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An alternate induction argument in Simons' proof of holonomy theorem

Abstract: The paper gives an exposition and an alternate argument of Simons algebraic proof [7] of the holonomy theorem via the holonomy system.

Keywords: Curvature, Lie groups, holonomy groups, symmetric spaces

MSC 2010: 53C29, 53C05, 58D19, 57S25

Contents

1 Introduction

Berger's classification [2] of Riemannian holonomy groups is very important in Riemannian geometry. The proof utilized Cartan's classification of simple Lie groups. An intrinsic proof was later discovered by J. Simons [7]. In fact, Simons proved the following result without appealing to Cartan's classification results.

Theorem 1.1 (Berger). *Assume that H_p^0, the restricted holonomy group of a Riemannian manifold (M^n, g), acts irreducibly on the tangent space M_p. Then either H_p^0 acts transitively on $\mathbb{S}^{n-1} \subset M_p$ or (M^n, g) is a locally symmetric space with rank greater than or equal to* 2.

This result implies Berger's list for possible holonomy groups of Riemannian manifolds which are not locally symmetric due to the earlier work of [4] on transformation

Acknowledgement: We would like to thank the organizers of Nankai conference "Analysis and PDEs on Manifolds and Fractals" for the invitation to the event and this contribution. The author thanks Professors N. Wallach and H. Wu for their interests, Y. Niu, F. Zheng for helpful comments. He also thanks the participating students of the graduate course Math 250C (UCSD, Spring of 2020) for their patience. The alternate argument was cleaned up during the course.

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https://doi.org/10.1515/9783110700763-016

groups of the sphere (cf. [3]). Simons obtained the above result by solving an algebraic problem in terms of the holonomy system:

Theorem 1.2 (Simons). *Assume that S is an irreducible Riemannian holonomy system. Assume that G acts nontransitively on $\mathbb{S}^{n-1} \subset V$. Then S is symmetric.*

Here a *Riemannian holonomy system* $S = \{V, \mathrm{R}, G\}$ consists of a Euclidean space V of dimension n (we call it the degree of S) endowed with an inner product, a connected compact subgroup G of $\mathsf{SO}(n)$, and an algebraic curvature operator R (defined on V) satisfying the first Bianchi identity and such that $\mathrm{R}_{x,y} \in \mathfrak{g}$, $\forall x, y \in V$ with $\mathfrak{g} \subset \mathfrak{so}(n)$ being the Lie algebra of G. The system S is called irreducible if G acts irreducibly on V. That a holonomy system $S = \{V, \mathrm{R}, G\}$ is symmetric means $g(\mathrm{R}) = \mathrm{R}$, $\forall g \in G$. Let $S^2_B(\wedge^2 V)$ denote the space of algebraic curvature operators after identifying $\mathfrak{so}(n)$ with $\wedge^2 V$. Here $S^2(\wedge^2 V)$ denotes the symmetric transformations of $\wedge^2 V$. The $S^2_B(\cdot)$ denotes the subspace satisfying the first Bianchi identity. The action $g(\mathrm{R})$ is the natural extension of the action of $\mathsf{SO}(n)$ on V to $S^2_B(\wedge^2 V)$ (see Section 2 for details).

What was proved in [7] is slightly stronger. To state that result, we need to introduce additional notions. Let $G(\mathrm{R}) \subset S^2_B(\wedge^2 V)$ be the linear subspace spanned by $\{g(\mathrm{R}), g \in G\}$. Namely, $G(\mathrm{R})$ is the subspace generated by the orbit of R under the action of G (also see Section 2 for more details). Define $\mathfrak{g}^{\mathrm{R}} \subset \mathfrak{g}$ as the subspace spanned by $\{Q(\wedge^2 V), Q \in G(\mathrm{R})\}$. One may check that $\mathfrak{g}^{\mathrm{R}}$ (see Lemma 4.1) is an ideal of $\mathfrak{g}$. Now $G^{\mathrm{R}} \subset G$ is defined as the Lie (closed) subgroup of G generated by $\mathfrak{g}^{\mathrm{R}}$. Clearly the nontransitivity of G implies the nontransitivity of G^{R}.

