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Positivity and the Kodaira embedding theorem

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The Kodaira embedding theorem provides an effective characterization of projectivity of a Kähler manifold in terms the second cohomology. X Yang (2018) proved that any compact Kähler manifold with positive holomorphic sectional curvature must be projective. This gives a metric criterion of the projectivity in terms of its curvature. We prove that any compact Kähler manifold with positive 2nd scalar curvature (which is the average of holomorphic sectional curvature over 2–dimensional subspaces of the tangent space) must be projective. In view of generic 2–tori being nonabelian, this new curvature characterization is sharp in certain sense.

53C55; 53C44

1 Introduction

Let (M^m, g) be a Kähler manifold with complex dimension m. For $x \in M$, denote by $T'_x M$ the holomorphic tangent space at x. Let R denote the curvature tensor. For $X \in T'_x M$ let $H(X) = R(X, \overline{X}, X, \overline{X})/|X|^4$ be the holomorphic sectional curvature. Here $|X|^2 = \langle X, \overline{X} \rangle$, and we extended the Riemannian product $\langle \cdot, \cdot \rangle$ and the curvature tensor R linearly over $\mathbb{C}$, following the convention of Ni and Zheng [11]. We say that (M, g) has positive holomorphic sectional curvature if $H(X) > 0$ for any $x \in M$ and any $0 \neq X \in T'_x M$. It was known that compact manifolds with positive holomorphic sectional curvature must be simply connected; see Tsukamoto [13]. A three-circle property was established for noncompact complete Kähler manifolds with nonnegative holomorphic sectional curvature; see Liu [6]. On the other hand, it was known that such metrics may not even have positive Ricci curvature; see Hitchin [2].

The following result was recently proved by X Yang in [16] , which answers affirmatively a question in Yau [17]:

Theorem *If the compact Kähler manifold M has positive holomorphic sectional curvature, then M is projective. Namely, M can be embedded into a complex projective space via a holomorphic map.*

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The key step is to show that the Hodge number $h^{2,0}$ equals 0. Then a well-known result of Kodaira (see Morrow and Kodaira [7, Chapter 3, Theorem 8.3]) implies the projectivity.

The purpose of this paper is to prove a generalization of the above result of Yang. First we introduce some notation after recalling:

Lemma 1.1 (Berger) *If $S(p)=\sum_{i,j=1}^{m} R(E_i,\overline{E}_i,E_j,\overline{E}_j)$, where $\{E_i\}$ is a unitary basis of T'_pM, denotes the scalar curvature of M, then*

$$2S(p)=\frac{m(m+1)}{\mathrm{Vol}(\mathbb{S}^{2m-1})}\int_{|Z|=1,\, Z\in T'_pM} H(Z)\,d\theta(Z). \tag{1-1}$$

Proof Direct calculation shows that

$$\frac{1}{\mathrm{Vol}(\mathbb{S}^{2m-1})}\int_{\mathbb{S}^{2m-1}}|z_i|^4=\frac{2}{m(m+1)}\quad\text{for each } i,$$

$$\frac{1}{\mathrm{Vol}(\mathbb{S}^{2m-1})}\int_{\mathbb{S}^{2m-1}}|z_i|^2|z_j|^2=\frac{1}{m(m+1)}\quad\text{for each } i\neq j.$$

Equation (1-1) then follows by expanding $H(Z)$ in terms of $Z=\sum_i z_iE_i$ and the above formulas. □

For any integer k with $1\le k\le m$ and any k–dimensional subspace $\Sigma\subset T'_xM$, one can define the k–scalar curvature as

$$S_k(x,\Sigma)=\frac{k(k+1)}{2\,\mathrm{Vol}(\mathbb{S}^{2k-1})}\int_{|Z|=1, Z\in\Sigma} H(Z)\,d\theta(Z).$$

By Berger's lemma $\{S_k(x,\Sigma)\}$ interpolates between the holomorphic sectional curvature, which is $S_1(x,\{X\})$, and scalar curvature, which is $S_m(x,T_xM)$.

We say that (M,g) has positive 2^{nd} scalar curvature if $S_2(x,\Sigma)>0$ for any x and any 2–dimensional complex plane Σ.

Clearly the positivity of the holomorphic sectional curvature implies the positivity of the 2^{nd} scalar curvature, and the positivity of S_k implies the positivity of S_l if $k\le l$. We shall prove the following generalization of the above result of Yang:

Theorem 1.2 *Any compact Kähler manifold M^m with positive 2^{nd} scalar curvature must be projective. In fact, $h^{2,0}(M)=0$.*

Recall that a projective manifold M is said to be *rationally connected* if any two generic points can be connected by a chain of rational curves. By the work of Kollár, Miyaoka and Mori [5], any projective manifold M admits a rational map $f\colon M \dashrightarrow Z$ onto a projective manifold Z such that any generic fiber is rationally connected and, for any very general point (meaning away from a countable union of proper subvarieties) $z \in Z$, any rational curve in M which intersects the fiber $f^{-1}(z)$ must be contained in that fiber. Such a map is called a *maximal rationally connected fibration* for M, or *MRC fibration* for short. It is unique up to birational equivalence. The dimension of the fiber of an MRC fibration of M is called the *rational dimension* of M, and is denoted by $\mathrm{rd}(M)$.

