + Y_2)\left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right) d\mu_t\right) dt \geq 0. \tag{4.2}$$

Notice that (4.2) sharpens the original Li-Yau-Hamilton estimate of [**NT**], by encoding the rigidity (such as Hamilton-Cao's characterization on the singularity models), derived out of the equality case in the Li-Yau-Hamilton estimate $Z \geq 0$, into the integral of the right hand side. The result holds for the Riemannian case if one uses computation from [**CH**].

In [**N3**], the author discovered a new matrix Li-Yau-Hamilton inequality for the Kähler-Ricci flow. (We also showed a family of equations which connects this matrix inequality to Perelman's entropy formula.) More precisely we showed that for any positive solution u to the *forward conjugate heat equation* $\left(\frac{\partial}{\partial t} - \Delta - R\right) u = 0$, we have that

$$\Upsilon_{\alpha\bar\beta} := u\left(\nabla_\alpha \nabla_{\bar\beta} \log u + R_{\alpha\bar\beta} + \frac{1}{t} g_{\alpha\bar\beta}\right) \geq 0 \tag{4.3}$$

under the assumption that $(M, g(t))$ has bounded nonnegative bisectional curvature. Using the above argument we can also obtain a new monotonicity related to (4.3). Indeed, tracing (1.21) of [**N3**] gives that

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q = RQ - R_{\alpha\bar\beta}\Upsilon_{\beta\bar\alpha} - \frac{2}{t}Q + \frac{1}{u}|\Upsilon_{\alpha\bar\beta}|^2 + u\,|\nabla_\alpha \nabla_\beta \log u|^2 + Y_3$$

where $Q = g^{\alpha\bar\beta}\Upsilon_{\alpha\bar\beta}$ and

$$\begin{aligned} Y_3 &= u\left(\Delta R + |R_{\alpha\bar\beta}|^2 + \nabla_\alpha R \nabla_{\bar\alpha} \log u + \nabla_\alpha \log u \nabla_{\bar\alpha} R \right.\\ &\quad \left. + R_{\alpha\bar\beta}\nabla_{\bar\alpha}\log u \nabla_\beta \log u + \frac{1}{t}R\right)\\ &\geq 0. \end{aligned}$$

Hence we have the following monotonicity formula, noticing that

$$Y_4 := RQ - R_{\alpha\bar\beta}\Upsilon_{\beta\bar\alpha} \geq 0.$$

Proposition 4.3. *Let $(M, g(t))$ and (x_0, t_0) be as in Proposition* 4.2. *Then*

$$\begin{aligned}(4.4)\quad & \frac{d}{dt}\int_M t^2 Q\left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right) d\mu_t \\ &\geq t^2\int_M \left(\frac{1}{u}|\Upsilon_{\alpha\bar\beta}|^2 + u\,|\nabla_\alpha\nabla_\beta \log u|^2 + Y_3 + Y_4\right)\left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right) d\mu_t \\ &\geq 0.\end{aligned}$$

In particular,

$$\begin{aligned}(4.5)\quad & t_0^2 Q(x_0,t_0) \\ &\geq \int_0^{t_0} t^2\int_M \left(\frac{1}{u}|\Upsilon_{\alpha\bar\beta}|^2 + u\,|\nabla_\alpha\nabla_\beta \log u|^2 + Y_3 + Y_4\right)\left(\frac{\exp(-\ell)}{(\pi\tau)^m}\right).\end{aligned}$$

Again the advantage of the above monotonicity formula is that it encodes the consequence on equality case (which is that $(M, g(t))$ is an gradient expanding soliton) into the the right hand side integral.

Without Ricci flow, we can apply the similar argument to prove Li-Yau's inequality and obtain a monotonicity formula. More precisely, let (M, g) ($n = \dim_{\mathbb{R}} M$) be a complete Riemannian manifold with nonnegative Ricci curvature. Let $u(x, t)$ be a positive solution to the heat equation on $M \times [0, T]$. Li and Yau proved that

$$\Delta \log u + \frac{n}{2t} \geq 0.$$

Another way of proving the above Li-Yau's inequality is through the above integration by parts argument and the differential equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) Q = \frac{2}{u}|\Upsilon_{ij}|^2 - \frac{2}{t}Q + \frac{2}{u}R_{ij}\nabla_i u\nabla_j u$$

where

$$\Upsilon_{ij} = \nabla_i\nabla_j u + \frac{u}{2t}g_{ij} - \frac{u_i u_j}{u}$$

and $Q = g^{ij}\Upsilon_{ij} = u(\Delta \log u + \frac{n}{2t})$. This together with Cheeger-Yau's theorem [**CY**] on lower bound of the heat kernel, gives the following monotonicity formula, which also give characterization on the manifold if the equality holds somewhere for some positive u.

