POSITIVITY AND KODAIRA EMBEDDING THEOREM

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Abstract. Kodaira embedding theorem provides an effective characterization of projectivity of a Kähler manifold in terms of the second cohomology. Recently X. Yang [20] proved that any compact Kähler manifold with positive holomorphic sectional curvature must be projective. This gives a metric criterion of the projectivity in terms of its curvature. In this note, we prove that any compact Kähler manifold with positive 2nd scalar curvature (which is the average of holomorphic sectional curvature over 2-dimensional subspaces of the tangent space) must be projective. In view of generic 2-tori being non-Abelian, this new curvature characterization is sharp in certain sense. Vanishing theorems are also proved for the Hodge numbers when the condition is replaced by the positivity of the weaker interpolating k-scalar curvature. The proof uses a second variational consideration via orbits generated by one parameter family of group action in the spirit of B. Wilking [17], and a maximum principle argument involving partial directional derivatives in the spirit of B. Andrews [1].

1. Introduction

Let $(M^m, g)$ be a Kähler manifold with complex dimension $m$. For $x \in M$, denote by $T'_x M$ the holomorphic tangent space at $x$. Let $R$ denote the curvature tensor. For $X \in T'_x M$ let $H(X) = R(X, \bar{X}, X, \bar{X})/|X|^4$ be the holomorphic sectional curvature. Here $|X|^2 = \langle X, \bar{X} \rangle$, and we extended the Riemannian product $\langle \cdot, \cdot \rangle$ and the curvature tensor $R$ linearly over $\mathbb{C}$, following the convention of [13] as well as [15]. We say that $(M, g)$ has positive holomorphic sectional curvature, if $H(X) > 0$ for any $x \in M$ and any $0 \neq X \in T'_x M$. It was known that compact manifolds with positive holomorphic sectional curvature must be simply connected [16]. A three circle property was established for noncompact complete Kähler manifolds with nonnegative holomorphic sectional curvature [9]. On the other hand it was known that such metric may not even have positive Ricci curvature [4].

The following result was proved by X. Yang in [20] recently, which answers affirmatively a question in [22].

If the compact Kähler manifold $M$ has positive holomorphic sectional curvature, then $M$ is projective. Namely $M$ can be embedded holomorphically into a complex projective space.

The key step is to show that the Hodge number $h^{2,0} = 0$. Then a well-known result of Kodaira (cf. Chapter 3, Theorem 8.3 of [10]) implies the projectiveness.

The purpose of this paper is to prove a generalization of the above result of Yang. First of all we introduce some notations after recalling a lemma of Berger.

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If $E$ is a rationally connected manifold, then

$$2S(p) = \frac{m(m+1)}{\text{Vol}(S^{2m-1})} \int_{|Z|=1, Z \in T^*_p M} H(Z) \ d\theta(Z).$$  \hspace{1cm} (1.1)

Proof. Direct calculations shows that

$$\frac{1}{\text{Vol}(S^{2m-1})} \int_{S^{2m-1}} |z_i|^4 = \frac{2}{m(m+1)}, \quad \frac{1}{\text{Vol}(S^{2m-1})} \int_{S^{2m-1}} |z_i|^2 |z_j|^2 = \frac{1}{m(m+1)}$$

for each $i$ and each $i \neq j$. Equation (1.1) then follows by expanding $H(Z)$ in terms of $Z = \sum z_i E_i$, and the above formulae. \hfill \square

For any integer $k$ with $1 \leq k \leq m$ and any $k$-dimensional subspace $\Sigma \subset T^*_p M$, one can defined the $k$-scalar curvature as

$$S_k(x, \Sigma) = \frac{k(k+1)}{2\text{Vol}(S^{2k-1})} \int_{|Z|=1, Z \in \Sigma} H(Z) \ d\theta(Z).$$

By the above Berger’s lemma, $\{S_k(x, \Sigma)\}$ interpolate between the holomorphic sectional curvature, which is $S_1(x, \{X\})$, and scalar curvature, which is $S_m(x, T_x M)$.

We say that $(M, g)$ has positive 2nd-scalar curvature if $S_2(x, \Sigma) > 0$ for any $x$ and any two complex plane $\Sigma$.

Clearly, the positivity of the holomorphic sectional curvature implies the positivity of the 2nd-scalar curvature, and the positivity of $S_k$ implies the positivity of $S_l$ if $k \leq l$. We shall prove the following generalization of above mentioned result of Yang.

**Theorem 1.1.** Any compact Kähler manifold $M^m$ with positive 2nd-scalar curvature must be projective. In fact $h^{p,0}(M) = 0$ for any $2 \leq p \leq m$.