Heier and Wong [1, Theorem 1.7] proved that any projective manifold M^m with $S_k > 0$ satisfies $\mathrm{rd}(M) \geq m-(k-1)$. So, as a corollary of their result and Theorem 1.2 above, we have:

Corollary *If M^m is a compact Kähler manifold with positive 2^{nd} scalar curvature then $\mathrm{rd}(M) \geq m-1$. Namely, either M is rationally connected or there is a rational map $f : M \dashrightarrow C$ from M onto a curve C of positive genus such that, over the complement of a finite subset of C, the map f is a holomorphic submersion with compact, smooth fibers and each fiber is a rationally connected manifold.*

Note that the intrinsic criterion of the 2^{nd} scalar curvature can be used to imply that all compact Riemann surfaces (by taking a product with a very positive $\mathbb{P}^1$) are projective, while Yang's result (under the positivity of holomorphic sectional curvature) can only be applied to $\mathbb{P}^1$. Since a generic 2–dimensional complex torus is not algebraic, the projectivity *cannot* be implied by the positivity of S_k with $k \geq 3$ (taking the product of a nonalgebraic torus of complex dimension 2 with a very positive $\mathbb{P}^1$, one can endow a Kähler metric with $S_k > 0$ for $k \geq 3$ on such a nonalgebraic manifold). In view of these examples, our result is sharp in some sense. Moreover, the positivity of S_2 is stable (namely a open condition) under the holomorphic deformation of the complex manifolds along with the smooth deformation of the Kähler metrics specified by Kodaira and Spencer (see Morrow and Kodaira [7]). Hence, our result provides a condition invariant under small deformation of holomorphic structure. On the other hand, there are celebrated examples of Voisin [14] of Kähler manifolds of complex dimension 4 and above that cannot be deformed into algebraic ones via a complex holomorphic deformation, and the wildly open Kodaira's problem in complex dimension 3 asking whether or not a Kähler threefold can be deformed into a projective manifold.

It is well known that $h^{m,0} = 0$ if (M^m, g) has positive scalar curvature. The traditional Bochner formula also implies the vanishing of $h^{p,0} = 0$ for $k \leq p \leq m$ if the Ricci curvature of (M^m, g) is k–positive, namely the sum of the smallest k eigenvalues of the Ricci tensor is positive (see Kobayashi [4]).

Theorem 1.3 *Let (M^m, g) be a compact Kähler manifold. If the k^{th} scalar curvature is positive, then $h^{p,0} = 0$ for any $k \le p \le m$.*

It turns out that the original argument proving the above result contains an error. However, it can be proved using a maximum principle consideration via the comass (an operator norm) of differential forms; see Ni [9, Proposition 4.2 and Corollary 4.3].

As a counterpart to Theorem 1.7 of Heier and Wong [1], one can ask the question: for a given projective Kähler manifold M^m with $S_k < 0$, what is the maximal possible rational dimension? A naive conjecture which mimics the Heier–Wong theorem would be: $S_k < 0$ implies $\mathrm{rd}(M) < k$. Note that a recent result in Ni [10, Theorem 5.1] implies that there are neither projective planes nor 2–dimensional tori in a Kähler manifold (not necessarily compact) with $S_2 < 0$. For $k = m$, the conjecture says that having negative scalar curvature would imply the manifold cannot be rationally connected. This is still unknown even for $m = 2$ as far as we know. Masataka Iwai (personal communication, 2018) shared an example of a complex surface with a *Hermitian metric* of negative scalar curvature which is rationally connected. On the other hand, $S_m < 0$ (or just the integral of the scalar curvature being negative) does imply that $H^0(M, K_M^{-\otimes \ell}) = 0$ for any $\ell > 0$, where K_M^{-1} is the anticanonical line bundle, so M cannot be a Fano manifold when $S_k < 0$ for any k.

We should mention that there is also a recent work of Wu and Yau [15] on the ampleness of the canonical line bundle assuming the holomorphic sectional curvature is negative, which is another perfect example of getting algebraic geometric consequences in terms of the metric property via the holomorphic sectional curvature.

Generally speaking, we think it is interesting to obtain algebraic geometric characterizations of the conditions $S_k > 0$ or $S_k < 0$, as well as the conditions $\mathrm{Ric}^\perp > 0$ and $\mathrm{Ric}^\perp < 0$. The manifolds with $\mathrm{Ric}^\perp > 0$ were studied recently in Ni and Zheng [11], where a complementary metric criterion for projectivity was given in terms of $\mathrm{Ric}^\perp_{\frac{1}{2}} > 0$. A complete classification result for threefolds and a partial classification of fourfolds have been obtained (see Ni and Zheng [12]) for Kähler manifolds with $\mathrm{Ric}^\perp > 0$. The estimates developed in the proof of this paper have also been useful in proving the rational-connectedness of Kähler manifolds with $\mathrm{Ric}_k > 0$ (see Ni [9]). We refer the interested readers to [9] for these and other notions of curvature positivities as well as many related results and questions.

2 The projectivity of M with positive S_2

Here we adopt the argument of [11] to show that the dimension $h^{2,0}(M)$ of $\mathcal{H}^{2,0}(M)$, the space of harmonic $(2,0)$–forms, equals 0. Then Theorem 8.3 of [7] implies that M is projective.

First recall the formula below (see [4, Chapter III, Proposition 1.5], as well as [8, Proposition 2.1]).