Proposition 4.4. *Let (M, g) be a complete Riemannian manifold with non-negative Ricci curvature. Let (x_0, t_0) be a space-time point with $t_0 > 0$. Let $\tau = t_0 - t$. Then*

$$\begin{aligned}(4.6)\quad & \frac{d}{dt}\left(\int_M t^2 Q(y,t)\hat H(x_0, y, \tau)\, d\mu(x)\right) \\ &\geq 2t^2\int_M \left(\left|\nabla_i\nabla_j \log u + \frac{1}{2t}g_{ij}\right|^2 + R_{ij}\nabla_i \log u\nabla_j \log u\right) u\hat H\, d\mu \geq 0,\end{aligned}$$

where $\hat{H}(x_0,y,\tau)=\frac{1}{(4\pi\tau)^{\frac{n}{2}}}\exp\left(-\frac{d^2(x_0,y)}{4\tau}\right)$ *with* $d(x_0,y)$ *being the distance function between* x_0 *and* y. *In particular, we have that*

$$\begin{aligned}(4.7)\quad &\left(u\Delta\log u+\frac{n}{2t}u\right)(x_0,t_0)\\ \geq\ &\frac{2}{t_0^2}\int_0^{t_0}t^2\int_M\left(\left|\nabla_i\nabla_j\log u+\frac{1}{2t}g_{ij}\right|^2+R_{ij}\nabla_i\log u\nabla_j\log u\right)u\hat{H}.\end{aligned}$$

It is clear that (4.7) improves the estimate of Li-Yau slightly by providing the lower estimate, from which one can see easily that the equality (for Li-Yau's estimate) holding somewhere implies that $M=\mathbb{R}^n$ (this was first observed in [**N1**], with the help of an entropy formula). The expression in the right hand side of (4.6) also appears in the linear entropy formula of [**N1**].

One can write down similar improving results for the Li-Yau type estimate proved in [**N1**], which is a linear analogue of Perelman's estimate $v_H\leq 0$, and the one in [**N2**], which is a linear version of Theorem 4 above. For example, when M is a complete Riemannian manifold with the nonnegative Ricci curvature, if $u=H(x,y,t)=\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$, the fundamental solution to the heat equation centered at x at $t=0$, letting $W=t(2\Delta f-|\nabla f|)+f-n$, we have that $W\leq 0$. If $\hat{H}$ is the 'pseudo backward heat kernel' defined as in Proposition 4.4 we have that

$$\begin{aligned}&\frac{d}{dt}\int_M(-W)u\hat{H}(x_0,y,\tau)\,d\mu(y)\\ =\ &2t\int_M\left(\left|\nabla_i\nabla_j f-\frac{1}{2t}g_{ij}\right|^2+R_{ij}\nabla_i f\nabla_j f\right)u\hat{H}(x_0,y,\tau)\,d\mu(y)\geq 0\end{aligned}$$

and

$$(-Wu)(x_0,t_0)\geq 2\int_0^{t_0}t\int_M\left(\left|\nabla_i\nabla_j f-\frac{1}{2t}g_{ij}\right|^2+R_{ij}\nabla_i f\nabla_j f\right)u\hat{H}.$$

If we assume further that M is a complete Kähler manifold with nonnegative bisectional curvature and $u(y,t)$ is a strictly plurisubharmonic solution to the heat equation with $w=u_t$, then

$$\frac{d}{dt}\int_M t^2Z_m^w\hat{H}(x_0,y,t)\,d\mu(y)=t^2\int_M Y_5\hat{H}(x_0,y,t)\ d\mu(y)\geq 0,$$

where

$$Z_m^w(y,t)=\inf_{V\in T^{1,0}M}\left(w_t+\nabla_\alpha wV_{\bar{\alpha}}+\nabla_{\bar{\alpha}}wV_\alpha+u_{\alpha\bar{\beta}}V_{\bar{\alpha}}V_\beta+\frac{w}{t}\right)$$

and

$$\begin{aligned} Y_5 &= u_{\gamma\bar{\alpha}}\left[\nabla_p V_{\bar{\gamma}} - \frac{1}{t}g_{p\bar{\gamma}}\right]\left[\nabla_{\bar{p}} V_{\alpha} - \frac{1}{t}g_{\bar{p}\alpha}\right] \\ &\quad + u_{\gamma\bar{\alpha}}\nabla_{\bar{p}}V_{\bar{\gamma}}\nabla_p V_{\alpha} + R_{\alpha\bar{\beta}s\bar{t}}u_{\bar{s}t}V_{\beta}V_{\bar{\alpha}} \\ &\geq 0 \end{aligned}$$

with V being the minimizing vector in the definition of Z_m^w. In particular,

$$\left(\frac{\partial^2}{\partial(\log t)^2}u(y,t)\right)(x_0,t_0) \geq \int_0^{t_0} t^2 \int_M Y_5 \hat{H}(x_0,y,t)\, d\mu(y)\, dt.$$

This sharpens the logarithmic-convexity of $u(y,t)$ proved in [**N2**].