Theorem 1.3 (Simons). *Let $S = \{V, \mathrm{R}, G\}$ be an irreducible Riemannian holonomy system. Assume that G^{R} acts nontransitively on $\mathbb{S}^{n-1}$. Then S is symmetric with rank ≥ 2.*

The proof of Simons [7] is via a double induction on $\dim(V)$ and $\dim(\mathfrak{g}^{\mathrm{R}})$. The purpose of this paper is to give an exposition of Simons' proof via an alternate induction on $\dim(G(\mathrm{R}))$. Since it was believed that (cf. [5]) "… the proof of Simons is long and involved, except for the first general part. At some step he used case by case arguments, combined with induction on the dimension. Few mathematicians went through all the details of this proof…", our hope is that the exposition here and this alternate induction can offer some enhancement in understanding the important work [7] which contains many ingenious ideas.

There exist several expositions, e. g., [3, 6, 10], on holonomy theorem and Simons' proof. Due to the importance of Theorem 1.1, our presentation includes basic definitions and a derivation of Theorem 1.1 using Theorem 1.2. The argument here does not use any result from the theory of symmetric spaces. Precisely, it does not use the correspondence between the orthogonal symmetric Lie algebras and symmetric spaces, or the full Ambrose–Singer's theorem [1]. The presentation is completely self-contained except very basic results, such as the Schur's lemma.

2 Preliminaries

In this section we recall basic concepts and definitions. A *fiber bundle* is a triple (E, F, M) with a projection map $\pi : E \to M$ such that π is regular with $\pi^{-1}(x)$ (denoted as E_x) being diffeomorphic to the space F such that for any point $p \in M$, there exists a neighborhood U_α and a diffeomorphism $\varphi_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ such that $\varphi_\alpha(x, f) \in \pi^{-1}(x)$. We say that it has a *structure group* G, if the transition functions $T_{\alpha\beta}(x)$ (where $\varphi_\beta^{-1} \circ \varphi_\alpha(x, f) = (x, T_{\alpha\beta}(x)(f))$) is in G. Here E, F, M are all smooth manifolds and we also require $T_{\alpha\beta}(x)$ be smooth in x. *A connexion* of (E, F, M) is a mapping P, defined for any piecewisely smooth path $\gamma : (0, 1) \to M$, a $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ such that it satisfies that (i) P_γ depends on γ smoothly, (ii) $P_{\gamma_1 \circ \gamma_2} = P_{\gamma_1} \circ P_{\gamma_2}$, and (iii) $P_{\gamma^{-1}} = (P_\gamma)^{-1}$. Such P_γ is called the *parallel transport* along γ. In general, P_γ is in $\mathrm{Diff}(F)$. When F is a linear space and G is a subgroup of general linear transformations, namely (E, M) is a vector bundle, P_γ is required to be a linear map.

Let $\Omega(x_0, M)$ be the *loop space* at x_0. Then $P_{(\cdot)} : \Omega(x_0, M) \to \mathrm{Diff}(E_{x_0})$ (or G) defined by P_γ is a homomorphism. The image (denoted by H_{x_0}) is called the *holonomy group*. For most of our discussion, emphasis is given to the image of the connected component of the trivial loop $\gamma(t) \equiv x_0$, namely the loops which are homotopically trivial. The corresponding image is called the restricted holonomy group, denoted by $H^0_{x_0}$. Its Lie algebra is denoted by $\mathfrak{h}$. It is easy to see that for a different base point x_1, if γ is a path from x_1 to x_0, then $H_{x_0} = P_\gamma H_{x_1} P_{\gamma^{-1}}$, and $H^0_{x_0} = P_\gamma H^0_{x_1} P_{\gamma^{-1}}$.