Lemma 2.1 *Let s be a global holomorphic p–form on M^m which locally is expressed as $s = \frac{1}{p!}\sum_{I_p} f_{I_p}\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_p}$, where $I_p=(i_1,\dots,i_p)$ and $\{\varphi_1,\dots,\varphi_m\}$ is a local unitary coframe. Then*

$$\partial\bar\partial\,|s|^2 = \langle\nabla s,\overline{\nabla} s\rangle - \widetilde{R}(s,\bar s,\cdot,\cdot)$$

where $\widetilde{R}$ stands for the curvature of the Hermitian bundle $\bigwedge^p\Omega$, where $\Omega=(T'M)^*$ is the holomorphic cotangent bundle of M. The metric on $\bigwedge^p\Omega$ is derived from the metric of M^m. Then, for any unitary coframe $\{\varphi_i\}$,

$$\left\langle\sqrt{-1}\partial\bar\partial|s|^2,\frac{1}{\sqrt{-1}}v\wedge\bar v\right\rangle = \langle\nabla_v s,\overline{\nabla}_v s\rangle + \frac{1}{p!}\sum_{I_p}\sum_{k=1}^{p}\sum_{l=1}^{m} R_{v\bar v i_k\bar l}\, f_{I_p}\overline{f_{i_1\dots(l)_k\dots i_p}}. \tag{2-1}$$

Also, given any x_0 and $v\in T'_{x_0}M$, there exists a unitary coframe $\{\varphi_i\}$ at x_0, which may depend on v, such that

$$\left\langle\sqrt{-1}\partial\bar\partial|s|^2,\frac{1}{\sqrt{-1}}v\wedge\bar v\right\rangle = \langle\nabla_v s,\overline{\nabla}_v s\rangle + \frac{1}{p!}\sum_{I_p}\sum_{k=1}^{p} R_{v\bar v i_k\bar i_k}|f_{I_p}|^2. \tag{2-2}$$

Recall that for any given skew-symmetric $m\times m$ matrix A, there always exists a unitary matrix U such that such that ${}^tUAU$ is in block diagonal form where each nonzero diagonal block is a constant multiple of F with

$$F=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix};$$

see [3, Corollary 4.4.19] for a proof. In particular, given any $(2,0)$–form ψ and at any given point x_0, there always exists a local unitary coframe $\{\varphi_i\}$ such that, at x_0,

$$\psi = \lambda_1\varphi_1\wedge\varphi_2+\lambda_2\,\varphi_3\wedge\varphi_4+\cdots+\lambda_k\,\varphi_{2k-1}\wedge\varphi_{2k},$$

where $2k$ is the rank of the coefficient matrix A of ψ expressed under any unitary coframe. Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2 We prove the result by contradiction. Assume $\mathcal{H}^{2,0}(M) \neq \{0\}$. Let $\psi \in \mathcal{H}^{2,0}(M)$ be a nonzero harmonic form. It is well known that it is holomorphic. Let $k \leq m$ be the largest integer such that $\psi^{k+1} \equiv 0$ but ψ^k is not identically zero. Then $s = \psi^k$ is a nontrivial holomorphic $2k$–form. Let x_0 be a point where $|s|^2$ attains its maximum. Under any local unitary coframe $\{\varphi_i\}$, write $\psi = \sum_{i,j} a_{ij}\, \varphi_i \wedge \varphi_j$. The matrix $A = (a_{ij})$ at x_0 is skew-symmetric. So, replacing φ by another local unitary coframe if necessary, one may assume that, at x_0,

$$\psi = \lambda_1 \varphi_1 \wedge \varphi_2 + \lambda_2\, \varphi_3 \wedge \varphi_4 + \cdots + \lambda_k\, \varphi_{2k-1} \wedge \varphi_{2k},$$

where $\lambda_i \neq 0$ for $1 \leq i \leq k$. Write $s = \frac{1}{p!} \sum_{I_p} f_{I_p} \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p}$ with $p = 2k$. We see that, at the point x_0, the coefficients f_{I_p} of s are

$$f_{12\ldots p} = \lambda := \lambda_1 \lambda_2 \cdots \lambda_k \neq 0$$

while all other $f_{I_p} = 0$. By formula (2-1) in Lemma 2.1, we get

$$0 \geq \left\langle \sqrt{-1}\partial\bar{\partial}|\sigma|^2, \frac{1}{\sqrt{-1}} v \wedge \bar{v} \right\rangle \geq \frac{|\lambda|^2}{(2k)!} \sum_{i=1}^{2k} R_{v\bar{v}i\bar{i}}$$

for any v. Taking $v = e_j$, where $\{e_1, \ldots, e_m\}$ is the unitary tangent frame dual to $\{\varphi_i\}$, and summing over j, we have that, at x_0,

$$\sum_{i,j=1}^{2k} R_{i\bar{i}j\bar{j}} \leq 0. \tag{2-3}$$

On the other hand, it is easy to see that $S_2 > 0$ implies that $S_{2k} > 0$. This is a contradiction to (2-3). Hence there is no nonzero $\psi \in \mathcal{H}^{2,0}(M)$. □

In [9], via a different technique, the result has been extended to Kähler manifolds with so-called RC-2 positivity; namely, for any two unitary vectors $\{E_1, E_2\}$, there exists v such that $R(v, \bar{v}, E_1, \overline{E}_1) + R(v, \bar{v}, E_2, \overline{E}_2) > 0$.