Finally we should remark that in all the discussions above one can replace the 'pseudo backward heat kernel' $\hat{H}(y,t;x_0,t_0) = \frac{\exp(-\frac{d^2(x_0,y)}{4(t_0-t)})}{(4\pi(t_0-t))^{\frac{n}{2}}}$ (or $\frac{\exp(-\ell(y,\tau))}{(4\pi\tau)^{\frac{n}{2}}}$, centered at (x_0,t_0) in the case of Ricci flow), which we wrote before as $\hat{H}(y,x_0,\tau)$ by abusing the notation, by the fundamental solution to the backward heat equation (even by constant 1 in the case of compact manifolds). Also it still remains interesting on how to make effective uses of these improved estimates, besides the rigidity results out of the inequality being equality somewhere. There is also a small point that should not be glossed over. When the manifold is complete noncompact, one has to justify the validity of the Green's second identity (for example in Proposition 4.4 we need to justify that $\int_M \left(\hat{H}\Delta Q - Q\Delta\hat{H}\right) d\mu = 0$). This can be done when t_0 is sufficiently small together with integral estimates on the Li-Yau-Hamilton quantity (cf. [**CLN**]). The local monotonicity formula that shall be discussed in the next section provides another way to avoid possible technical complications caused by the non-compactness.

5. Local monotonicity formulae

In [**EKNT**], a very general scheme on localizing the monotonicity formulae is developed. It is for any family of metrics evolved by the equation $\frac{\partial}{\partial t}g_{ij} = -2\kappa_{ij}$. The localization is through the so-called 'heat ball'. More precisely for a smooth positive space-time function v, which often is the fundamental solution to the *backward conjugate heat equation* or the 'pseudo backward heat kernel' $\hat{H}(x_0,y,\tau) = \frac{e^{-\frac{r^2(x_0,y)}{4\tau}}}{(4\pi\tau)^{\frac{n}{2}}}$ (or $\frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{\frac{n}{2}}}$ in the case of Ricci flow), with $\tau = t_0 - t$, one defines the 'heat ball' by $E_r = \{(y,t)|\, v \geq r^{-n}; t < t_0\}$. For all interesting cases we can check that E_r is compact for small r (cf. [**EKNT**]). Let $\psi_r = \log v + n\log r$.

For any 'Li-Yau-Hamilton' quantity $\mathcal{Q}$ we define the local quantity:

$$P(r) := \int_{E_r} \left(|\nabla \psi_r|^2 + \psi_r(\mathrm{tr}_g \kappa)\right) \mathcal{Q}\, d\mu_t\, dt.$$

The finiteness of the integral can be verified via the localization of Lemma 2.2, a local gradient estimate. The general form of the theorem, which is proved in Theorem 1 of [**EKNT**], reads as the following.

Theorem 5.1. *Let $I(r) = \frac{P(r)}{r^n}$. Then*

$$\begin{aligned} I(r_2) - I(r_1) &= -\int_{r_1}^{r_2} \frac{n}{r^{n+1}} \int_{E_r} \left[\left(\left(\frac{\partial}{\partial t} + \Delta - \mathrm{tr}_g \kappa \right) v \right) \frac{\mathcal{Q}}{v} \right. \\ &\quad \left. + \psi_r \left(\frac{\partial}{\partial t} - \Delta \right) \mathcal{Q} \right] d\mu_t\, dt\, dr. \end{aligned} \tag{5.1}$$

It gives the monotonicity of $I(r)$ in the cases that $\mathcal{Q} \geq 0$, which is ensured by the Li-Yau-Hamilton estimates in the case we shall consider, and both $\left(\frac{\partial}{\partial t} + \Delta - \mathrm{tr}_g \kappa\right) v$ and $\left(\frac{\partial}{\partial t} - \Delta\right) \mathcal{Q}$ are nonnegative. The nonnegativity of $\left(\frac{\partial}{\partial t} + \Delta - \mathrm{tr}_g \kappa\right) v$ comes for free if we chose v to be the 'pseudo backward heat kernel'. The nonnegativity of $\left(\frac{\partial}{\partial t} - \Delta\right) \mathcal{Q}$ follows from the computation, which we may call as in [**N3**] the *pre-Li-Yau-Hamilton equation*, during the proof of the corresponding Li-Yau-Hamilton estimate. Below we illustrate examples corresponding to the monotonicity formulae derived in the previous section. These new ones expand the list of examples given in [**EKNT**].