Recall that a projective manifold $M$ is said to be *rationally connected*, if any two generic points in $M$ can be connected by a chain of rational curves. By the work of [7], any projective manifold $M$ admits a rational map $f : M \dashrightarrow Z$ onto a projective manifold $Z$ such that any generic fiber is rationally connected, and for any very general point (meaning away from a countable union of proper subvarieties) $z \in Z$, any rational curve in $M$ which intersects the fiber $f^{-1}(z)$ must be contained in that fiber. Such a map is called a maximal rationally connected fibration for $M$, or MRC fibration for short. It is unique up to birational equivalence. The dimension of the fiber of a MRC fibration of $M$ is called the rational dimension of $M$, denoted by $rd(M)$.

Heier and Wong (Theorem 1.7 of [3]) proved that any projective manifold $M^m$ with $S_k > 0$ satisfies $rd(M) \geq m - (k - 1)$. So as a corollary of their result and Theorem 1.1 above, we have the following consequence.

**Corollary 1.2.** Let $M^m$ be a compact Kähler manifold with positive 2nd scalar curvature. Then $rd(M) \geq m - 1$, namely, either $M$ is rationally connected, or there is a rational map $f : M \dashrightarrow C$ from $M$ onto a curve $C$ of positive genus, such that over the complement of a finite subset of $C$, $f$ is a holomorphic submersion with compact, smooth fibers, each fiber is a rationally connected manifold.
Clearly the positivity of $S_2$ is stable (namely a open condition) under the deformation of the complex manifolds (along with the smoothly deformation of the Kähler metrics specified in [10]). Hence the result proved here provide a stable condition on the projectivity.

It is well known that $h^{m,0} = 0$ if $(M^m, g)$ has positive scalar curvature. The traditional Bochner formula also implies the vanishing of $h^{p,0} = 0$ for $k \leq p \leq m$ if the Ricci curvature of $(M^m, g)$ is $k$-positive, namely the sum of the smallest $k$ eigenvalues of the Ricci tensor is positive (cf. [8]). The following provides an analogue of this result.

**Theorem 1.3.** Let $(M^m, g)$ be a compact Kähler manifold. If the $k$-th scalar curvature is positive, then $h^{p,0} = 0$ for any $k \leq p \leq m$.

The proof of these result uses a $\partial \bar{\partial}$-Bochner formula and applying the maximum principle to part of directions, which was revived recently by the work of Andrews-Clutterbuck [2] (cf. also [12]), Andrews [1], as well as G. Liu [9], X. Yang [20] in the Kähler setting.

Note that the proof of Theorem 1.3 goes verbatim in the negative curvature case, namely, the same proof gives the following

**Theorem 1.4.** Let $(M^m, g)$ be a compact Kähler manifold. If the $k$-th scalar curvature is negative, then $H^0(M, \Lambda^p T^*M) = 0$ for any $k \leq p \leq m$.

As a counterpart to Theorem 1.7 of [3], one can ask the question that, for a given projective Kähler manifold $M^m$ with $S_k < 0$, what is the maximal possible rational dimension? We should mention that there is also a recent work of Wu and Yau [18] on the ampleness of the canonical line bundle assuming the holomorphic sectional curvature being negative, which gives another algebraic geometric consequence in terms of the metric property via the holomorphic sectional curvature.

Generally speaking, we think it is interesting to obtain algebraic geometric characterizations of condition $S_k > 0$ or $S_k < 0$, as well as the conditions $\text{Ric}^+ > 0$, $\text{Ric}^+ < 0$ studied recently in [13] by the authors, where an complementary metric criterion of the projectivity was given in terms of $\text{Ric}^+ > 0$.

## 2. Proof of Theorem 1.1

First recall the formula below (cf. Ch III, Proposition 1.5 of [8], as well as Proposition 2.1 of [11]).