3 Some related estimates

Let Σ be a 2–plane with $S_2(x_0, \Sigma) = \inf_{\Sigma'} S_2(x_0, \Sigma')$. Denote by $⨍ h(Z)$ the average of the integral of the function h over $\mathbb{S}^3 \subset \Sigma$. Choose a local unitary frame e at x_0 so that $⨍ R(v, \bar{v}, \cdot, \overline{(\cdot)})$ is diagonalized. Then, for any holomorphic 2–form $s = \sum_{i \neq j} f_{ij} \varphi_i \wedge \varphi_j$, where $\{\varphi_i\}$ is dual to e, by integrating the Bochner formula (2-1) of

Lemma 2.1 for $v \in \mathbb{S}^3 \subset \Sigma$, we have

$$\fint \partial_v \bar\partial_{\bar v} |s|^2 = \fint \langle \nabla_v s, \overline{\nabla}_v s\rangle + \frac{1}{2} \sum_{i,j=1}^{m} |f_{ij}|^2 \fint (R_{v\bar v i \bar i} + R_{v \bar v j \bar j}). \tag{3-1}$$

This suggests a possible alternative approach to Theorem 1.2, which is to apply the maximum principle at x_0 where $|s|^2$ attains its maximum in the above integral form. In view of the compactness of the Grassmannians one can always find a complex 2–plane Σ in $T'_{x_0}M$ such that $S_2(x_0, \Sigma) = \inf_{\Sigma'} S_2(x_0, \Sigma') > 0$. We prove the following estimates, some of which were used in establishing the rational-connectedness of algebraic manifolds under the $\mathrm{Ric}_k > 0$ condition in [9]:

Proposition 3.1 *For any $E \in \Sigma$, any $E' \perp \Sigma$ with $|E| = |E'| = 1$ and any 2–dimensional plane $\Sigma' \subset T'_p M$ with $\Sigma' \neq \Sigma$ and unitary frame $\{v_1, v_2\}$, we have*

$$\fint R(E, \bar E', Z, \bar Z)\, d\theta(Z) = \fint R(E', \bar E, Z, \bar Z)\, d\theta(Z) = 0, \tag{3-2}$$

$$\begin{aligned} \fint R(v_1, \bar v_1, Z, \bar Z) + R(v_2, \bar v_2, Z, \bar Z)\, d\theta(Z) &\geq \tfrac{1}{3} S_2(x_0, \Sigma) + \tfrac{1}{12}(|\mu_1|^2 + |\mu_2|^2) S_2(x_0, \Sigma) \\ &\quad + \tfrac{1}{4}(|\mu_1|^2 - |\mu_2|^2)(R_{1\bar 1 1 \bar 1} - R_{2 \bar 2 2 \bar 2}), \end{aligned} \tag{3-3}$$

$$\fint R(E', \bar E', Z, \bar Z)\, d\theta(Z) \geq \tfrac{1}{6} S_2(x_0, \Sigma). \tag{3-4}$$

Here μ_1 and μ_2 are the singular values of the projection P from Σ' to Σ, and $\{E_1, E_2\}$ is a unitary basis of Σ such that $P v_1 = \mu_1 E_1$ and $P v_2 = \mu_2 E_2$.

The relevance to Theorem 1.2 is that, at x_0 where $|s|^2$ attains its maximum, we have

$$0 \geq \fint \partial_v \bar\partial_{\bar v} |s|^2\, d\theta(v) = \fint \langle \nabla_v s, \bar\nabla_{\bar v} \bar s\rangle + \frac{1}{2} \sum_{i,j=1}^{m} |f_{ij}|^2 (R_{v \bar v i \bar i} + R_{v \bar v j \bar j})\, d\theta(v).$$

The integral is clearly independent of the choice of unitary frame of the 2–dimensional space spanned by $\{e_i, e_j\}$ and the choice of unitary frame $\{E_1, E_2\}$ of Σ. If the right-hand side of (3-3) has a positive lower bound, the maximum principle shows that $|s|^2 = 0$ at x_0, and thus $|s|^2 = 0$ everywhere, which gives another proof Theorem 1.2.

Since the estimates of Proposition 3.1 have other applications, we include a proof here. The proof needs some basic algebra and computations. Let $a \in \mathfrak{u}(m)$ be an element of the Lie algebra of $\mathsf{U}(m)$. Consider the function

$$f(t) = \fint H(e^{ta}X)\,d\theta(X).$$

By the choice of Σ, $f(t)$ attains its minimum at $t=0$. This implies that $f'(0)=0$ and $f''(0)\ge 0$. Hence,

$$\fint (R(a(X),\bar{X},X,\bar{X}) + R(X,\bar{a}(\bar{X}),X,\bar{X}))\,d\theta(X) = 0, \tag{3-5}$$

$$\begin{aligned}\fint \bigl(R(a^2(X),\bar{X},X,\bar{X}) + R(X,\bar{a}^2(\bar{X}),X,\bar{X}) \\ + 4R(a(X),\bar{a}(\bar{X}),X,\bar{X})\bigr)\,d\theta(X) \\ + \fint (R(a(X),\bar{X},a(X),\bar{X}) + R(X,\bar{a}(\bar{X}),X,\bar{a}(\bar{X})))\,d\theta(X) \ge 0.\end{aligned} \tag{3-6}$$

We exploit these by looking into some special cases of a. Let $W\perp\Sigma$ and $Z\in\Sigma$ be two fixed vectors with $|W|=1$. Let $a=\sqrt{-1}(Z\otimes\overline{W}+W\otimes\bar{Z})$. Then

$$a(X) = \sqrt{-1}\langle X,\bar{Z}\rangle W \quad\text{and}\quad a^2(X) = -\langle X,\bar{Z}\rangle Z.$$