For the case of Ricci/Kähler-Ricci flow, for a fixed (x_0, t_0), let $v = \frac{e^{-\ell(y,\tau)}}{(4\pi\tau)^{\frac{n}{2}}}$, the 'pseudo backward heat kernel', where ℓ is the reduced distance centered at (x_0, t_0).

Example 5.2. Let Z_m, Y_1 and Y_2 be as in Proposition 4.2. Let $\mathcal{Q} = t^2 Z_m$. Then

$$\frac{d}{dr} I(r) \leq -\frac{n}{r^{n+1}} \int_{E_r} \left[t^2 \psi_r \left(Y_1 + Y_2 \right) \right] d\mu_t\, dt \leq 0$$

and

$$\mathcal{Q}(x_0, t_0) \geq I(\bar{r}) + \int_0^{\bar{r}} \frac{n}{r^{n+1}} \int_{E_r} \left[t^2 \psi_r \left(Y_1 + Y_2 \right) \right] d\mu_t\, dt\, dr.$$

Example 5.3. Let u, $\mathcal{Q} = t^2 Q$, $\Upsilon_{\alpha\bar{\beta}}$, Y_3 and Y_4 be as in Proposition 4.3. Then

$$\frac{d}{dr} I(r) \leq -\frac{n}{r^{n+1}} \int_{E_r} t^2 \psi_r \left(\frac{1}{u} |\Upsilon_{\alpha\bar{\beta}}|^2 + u \left| \nabla_\alpha \nabla_\beta \log u \right|^2 + Y_3 + Y_4 \right) \leq 0$$

and

$$\mathcal{Q}(x_0, t_0) \geq I(\bar{r}) + \int_0^{\bar{r}} \frac{n}{r^{n+1}} \int_{E_r} t^2 \psi_r \left(\frac{1}{u} |\Upsilon_{\alpha\bar{\beta}}|^2 + u \left| \nabla_\alpha \nabla_\beta \log u \right|^2 + Y_3 + Y_4 \right).$$

For the fixed metric case, we may choose either $v = H(x_0, y, \tau)$, the *backward heat kernel* or $v = \hat{H}(x_0, y, \tau) = \frac{e^{-\frac{d^2(x_0,y)}{4\tau}}}{(4\pi\tau)^{\frac{n}{2}}}$, the 'pseudo backward heat kernel'.

Example 5.4. Let u and Q be as in Proposition 4.4. Let $\mathcal{Q} = t^2 Q$ and $f = \log u$. Then

$$\frac{d}{dr} I(r) \leq -\frac{2n}{r^{n+1}} \int_{E_r} t^2 u \psi_r \left(\left| \nabla_i \nabla_j f + \frac{1}{2t} g_{ij} \right|^2 + R_{ij} \nabla_i f \nabla_j f \right) d\mu \, dt \leq 0$$

and

$$\mathcal{Q}(x_0, t_0) \geq I(\bar{r}) + \int_0^{\bar{r}} \frac{2n}{r^{n+1}} \int_{E_r} t^2 u \psi_r \left(\left| \nabla_i \nabla_j f + \frac{1}{2t} g_{ij} \right|^2 + R_{ij} \nabla_i f \nabla_j f \right).$$

Example 5.5. Let $u = \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}$ be the fundamental solution to the (regular) heat equation. Let $W = t(2\Delta f - |\nabla f|^2) + f - n$ and $\mathcal{Q} = -uW$. Then

$$\frac{d}{dr} I(r) \leq -\frac{2n}{r^{n+1}} \int_{E_r} t u \psi_r \left(\left| \nabla_i \nabla_j f - \frac{1}{2t} g_{ij} \right|^2 + R_{ij} \nabla_i f \nabla_j f \right) d\mu \, dt \leq 0$$

and

$$\mathcal{Q}(x_0, t_0) \geq I(\bar{r}) + \int_0^{\bar{r}} \frac{2n}{r^{n+1}} \int_{E_r} t u \psi_r \left(\left| \nabla_i \nabla_j f - \frac{1}{2t} g_{ij} \right|^2 + R_{ij} \nabla_i f \nabla_j f \right).$$

Note that this provides another localization of entropy other than the one in [**N3**] (see also [**CLN**]).

Example 5.6. Let M be a complete Kähler manifold with nonnegative bisectional curvature. Let u, Z_m^w and Y_5 be as in the last case considered in Section 4. Let $\mathcal{Q} = t^2 Z_m^w$. Then

$$\frac{d}{dr} I(r) \leq -\frac{n}{r^{n+1}} \int_{E_r} t^2 Y_5 \psi_r \, d\mu \, dt$$

and

$$\left(\frac{\partial^2}{\partial (\log t)^2} u(x,t) \right)(x_0, t_0) \geq I(\bar{r}) + \int_0^{\bar{r}} \frac{n}{r^{n+1}} \int_{E_r} t^2 Y_5 \psi_r \, d\mu \, dt \, dr.$$

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