**Lemma 2.1.** Let $s$ be a global holomorphic $p$-form on $M^m$ which locally is expressed as $s = \frac{1}{p!} \sum_{I_p} a_{I_p} dz^{i_1} \wedge \cdots \wedge dz^{i_p}$, where $I_p = (i_1, \cdots, i_p)$. Then

$$\partial \bar{\partial} |s|^2 = \langle \nabla s, \overline{s} \rangle - \tilde{R}(s, \tilde{s}, \cdot, \cdot)$$

where $\tilde{R}$ stands for the curvature of the Hermitian bundle $\Lambda^p \Omega$, where $\Omega = (T^*M)^*$ is the holomorphic cotangent bundle of $M$. The metric on $\Lambda^p \Omega$ is derived from the metric of $M^m$. Then for any unitary frame $\{dz^j\}$,

$$\langle \sqrt{-1} \partial \bar{\partial} |s|^2, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \rangle = \langle \nabla_v s, \overline{s} \rangle + \frac{1}{p!} \sum_{I_p} \sum_{k=1}^p \sum_{l=1}^m (R_{\bar{v}i_1, ..., i_p} a_{I_p}, a_{I_1} - (i_1 \cdots i_p)).$$  \hspace{1cm} (2.1)
Given any \( x_0 \) and \( v \in T'_{x_0} M \), there exists a unitary frame \( \{dz^i\} \) at \( x_0 \), which may depends on \( v \), such that

\[
\langle \sqrt{-1} \partial \bar{\partial} |s|^2, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \rangle = \langle \nabla_v s, \nabla_{\bar{v}} \bar{s} \rangle + \frac{1}{p!} \sum_{i=1}^{p} \sum_{k=1}^{m} R_{v \bar{v} \cdot i \cdot k} |a_{i \bar{k}}|^2. \tag{2.2}
\]

First let us focus on the case \( p = 2 \). Suppose that \( |s|^2 \) attains its maximum at the point \( x_0 \). Write \( s = \sum_{i,j} f_{ij} \varphi_i \wedge \varphi_j \) under any unitary coframe \( \{\varphi_j\} \) which is dual to a local unitary tangent frame \( \{\frac{\partial}{\partial z^i}\} \). The \( m \times m \) matrix \( A = (f_{ij}) \) is skew-symmetric. Note that there exists a normal form for any holomorphic \((2,0)\)-form \( s \) at a given point \( x_0 \), (cf. [5]).

More precisely, given any skew-symmetric matrix \( A \), there exists a unitary matrix \( U \) such that \( UAU \) is in the block diagonal form where each non-zero diagonal block is a constant multiple of \( F \), with

\[
F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.
\]

In other words, we can choose a unitary coframe \( \varphi \) at \( x_0 \) such that

\[
s = \lambda_1 \varphi_1 \wedge \varphi_2 + \lambda_2 \varphi_3 \wedge \varphi_4 + \cdots + \lambda_k \varphi_{2k-1} \wedge \varphi_{2k},
\]

where \( k \) is a positive integer and each \( \lambda_i \neq 0 \). Then at \( x_0 \) for \( \{\frac{\partial}{\partial z^i}\} \) dual to \( \varphi_i \),

\[
\partial_v \partial_{\bar{v}} |s|^2 = \langle \nabla_v s, \nabla_{\bar{v}} \bar{s} \rangle + \sum_{i=1}^{k} (R_{v \bar{v} \cdot 2i} + R_{v \bar{v} \cdot 2i+1}) |\lambda_i|^2. \tag{2.3}
\]

To prove the theorem we will apply the maximum principle at \( x_0 \), where \( |s|^2 \) attains its maximum. In view of the compactness of the Grassmannians we can also find a complex two plane \( \Sigma \) in \( T'_{x_0} M \) such that \( S_2(x_0, \Sigma) = \inf_{\Sigma} S_2(x_0, \Sigma') > 0 \). In the following we denote \( \int f(Z) \) to be the average of the integral of the function \( f \) over \( \Sigma \). Theorem 1.1 will then follows from the following result.

**Proposition 2.1.** For any \( E \in \Sigma \) and \( E' \perp \Sigma \) with \( |E| = |E'| = 1 \), we have that

\[
\int R(E, E', Z, \bar{Z}) d\theta(Z) = \int R(E', E, Z, \bar{Z}) d\theta(Z) = 0, \tag{2.4}
\]

\[
\int R(E, E', Z, \bar{Z}) + R(E', E', Z, \bar{Z}) d\theta(Z) \geq \frac{1}{6} S_2(x_0, \Sigma), \tag{2.5}
\]

\[
\int R(E', E', Z, \bar{Z}) d\theta(Z) \geq \frac{1}{6} S_2(x_0, \Sigma). \tag{2.6}
\]

To prove Theorem 1.1, (2.3) implies that at \( x_0 \)

\[
0 \geq \int \partial_v \partial_{\bar{v}} |s|^2 d\theta(v) = \int \langle \nabla_v s, \nabla_{\bar{v}} \bar{s} \rangle + \sum_{i=1}^{k} (R_{v \bar{v} \cdot 2i} + R_{v \bar{v} \cdot 2i+1}) |\lambda_i|^2 d\theta(v)
\]

\[
\geq \sum_{i=1}^{k} |\lambda_i|^2 \int (R_{v \bar{v} \cdot 2i} + R_{v \bar{v} \cdot 2i+1}) d\theta(v).
\]

The integral is clearly independent of the choice of a unitary frame of the two dimensional space spanned by \( \{\frac{\partial}{\partial z^{2i}}, \frac{\partial}{\partial z^{2i+1}}\} \), or the choice of a unitary frame \( \{E_1, E_2\} \) of \( \Sigma \). Let \( j = 2i \).