To show (3-2), let us apply (3-5) to a and also to the element of $\mathfrak{u}(m)$ with W replaced by $\sqrt{-1}W$, and add the resulting estimates together to get

$$\fint \langle X,\bar{Z}\rangle R(W,\bar{X},X,\bar{X})\,d\theta(X) = 0.$$

Taking $Z=E_1$, we have

$$\begin{aligned} 0 &= \fint x_1 R(W,\bar{X},X,\bar{X})\,d\theta(X) \\ &= \fint \bigl(|x_1|^4 R(W,\bar{E}_1,E_1,\bar{E}_1) + 2|x_1x_2|^2 R(W,\bar{E}_1,E_2,\bar{E}_2)\bigr)\,d\theta(X) \\ &= \tfrac{1}{3}\bigl(R(W,\bar{E}_1,E_1,\bar{E}_1) + R(W,\bar{E}_1,E_2,\bar{E}_2)\bigr) \\ &= \frac{2}{3}\fint \bigl(|x_1|^2 R(W,\bar{E}_1,E_1,\bar{E}_1) + |x_2|^2 R(W,\bar{E}_1,E_2,\bar{E}_2)\bigr)\,d\theta(X) \\ &= \frac{2}{3}\fint R(W,\bar{E}_1,X,\bar{X})\,d\theta(X). \end{aligned}$$

Similarly, $\fint R(W,\bar{E}_2,X,\bar{X})\,d\theta(X)=0$; hence, (3-2) holds.

Next we prove (3-4). Applying (3-6) to a and also to the element with W replaced by $\sqrt{-1}W$, and adding the resulting estimates together, we have that

$$(3\text{-}7)\quad 4\fint |\langle X,\bar Z\rangle|^2 R(W,\overline W,X,\bar X)\,d\theta(X) \ge \fint \langle X,\bar Z\rangle R(Z,\bar X,X,\bar X)+\langle Z,\bar X\rangle R(X,\bar Z,X,\bar X).$$

Letting $Z=E_i$, we get

$$4\fint |x_i|^2 R(W,\overline W,X,\bar X)\,d\theta(X) \ge \fint x_i R(E_i,\bar X,X,\bar X)+\bar x_i R(X,\bar E_i,X,\bar X)\,d\theta.$$

Adding up for $i=1,2$ yields

$$4\fint R(W,\overline W,X,\bar X)\,d\theta(X) \ge 2\fint R(X,\bar X,X,\bar X)\,d\theta = \tfrac{2}{3}S_2(x_0,\Sigma);$$

thus, formula (3-4) holds.

To prove (3-3) we need to consider general W which may not be perpendicular to Σ. In other words, we consider the case $|Z|=|W|=1$ and $Z\in\Sigma$:

$$a(X)=\sqrt{-1}(\langle X,\bar Z\rangle W+\langle X,\overline W\rangle Z),$$
$$a^2(X)=-\langle X,\bar Z\rangle(Z+\langle W,\bar Z\rangle W)-\langle X,\overline W\rangle(W+\langle Z,\overline W\rangle Z).$$

Substituting this and the element with W replaced by $\sqrt{-1}W$ into (3-6) and adding the results up, we get the estimate

$$\begin{aligned}(3\text{-}8)\quad 4\fint |\langle X,\bar Z\rangle|^2 R(W,\overline W,X,\bar X)+|\langle X,\overline W\rangle|^2 R(Z,\bar Z,X,\bar X)\,d\theta(X)\\ \ge \fint \langle X,\bar Z\rangle R(Z,\bar X,X,\bar X)+\langle Z,\bar X\rangle R(X,\bar Z,X,\bar X)\,d\theta(X)\\ +\fint \langle X,\overline W\rangle R(W,\bar X,X,\bar X)+\langle W,\bar X\rangle R(X,\overline W,X,\bar X)\,d\theta(X)\\ +2\fint \langle X,\bar Z\rangle\langle X,\overline W\rangle R(W,\bar X,Z,\bar X)\\ +\langle Z,\bar X\rangle\langle W,\bar X\rangle R(X,\overline W,X,\bar Z)\,d\theta(X).\end{aligned}$$

Applying the above to $Z=E_i$ $(i=1,2)$ and summing the results we have

$$\begin{aligned}(3\text{-}9)\quad 4\fint R(W,\overline W,X,\bar X)+|\langle X,\overline W\rangle|^2(R_{1\bar 1X\bar X}+R_{2\bar 2X\bar X})\,d\theta(X)\\ \ge \tfrac{2}{3}S_2(x_0,\Sigma)+4\fint \langle X,\overline W\rangle R(W,\bar X,X,\bar X)+\langle W,\bar X\rangle R(X,\overline W,X,\bar X)\,d\theta(X).\end{aligned}$$

Now we want to apply the above to all unit vectors $W\in\Sigma'$ and take the average. Denote by P the orthogonal projection to Σ. Let $\{v_1,v_2\}$ be a unitary basis of Σ'. Replacing