By unitary transformation of \( \{\frac{\partial}{\partial z^{2j}}, \frac{\partial}{\partial z^{2j+1}}\} \), and a choice of a unitary frame of \( \Sigma \) we can write
\[ \frac{\partial}{\partial s_j} = \mu_1 E_1 + \beta_1 E' \quad \text{and} \quad \frac{\partial}{\partial s_j} = \mu_2 E_2 + \beta_2 E'', \]

where \( \{ E_i \} \) is a unitary frame of \( \Sigma \) and \( E', E'' \perp \Sigma \) with \( |E'| = |E''| = 1 \). It is clear that \( |\mu_1|^2 + |\beta_1|^2 = 1 = |\mu_2|^2 + |\beta_2|^2 \). By (2.4)

\[
\int R_{\nu \nu j} \, d\theta(v) = |\mu_1|^2 \int R_{\nu \nu E_1 E_1} \, d\theta(v) + |\beta_1|^2 \int R_{\nu \nu E E''} \, d\theta(v).
\]

Similarly

\[
\int R_{\nu \nu j + 1 j + 1} \, d\theta(v) = |\mu_2|^2 \int R_{\nu \nu E_2 E_2} \, d\theta(v) + |\beta_2|^2 \int R_{\nu \nu E E'''} \, d\theta(v).
\]

The sum is positive due to (2.5), (2.6) and the following reasoning: If \( |\mu_1| \geq |\mu_2| \), then

\[
\int R_{\nu \nu j} + R_{\nu \nu j + 1 j + 1} \, d\theta(v) = |\mu_2|^2 \int R_{\nu \nu E_1 E_1} + R_{\nu \nu E_2 E_2} \, d\theta(v) + (|\mu_1|^2 - |\mu_2|^2) \int R_{\nu \nu E_1 E_1} + R_{\nu \nu E E''} \, d\theta(v) + |\beta_1|^2 \int R_{\nu \nu E E''} + R_{\nu \nu E E''' \nu} \, d\theta(v) > 0.
\]

The case \( |\mu_2| \geq |\mu_1| \) works similarly. Hence \( |\lambda_i|^2 = 0 \) for all \( 1 \leq i \leq k \). This shows that \( |s|^2 = 0 \) at \( x_0 \), thus \( |s|^2 = 0 \) everywhere, which proves Theorem 1.1.

We shall devote the rest of the section to the proof of Proposition 2.1. The proof needs some basic algebra and computations. Let \( a \in \mathfrak{u}(m) \) be an element of the Lie algebra of \( \mathbb{U}(m) \). Consider the function:

\[
f(t) = \int H(e^{t a} X) \, d\theta(X).
\]

By the choice of \( \Sigma \), \( f(t) \) attains its minimum at \( t = 0 \). This implies that \( f'(0) = 0 \) and \( f''(0) \geq 0 \). Hence

\[
\int (R(a(X), X, X) + R(X, \tilde{a}(X), X, X)) \, d\theta(X) = 0; \quad (2.7)
\]

\[
\int (R(a^2(X), X, X) + R(X, \tilde{a}^2(X), X, X) + 4R(a(X), \tilde{a}(X), X, X)) \, d\theta(X)
+ \int (R(a(X), \tilde{a}(X), X, X) + R(X, \tilde{a}(X), X, \tilde{a}(X)) \, d\theta(X) \geq 0. \quad (2.8)
\]

We exploit these by looking into some special cases of \( a \). Let \( W \perp \Sigma \) and \( Z \in \Sigma \) be two fixed vectors. Let \( a = \sqrt{-1}(Z \otimes W + W \otimes Z) \). Then

\[
a(X) = \sqrt{-1}(X, Z)W; \quad a^2(X) = -(X, Z)Z.
\]

Applying (2.8) to the above \( a \) and also the one with \( W \) being replaced by \( \sqrt{-1}W \), and add the resulting two estimates together, we have that

\[
4 \int \langle X, Z \rangle^2 R(W, W, X, X) \, d\theta(X) \geq \int \langle X, Z \rangle R(Z, X, X, Z) + \langle Z, X \rangle R(X, Z, X, X). \quad (2.9)
\]

Now we may pick a unitary basis \( \{ E_i \} \) such that the linear span of \( \{ E_1, E_2 \} \) is \( \Sigma \). Additionally we can assume that \( E_i \) (\( i = 1, 2 \)) is so chosen that

\[
R_{11(\cdot)} + R_{22(\cdot)} \quad (2.10)
\]

is diagonal. Namely the restricted (to \( \Sigma \)) Ricci tensor is diagonal.
Apply the above to \( Z = E_i \) \((i = 1, 2)\) and sum the results up we get (2.6). We also exploit (2.7) for the above choice of \( W \) for our later use. By combining (2.8) (with \( a \) as above) and the one with \( W \) being replaced by \( \sqrt{-1}W \), we obtain two equalities:

\[
\oint (X, Z) R(W, X, X, X) = \oint (Z, X) R(W, W, X, X) = 0.
\]

Now write \( X = x_1 E_1 + x_2 E_2 \). Let \( Z = E_i, W = E_k \) \((i = 1, 2, k \geq 3)\). Direct calculation (with \( Z = E_1 \)) shows that

\[
\oint R_{k111}|x_1|^4 + R_{k123}|x_1|^2|x_2|^2 + R_{k221}|x_1|^2|x_2|^2 = 0.
\]

Applying the integral identities in the proof of the Berger’s lemma, the above equation (together with the case \( Z = E_2 \)) implies that

\[
R_{k111} + R_{k123} = 0 = R_{k221} + R_{k211}.
\]

(2.11)

These imply (2.4).

To prove (2.5) we need to consider general \( W \). In other words, we consider the case \(|Z| = |W| = 1\) and \( Z \in \Sigma \).

\[
a(X) = \sqrt{-1} \left( \langle X, Z \rangle W + \langle X, W \rangle Z \right)
\]

\[
a^2(X) = -\langle X, Z \rangle (Z + \langle W, Z \rangle W) - \langle X, W \rangle (W + \langle Z, W \rangle Z).
\]

Apply this to (2.8) and also apply to \( a \) with \( W \) being replaced by \( \sqrt{-1}W \), add the results up we get the estimate:

\[
4 \oint |\langle X, Z \rangle|^2 R(W, W, X, X) + |\langle X, W \rangle|^2 R(Z, Z, X, X) d\theta(X)
\]

\[
\geq \oint \langle X, Z \rangle R(Z, X, X, X) + \langle Z, X \rangle R(X, Z, X, X) d\theta(X)
\]

\[
\quad + \oint \langle X, W \rangle R(W, Z, X, X) + \langle Z, W \rangle R(X, W, X, X) d\theta(X).
\]

(2.12)

Apply the above to \( Z = E_i \) \((i = 1, 2)\) and sum the results together we have

\[
4 \oint R(W, W, X, X) + |\langle X, W \rangle|^2 (R_{11XX} + R_{22XX}) d\theta(X)
\]

\[
\geq \frac{2}{3} S_2(x_0, \Sigma) + \oint \langle X, W \rangle R(W, Z, X, X) + \langle Z, W \rangle R(X, W, X, X) d\theta(X).
\]

(2.13)

Now we apply the above to \( W = \frac{1}{\sqrt{2}}(E_i + E_k) \) with \( i = 1, 2 \) and \( k \geq 3 \). We shall compute each terms below. The first term of the left can be simplified as

\[
4 \oint R(W, W, X, X) d\theta(X) = 2 \oint (R_{i1XX} + R_{kkXX}) d\theta(X) + 2 \oint (R_{kkXX} + R_{i1XX}) d\theta(X)
\]

\[
= 2 \oint R_{i1XX} + R_{kkXX} d\theta(X).
\]

Here we have used equations in (2.11) and their conjugations to eliminate the last two terms. Express \( X = x_1 E_1 + x_2 E_2 \) as before. If \( W = \frac{1}{\sqrt{2}}(E_i + E_k) \) \((i = 1, 2, k \geq 3)\) the second
term of the left hand side of (2.13) can be computed as

$$4 \int \langle X; W \rangle (R_{11XX} + R_{22XX}) d\theta(X) = 2 \int |x_1|^2 (R_{11XX} + R_{22XX})$$