$\{v_1, v_2\}$ by a new unitary basis $\{av_1 + bv_2, -\bar{a}v_1 + \bar{b}v_2\}$ (where $|a|^2 + |b|^2 = 1$) if necessary, we may assume that $Pv_1 \perp Pv_2$. So we can choose a unitary basis $\{E_1, E_2\}$ of Σ such that $v_1 = \mu_1 E_1 + \alpha E'$ and $v_2 = \mu_2 E_2 + \beta E''$ with μ_i being the singular value of the projection to Σ restricted to Σ', and with $E', E'' \in \Sigma^{\perp}$. Now we apply (3-9) to $W \in \mathbb{S}^3 \subset \Sigma'$. First we observe that

$$2 \fint R(v_1, \bar{v}_1, X, \overline{X}) + R(v_2, \bar{v}_2, X, \overline{X})\, d\theta(X) = 4 \fint_{\mathbb{S}^3 \subset \Sigma'} \fint R(W, \overline{W}, X, \overline{X})\, d\theta(X)\, d\theta(W).$$

The second term on the left-hand side of (3-9) has average value

$$\begin{aligned} L_2 &= 4 \fint_{\mathbb{S}^3 \subset \Sigma'} \fint |\langle X, \overline{W} \rangle|^2 (R_{1\bar{1}X\overline{X}} + R_{2\bar{2}X\overline{X}})\, d\theta(X)\, d\theta(W) \\ &= 2 \fint (|\langle X, \bar{v}_1 \rangle|^2 + |\langle X, \bar{v}_2 \rangle|^2)(R_{1\bar{1}X\overline{X}} + R_{2\bar{2}X\overline{X}})\, d\theta(X). \end{aligned}$$

Expressing $X = x_1 E_1 + x_2 E_2$, we have

$$\begin{aligned} &2 \fint |\langle X, \bar{v}_1 \rangle|^2 (R_{1\bar{1}X\overline{X}} + R_{2\bar{2}X\overline{X}})\, d\theta(X) \\ &\quad = 2|\mu_1|^2 \fint |x_1|^2 (R_{1\bar{1}X\overline{X}} + R_{2\bar{2}X\overline{X}})\, d\theta \\ &\quad = 2|\mu_1|^2 \fint (|x_1|^4 R_{1\bar{1}1\bar{1}} + R_{1\bar{1}2\bar{2}} |x_1|^2 |x_2|^2)\, d\theta \\ &\qquad\qquad + 2|\mu_1|^2 \fint (|x_1|^4 R_{1\bar{1}2\bar{2}} + R_{2\bar{2}2\bar{2}} |x_1|^2 |x_2|^2)\, d\theta \\ &\quad = \tfrac{2}{3}|\mu_1|^2 R_{1\bar{1}1\bar{1}} + |\mu_1|^2 R_{1\bar{1}2\bar{2}} + \tfrac{1}{3}|\mu_1|^2 R_{2\bar{2}2\bar{2}}. \end{aligned}$$

Similarly, we have

$$2 \fint |\langle X, \bar{v}_2 \rangle|^2 (R_{1\bar{1}X\overline{X}} + R_{2\bar{2}X\overline{X}})\, d\theta(X) = \tfrac{2}{3}|\mu_2|^2 R_{2\bar{2}2\bar{2}} + |\mu_2|^2 R_{1\bar{1}2\bar{2}} + \tfrac{1}{3}|\mu_2|^2 R_{1\bar{1}1\bar{1}}.$$

The second term on the right-hand side of (3-9) has average value

$$\begin{aligned} R_2 &= 4 \fint_{\mathbb{S}^3 \subset \Sigma'} \fint \langle X, \overline{W} \rangle R(W, \overline{X}, X, \overline{X}) + \langle W, \overline{X} \rangle R(X, \overline{W}, X, \overline{X})\, d\theta(X)\, d\theta(W) \\ &= 2 \fint \langle X, \bar{v}_1 \rangle R(v_1, \overline{X}, X, \overline{X}) + \langle v_1, \overline{X} \rangle R(X, \bar{v}_1, X, \overline{X})\, d\theta(X) \\ &\qquad + 2 \fint \langle X, \bar{v}_2 \rangle R(v_2, \overline{X}, X, \overline{X}) + \langle v_2, \overline{X} \rangle R(X, \bar{v}_2, X, \overline{X})\, d\theta(X). \end{aligned}$$

We compute

$$2\fint \langle X,\bar v_1\rangle R(v_1,\bar X,X,\bar X) = 2\fint x_1(|\mu_1|^2 R_{1\bar X X\bar X} + \bar\mu_1\alpha R_{E'\bar X X\bar X})\,d\theta$$
$$= 2|\mu_1|^2\fint x_1 R_{1\bar X X\bar X}\,d\theta + \tfrac{2}{3}\bar\mu_1\alpha(R_{E'\bar 1 1\bar 1} + R_{E'\bar 1 2\bar 2})$$
$$= 2|\mu_1|^2\fint (|x_1|^4 R_{1\bar 1 1\bar 1} + 2|x_1|^2|x_2|^2 R_{1\bar 1 2\bar 2})\,d\theta$$
$$= \tfrac{2}{3}|\mu_1|^2(R_{1\bar 1 1\bar 1} + R_{1\bar 1 2\bar 2}).$$

Hence, after adding the result with its conjugation, we have

$$2\fint \langle X,\bar v_1\rangle R(v_1,\bar X,X,\bar X) + \langle v_1,\bar X\rangle R(X,\bar v_1,X,\bar X)\,d\theta(X)$$
$$= \tfrac{4}{3}|\mu_1|^2(R_{1\bar 1 1\bar 1} + R_{1\bar 1 2\bar 2}).$$