$$= 2 \int (|x_1|^4 R_{1111} + R_{1122} |x_1|^2 |x_2|^2) d\theta$$

$$+ 2 \int (|x_1|^4 R_{1111} + R_{1122} |x_1|^2 |x_2|^2) d\theta$$

$$= \frac{2}{3} R_{1111} + R_{1122} + \frac{1}{3} R_{2222}.$$  

Starting in the second line above (and the computation below) we fix $i = 1$ (the case $i = 2$ is similar). The last two terms of the right hand side of (2.13) are conjugate to each other. The first one can be computed as

$$\int \langle X; W \rangle R(W; X; X; X) d\theta(X) = \frac{1}{2} \int x_1 (R_{11XX} + R_{kXX}) d\theta$$

$$= \frac{1}{2} \int x_1 R_{11XX} d\theta$$

$$= \frac{1}{2} \int |x_1|^4 R_{1111} + 2 |x_1|^2 |x_2|^2 R_{1122} = \frac{1}{6} (R_{1111} + R_{1122}).$$

Hence the last two terms of the right hand side of (2.13) becomes

$$\int \langle X; W \rangle R(W; X; X; X) + \langle W; X \rangle R(X; W; X; X) d\theta(X) = \frac{1}{3} (R_{1111} + R_{1122}).$$

Putting them all together and noting that $S_2(x_0, \Sigma) = R_{1111} + 2 R_{1122} + R_{2222}$ we have arrived at (2.5) for the case $i = 1$. The case for $i = 2$ is exactly the same. This completes the proof of Proposition 2.1.

For the general $p$, for any holomorphic $(p; 0)$-form, applying (2.1) to any $v$ at the maximum point $x_0$ of $|s|^2$ that for any unitary basis $\{dz^j\}$

$$0 \geq \frac{1}{p!} \sum_{I_p} \sum_{k=1}^m \sum_{l=1}^n \int \langle \sum_{k=1}^p R_{v\bar{v}_i} a_{I_p}, u_{i_1 \cdots (i)_{k-1} i_p} \rangle d\theta(v).$$

If we choose the unitary basis such that the Hermitian form

$$\int R_{v\bar{v}(\cdot)} d\theta(v)$$

is diagonal we have that

$$0 \geq \sum_{i_1 < \cdots < i_p} |a_{I_p}|^2 \int \left( \sum_{k=1}^p R_{v\bar{v}_i} \right) d\theta(v).$$

The result follows if we can show that

$$\int \left( \sum_{k=1}^p R_{v\bar{v}_i} \right) d\theta(v) > 0.$$  

Without the loss of the generality we may assume that $(i_1, \cdots, i_p) = (1, \cdots, p)$. Clearly (*) does not depend on the choice of unitary frame which spans $\{ \frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^p} \}$, nor on the choice of unitary frame of $\Sigma$. By the singular value decomposition, we may choose a unitary
frame of \( \Sigma, \{E_1, E_2\} \), as well as a unitary frame for the span of \( \{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_p}\} \) (which we shall still denote with the same notation) such that

\[
\frac{\partial}{\partial z_1} = \mu_1 E_1 + \beta_1 E_1', \quad \frac{\partial}{\partial z_2} = \mu_2 E_2 + \beta_2 E_2', \quad \frac{\partial}{\partial z_k} = \delta_k \quad \Sigma
\]

with \( E_i' \perp \Sigma \) and \( |E_i'| = 1 \) \((i = 1, 2)\). Now Proposition 2.1 together with the same argument for the \( p = 2 \) case lead to a proof of the vanishing of \( h^{p,0} \) for any \( 2 < p \leq m \).

3. Proof of Theorem 1.3

We adapt the argument in the proof of Theorem 1.1 to this more general case. Apply the maximum principle at the point \( x_0 \), where \( |s|^2 \) attains its maximum. Let \( \Sigma \) be the \( k \)-dimensional subspace such that \( S_k(x_0, \Sigma') \) attains its minimum among all \( k \)-dimensional subspaces. The key is to extend estimates of Proposition 2.1 to cover the \( \Sigma \) case. As in the last section, we will denote the average of a function \( f(X) \) over the unit sphere \( S^{2k-1} \) in \( \Sigma \) as \( \bar{f}(X) \)

**Proposition 3.1.** Let \( \{E_1, \ldots, E_m\} \) be a unitary frame at \( x_0 \) such that \( \{E_i\}_{1 \leq i \leq k} \) spans \( \Sigma \). Let \( I \) be any non-empty subset of \( \{1, 2, \ldots, k\} \). Then for any \( E \in \Sigma, E' \perp \Sigma, \) and any \( 1 \leq p \leq m \), we have

\[
\int R(E, E', Z, Z) d\theta(Z) = \int R(E', E, Z, Z) d\theta(Z) = 0, \quad (3.1)
\]

\[
\int \left( R(E_p, E_p, Z, Z) + \sum_{j \in I} R(E_j, E_j, Z, Z) \right) d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}, \quad (3.2)
\]

\[
\int R(E_p, E_p, Z, Z) d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}. \quad (3.3)
\]

**Proof.** Let \( f(t) \) be the function constructed by the variation under the 1-parameter family of unitary transformations. The equations (2.7) and (2.8), as well as their proofs, remain the same. The proof of (3.3) is exactly the same as the proof of (2.6).

To prove (3.1) first we exploit the equation (2.7) in the similar fashion by choosing \( Z = E_i \) and \( W = E_p \), and obtain for any \( p \geq k + 1 \) and \( 1 \leq i \leq k \)

\[
\sum_{j=1}^{k} R_{i p j} = 0. \quad (3.4)
\]

This proves (3.1). Note that (2.12) can be derived in exactly the same fashion. By applying that to \( Z = E_i \) and sum up \( i \) from 1 to \( k \), we get

\[
4 \int R(W, W, X, X) + |\langle X, W \rangle|^2 \left( \sum_{i=1}^{k} R_{i X X} \right) d\theta(X) \geq \frac{4}{k(k+1)} S_k(x_0, \Sigma) + \int |\langle X, W \rangle R(W, X, X) + (W, X) R(X, W, X, X) d\theta(X). \quad (3.5)
\]

For any \( p \geq k + 1 \) and \( 1 \leq i \leq k \), let \( W = \sqrt{2} (E_i + E_p) \). Without loss of the generality we can perform the similar calculation as in the last section to compute the terms involved in
(3.5) for $i = 1$. This implies that
\[
\int R(E_i, E_i, X, X) + R(E_p, E_p, X, X) \, d\theta(X) \geq \frac{1}{k(k+1)} S_k(x_0, \Sigma).
\]
This proves (3.2) for the special case of $|I| = 1$.

For the general case let us assume that $I = \{1, \cdots, l\}$. First we notice that Proposition 3.1 is independent of the choice of a unitary frame in $\Sigma$. So by a unitary change of $(E_1, \ldots, E_k)$ if necessary, we may assume that
\[
\sum_{j=1}^{k} R(E_a, E_b, E_j, E_j) = 0
\]
for any $1 \leq a \neq b \leq k$. Now let $W = \frac{1}{\sqrt{l+1}} (E_1 + \cdots + E_l + E_p)$. We shall apply (3.5) with such a $W$. We handle the three involved terms similarly by some calculation which involves computing integrals over $S^2 = 1$. First
\[
4 \int R(W, W, X, X) \, d\theta(X) = \frac{4}{l+1} \int \left( \sum_{j=1}^{l} R_{jj} X X + R_{pp} X X \right) d\theta(X)
\]
\[
+ \frac{4}{l+1} \sum_{j=1}^{l} \int (R_{jp} X X + R_{pj} X X) d\theta(X)
\]
\[
= \frac{4}{l+1} \int \left( \sum_{j=1}^{l} R_{jj} X X + R_{pp} X X \right) d\theta(X).
\]
In the first equality on the right, we used (3.6), and in the last equality we applied (3.4). Secondly,
\[
4 \int |(X, W)|^2 \left( \sum_{i=1}^{k} R_{i} X X \right) \, d\theta(X) = \frac{4}{l+1} \int \left( \sum_{j \in I} |x_j|^2 r_{j} X X + \sum_{j \neq s \in I} R_{ij} X X x_j x_s \right) d\theta
\]
\[
= \frac{4}{l+1} \left( \int \sum_{j=1}^{l} |x_j|^2 r_{jj} X X + \sum_{j \neq s \in I} |x_j|^2 x_s r_{sj} \right).
\]
Here we denote $r_{j}$ the restricted Ricci curvature (namely the Ricci curvature of $R$ restricted to $\Sigma$). Hence we have
\[
4 \int |(X, W)|^2 \left( \sum_{i=1}^{k} R_{i} X X \right) \, d\theta(X) = \frac{4}{l+1} \left( \frac{2}{k(k+1)} \sum_{j \in I} r_{jj} + \frac{1}{k(k+1)} \sum_{j \in I} \sum_{1 \leq s \neq j \leq k} r_{sj} \right)
\]
\[
+ \frac{4}{l+1} \frac{1}{k(k+1)} \sum_{j \neq s \in I} r_{js}
\]
\[
= \frac{4}{l+1} \frac{1}{k(k+1)} \left( \sum_{j \in I} r_{jj} + |I| \cdot S_k(x_0, \Sigma) \right)
\]
\[
+ \frac{4}{l+1} \frac{1}{k(k+1)} \sum_{j \neq s \in I} r_{js}.
\]
Similarly the two last terms in the right hand side of (3.5) gives
\[ 2 \int \langle X; W \rangle R(W, X, X, X) d\theta(X) = \frac{2}{l+1} \int \sum_{j,j' \in I} x_j(R_{j'}XX + R_kXX) d\theta \]
\[ = \frac{2}{l+1} \int \sum_{j,j' \in I} \sum_{s,t=1}^k x_jx_{j'}x_sx_t R_{j's't} d\theta \]
\[ = \frac{4}{l+1} \sum_{j,j' \in I} \left( \sum_{s=1}^k R_{j'ss} + \sum_{j \neq j' \in I} r_{jj'} \right). \]