Similarly, we have

$$2\fint \langle X,\bar v_2\rangle R(v_2,\bar X,X,\bar X) + \langle v_2,\bar X\rangle R(X,\bar v_2,X,\bar X)\,d\theta(X)$$
$$= \tfrac{4}{3}|\mu_2|^2(R_{2\bar 2 2\bar 2} + R_{1\bar 1 2\bar 2}).$$

Therefore, we have

$$R_2 = \tfrac{4}{3}|\mu_1|^2(R_{1\bar 1 1\bar 1} + R_{1\bar 1 2\bar 2}) + \tfrac{4}{3}|\mu_2|^2(R_{2\bar 2 2\bar 2} + R_{1\bar 1 2\bar 2}).$$

Putting them all together and noting that $S_2(x_0,\Sigma) = R_{1\bar 1 1\bar 1} + 2R_{1\bar 1 2\bar 2} + R_{2\bar 2 2\bar 2}$, we get

$$2\fint R(v_1,\bar v_1,X,\bar X) + R(v_2,\bar v_2,X,\bar X)\,d\theta(X)$$
$$\geq \tfrac{2}{3}S_2(x_0,\Sigma) + \tfrac{1}{6}(|\mu_1|^2+|\mu_2|^2)S_2(x_0,\Sigma) + \tfrac{1}{2}(|\mu_1|^2-|\mu_2|^2)(R_{1\bar 1 1\bar 1} - R_{2\bar 2 2\bar 2}).$$

This proves (3-3).

4 The high-dimensional case

Now, for a k–dimensional subspace $\Sigma \subset T'_{x_0}M$ with $S_k(x_0,\Sigma) = \inf_{\Sigma'} S_k(x_0,\Sigma')$, we derive estimates similar to Proposition 3.1.

Proposition 4.1 *Let Σ and Σ' be two k–dimensional subspaces of $T'_{x_0}M$. Assume that $S_k(x_0,\Sigma) = \inf_{\Sigma'} S_k(x_0,\Sigma')$, and that $\{v_1,\dots,v_k\}$ and $\{E_1,\dots,E_k\}$ are unitary frames at x_0 of Σ' and Σ, respectively. Let $\{\mu_i\}$ be the singular values of the projection* *of Σ' towards Σ. Then, for any $E \in \Sigma$ with $E' \perp \Sigma$, we have*

$$\fint R(E, \bar{E}', Z, \bar{Z})\, d\theta(Z) = \fint R(E', \bar{E}, Z, \bar{Z})\, d\theta(Z) = 0, \tag{4-1}$$

$$\fint \Big(\sum_{j=1}^{k} R(v_j, \bar{v}_j, Z, \bar{Z}) \Big)\, d\theta(Z) \geq \frac{1}{k(k+1)} \Big(\sum_{i=1}^{k} (1 - |\mu_i|^2) \Big) S_k(x_0, \Sigma) + \frac{1}{k} \sum_{i=1}^{k} \Big(|\mu_i|^2 \sum_{j=1}^{k} R_{i\bar{i}j\bar{j}} \Big), \tag{4-2}$$

$$\fint R(E', \bar{E}', Z, \bar{Z})\, d\theta(Z) \geq \frac{S_k(x_0, \Sigma)}{k(k+1)}. \tag{4-3}$$

Proof Let $f(t)$ be the function constructed by the variation under the 1–parameter family of unitary transformations. The equations (3-5) and (3-6), as well as their proofs, remain the same. The proofs of (4-1) and (4-3) are exactly analogous to those of (3-2) and (3-4), so we omit them.

To prove (4-2) we apply (3-8) with $Z = E_i$ and add the results up:

$$4 \fint R(W, \overline{W}, X, \bar{X}) + |\langle X, \overline{W} \rangle|^2 \Big(\sum_{j=1}^{k} R_{j\bar{j}X\bar{X}} \Big)\, d\theta(X) \geq \frac{4}{k(k+1)} S_k(x_0, \Sigma) + (k+2) \fint \langle X, \overline{W} \rangle R(W, \bar{X}, X, \bar{X}) + \langle W, \bar{X} \rangle R(X, \overline{W}, X, \bar{X})\, d\theta(X). \tag{4-4}$$

For the given k–planes Σ and Σ', we may always choose a unitary basis $\{v_1, \dots, v_k\}$ of Σ' and a unitary basis $\{E_1, \dots, E_k\}$ of Σ so that the restriction on Σ' of the projection map to Σ is given by a diagonal matrix under these bases. That is, $v_i = \mu_i E_i + \alpha_i E_i'$ for each i, with $E_i' \perp \Sigma$ and where the $\{\mu_i\}$ are the singular values of the projection from Σ' to Σ.

Now we apply (4-4) to $W \in \mathbb{S}^{2k-1} \subset \Sigma'$ and take the average of the result:

$$\frac{4}{k} \fint \sum_{i=1}^{k} R(v_i, \bar{v}_i, X, \bar{X})\, d\theta(X) = 4 \fint_{\mathbb{S}^{2k-1} \subset \Sigma'} \fint R(W, \overline{W}, X, \bar{X})\, d\theta(X)\, d\theta(W).$$