Here in the last equality we used (3.6) and the fact that \( r_{pj} = 0 \) (here we abuse the notation letting \( r_{AB} = \sum_{i=1}^k R(E_A, E_B, E_i, E_i) \)) which is just (3.1). Now by putting the above together, we get
\[ \sum_{j \in I} r_{jj} + r_{pp} \geq \frac{1}{k+1} S_k(x_0, \Sigma), \]
which is the claimed estimate of (3.2).

To prove Theorem 1.3 we follow the similar argument as before. At the maximum point \( x_0 \) of \( |s|^2 \) we apply (2.1) and integrate over the \( k \)-subspace, where \( S_k(x_0, \cdot) \) attains the minimum. Also adapt a unitary frame \( \{ \frac{\partial}{\partial z_i} \} \) so that
\[ \int R_{v \in \Sigma \cap \cdot} d\theta(v) \]
is diagonal. Then we have that
\[ 0 \geq \sum_{i_1 < \cdots < i_p} |a_{i_p}|^2 \int \left( \sum_{j=1}^p R_{v \in E_{i_j} \cdot} \right) d\theta(v). \]

For simplicity we focus on \( p = k \) case since the \( m \geq p > k \) cases are similar. As before it suffices to show that
\[ \sum_{j=1}^p \int R_{v \in E_j \cdot} d\theta(v) > 0. \] (3.7)

Again we may assume that \( (1, \cdots, i_k) = (1, \cdots, k) \). The above quantity does not depend on the choice of the unitary frame of \( \Sigma \), nor on the choice of an unitary frame of the space spanned by \( \{ \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_k} \} \). By the singular value decomposition again, after changes of frames if necessary, we can have a unitary frame \( \{ E_1, \cdots, E_k \} \) of \( \Sigma \) and can have
\[ \frac{\partial}{\partial z_1} = \mu_1 E_1 + \beta_1 E'_1, \quad \frac{\partial}{\partial z_2} = \mu_2 E_2 + \beta_2 E'_2, \quad \cdots, \quad \frac{\partial}{\partial z_k} = \mu_k E_k + \beta_k E'_k, \]
where \( E'_i \perp \Sigma, \quad |E'_i| = 1, \quad |\mu| + |\beta| = 1, \quad 1 \leq i \leq k. \)
Without loss of generality we may also assume that \(|\mu_i|\) forms an increasing sequence. Then
\[
\sum_{i=1}^{p} \int R_{vi} \tau_id\theta(v) = \sum_{i=1}^{k} \left( |\mu_i|^2 \int R_{vi} + |\beta_i|^2 \int R_{vE_1'} \right) = |\mu_1|^2 \sum_{i=1}^{k} \int R_{vi} + (|\mu_2|^2 - |\mu_1|^2) \int \left( \sum_{i=2}^{k} R_{vi} + R_{vE_1'} \right) + |\beta_2|^2 \int R_{vE_1'} + (|\mu_3|^2 - |\mu_2|^2) \int \left( \sum_{i=3}^{k} R_{vi} + R_{vE_2'} \right) + |\beta_3|^2 \int R_{vE_2'} + \cdots + (|\mu_k|^2 - |\mu_{k-1}|^2) \int \left( R_{vE_{k-1}'} + R_{vE_{k-1}'} \right) + |\beta_k|^2 \int R_{vE_{k-1}'} > 0
\]
by Proposition 3.1. Hence we established (3.7).

This implies the vanishing of all coefficients \(a_{I_p}(x_0)\) at the maximum point \(x_0\) of \(|s|^{2}\), hence Theorem 1.3.

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References


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