Similarly we can calculate

$$4\fint_{\mathbb{S}^{2k-1}\subset\Sigma'}\fint |\langle X,\overline{W}\rangle|^2\Big(\sum_{j=1}^k R_{j\bar{j}X\overline{X}}\Big)\,d\theta(X)\,d\theta(W)$$
$$=\frac{4}{k}\fint\Big(\sum_{i=1}^k|\langle X,\bar{v}_i\rangle|^2\Big)\Big(\sum_{j=1}^k R_{j\bar{j}X\overline{X}}\Big)\,d\theta(X)$$
$$=\frac{4}{k}\frac{1}{k(k+1)}\sum_{i=1}^k\Big(|\mu_i|^2\Big(S_k+\sum_{j=1}^k R_{i\bar{i}j\bar{j}}\Big)\Big),$$

while

$$(k+2)\fint_{\mathbb{S}^{2k-1}\subset\Sigma'}\fint\langle X,\overline{W}\rangle R(W,\overline{X},X,\overline{X})+\langle W,\overline{X}\rangle R(X,\overline{W},X,\overline{X})\,d\theta(X)\,d\theta(W)$$
$$=\frac{k+2}{k}\fint\sum_{i=1}^k\langle X,\bar{v}_i\rangle R(v_i,\overline{X},X,\overline{X})+\langle v_i,\overline{X}\rangle R(X,\bar{v}_i,X,\overline{X})\,d\theta(X).$$

Using (4-1), the first half in the equation above can be further simplified into

$$\frac{k+2}{k}\fint\sum_{i=1}^k\langle X,\bar{v}_i\rangle R(v_i,\overline{X},X,\overline{X})\,d\theta(X)$$
$$=\frac{k+2}{k}\fint\sum_{i=1}^k x_i(|\mu_i|^2R_{i\overline{X}X\overline{X}}+\bar{\mu}_i\alpha_i R_{E_i'\overline{X}X\overline{X}})\,d\theta(X)$$
$$=\frac{k+2}{k}\fint\sum_{i=1}^k x_i(|\mu_i|^2R_{i\overline{X}X\overline{X}})\,d\theta(X)$$
$$=\frac{k+2}{k}\sum_{i=1}^k|\mu_i|^2\fint\Big(|x_i|^4R_{i\bar{i}i\bar{i}}+2\sum_{j\neq i}|x_ix_j|^2R_{i\bar{i}j\bar{j}}\Big)\,d\theta(X)$$
$$=\frac{k+2}{k}\frac{2}{k(k+1)}\sum_{i=1}^k\Big(|\mu_i|^2\sum_{j=1}^k R_{i\bar{i}j\bar{j}}\Big).$$

Putting the above together we have (4-2). □

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References

[1] **G Heier**, **B Wong**, *On projective Kähler manifolds of partially positive curvature and rational connectedness*, Doc. Math. 25 (2020) 219–238 MR Zbl

[2] **N Hitchin**, *On the curvature of rational surfaces*, from "Differential geometry, II" (S S Chern, R Osserman, editors), Proc. Symp. Pure Math. 27, Amer. Math. Soc., Providence, RI (1975) 65–80 MR Zbl

[3] **R A Horn**, **C R Johnson**, *Matrix analysis*, 2nd edition, Cambridge Univ. Press (2013) MR Zbl

[4] **S Kobayashi**, *Differential geometry of complex vector bundles*, Publ. Math. Soc. Japan 15, Princeton Univ. Press (1987) MR Zbl

[5] **J Kollár**, **Y Miyaoka**, **S Mori**, *Rationally connected varieties*, J. Algebraic Geom. 1 (1992) 429–448 MR Zbl

[6] **G Liu**, *Three-circle theorem and dimension estimate for holomorphic functions on Kähler manifolds*, Duke Math. J. 165 (2016) 2899–2919 MR Zbl

[7] **J Morrow**, **K Kodaira**, *Complex manifolds*, Holt, Rinehart and Winston, New York (1971) MR Zbl

[8] **L Ni**, *Vanishing theorems on complete Kähler manifolds and their applications*, J. Differential Geom. 50 (1998) 89–122 MR Zbl

[9] **L Ni**, *The fundamental group, rational connectedness and the positivity of Kähler manifolds*, J. Reine Angew. Math. 774 (2021) 267–299 MR Zbl

[10] **L Ni**, *Liouville theorems and a Schwarz lemma for holomorphic mappings between Kähler manifolds*, Comm. Pure Appl. Math. 74 (2021) 1100–1126 MR Zbl

[11] **L Ni**, **F Zheng**, *Comparison and vanishing theorems for Kähler manifolds*, Calc. Var. Partial Differential Equations 57 (2018) art. id. 151 MR Zbl

[12] **L Ni**, **F Zheng**, *On orthogonal Ricci curvature*, from "Advances in complex geometry" (Y A Rubinstein, B Shiffman, editors), Contemp. Math. 735, Amer. Math. Soc., Providence, RI (2019) 203–215 MR Zbl

[13] **Y Tsukamoto**, *On Kählerian manifolds with positive holomorphic sectional curvature*, Proc. Japan Acad. 33 (1957) 333–335 MR Zbl

[14] **C Voisin**, *On the homotopy types of compact Kähler and complex projective manifolds*, Invent. Math. 157 (2004) 329–343 MR Zbl

[15] **D Wu**, **S-T Yau**, *Negative holomorphic curvature and positive canonical bundle*, Invent. Math. 204 (2016) 595–604 MR Zbl

[16] **X Yang**, *RC-positivity, rational connectedness and Yau's conjecture*, Camb. J. Math. 6 (2018) 183–212 MR Zbl

[17] **S T Yau**, *Problem section*, from "Seminar on differential geometry" (S T Yau, editor), Ann. of Math. Stud. 102, Princeton Univ. Press (1982) 669–706 MR Zbl

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