A *covariant derivative* at point p is a map $\nabla : T_pM \times \mathcal{T}_pM \to T_pM$ ($\mathcal{T}_pM$ denotes the germs of tangent vectors) satisfying axioms: (i) $\nabla_{\alpha\xi+\beta\eta} Y = \alpha\nabla_\xi Y + \beta\nabla_\eta Y$; (ii) linear in the second component; (iii) $\nabla_\xi(fY) = (\xi f)Y + f\nabla_\xi Y$. This is also called an *affine connection*. A *global affine connection* is that defined for all $p \in M$ and such that if X, Y are smooth $\nabla_X Y$ is smooth. Once M is endowed with a global affine connection, one can define the *covariant derivative along a curve* $c(t) : (a, b) \to M$ for a *vector field* $X(t)$ along $c(t)$ by $\frac{D}{dt}X(c(t)) = \nabla_{\dot c(t)} X$, if X is defined on $c(t)$. This leads to a connexion defined above via the *parallel transport* along $c(t)$ by solving an ordinary differential equation: For any $X_{x_0} \in T_{x_0}M$ and a curve $\gamma(t)$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, $X(t) \in T_{\gamma(t)}M$ can be constructed by solving $\frac{D}{dt}X(t) = 0, X(0) = X_{x_0}$. Then one defines $P_\gamma(X_{x_0}) \doteqdot X(1)$. In general, $P^{t_1,t_2}_\gamma : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$ can be defined as $P^{t_1,t_2}_\gamma(\xi) = X(t_2)$ with $X(t)$ being the parallel vector along $\gamma(t)$ with $X(t_1) = \xi$. Note that the above discussion makes sense for any smooth vector bundle (E, M) of rank k as well. A basic result below asserts that a connexion on a vector bundle (with linear structure group) is equivalent to an affine connection.

Lemma 2.1. $\frac{D}{dt}X(t)\Big|_{t_0} = \lim_{t\to t_0} \frac{P^{t,t_0}_\gamma(X(t)) - X(t_0)}{t - t_0}$.

We focus on the case that $F = E_x = \pi^{-1}(x)$ is a vector space endowed with a smoothly depended inner product. Now P_γ is required to preserve this inner product (namely the metric is invariant w. r. t. the D above). Theorem 1.1 concerns the Levi-

Civita connection, namely the canonical affine connection of the Riemannian structure (M^n, g) on its tangent bundle. For $p \in M$, let γ be a loop at p or a path towards p. We also use γ to denote the parallel transport along γ, which is an isometry of M_p, the tangent space at p. In this setting $H_p^0 \subset \mathsf{SO}(n)$ is compact. Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be its Lie algebra. Let R be the curvature tensor of Levi-Civita connection. First we show that

Lemma 2.2. *$\forall x, y \in M_p$ and $\forall \gamma$ from q to p, $\gamma(\mathrm{R}^q)_{x,y} \in \mathfrak{h}$. Here*

$$\langle \gamma(\mathrm{R}^q)_{x,y} z, w\rangle \doteqdot \langle \mathrm{R}_{\gamma^{-1}(x),\gamma^{-1}(y)}\gamma^{-1}(z), \gamma^{-1}(w)\rangle, \quad \forall x, y, z, w \in M_p. \tag{2.1}$$

Proof. We start with the case that γ is trivial, namely $\gamma = \{p\}$. Extend x and y to a neighborhood of p and denote them by X, Y. We can extend in such a way that $[X, Y] = 0$. For a vector field Z, recall that $\nabla_X Z$ at p can be computed by

$$\lim_{t\to 0} \frac{P^{t,0}_{\varphi_p}(Z(\varphi_p(t))) - Z(p)}{t}.$$

Here $\varphi_q(t)$ denotes the integral curve of X originated at q (also abbreviated as α_q), $P^{t,0}_{\varphi}$ denotes the parallel transport from $\varphi_p(t)$ to $\varphi_p(0) = p$. Similarly, we let $\psi_q(s)$ denote the integral curve of Y originated from q (also denoted as β_q). The assumption $[X, Y] = 0$ ensures that φ_t and ψ_s commute, namely $\psi_{\varphi_p(t)}(s) = \varphi_{\psi_p(s)}(t)$. For any $z \in M_p$, let

$$Z(t, s) \doteqdot P^{0,s}_{\beta_{\alpha_p(t)}} \cdot P^{0,t}_{\alpha_p}(z).$$

From the definition it is easy to see that $\nabla_{\frac{\partial}{\partial s}} Z|_{(t,s)} = 0$ and $\nabla_{\frac{\partial}{\partial t}} Z|_{(t,0)} = 0$. Also define the mapping $\Psi(t, s) : M_p \to M_p$ as

$$\Psi(t, s) = P^{s,0}_{\beta_p} \cdot P^{t,0}_{\alpha_{\beta_p(s)}} \cdot P^{0,s}_{\beta_{\alpha_p(t)}} \cdot P^{0,t}_{\alpha_p}.$$

For sufficiently small t and s, we have that $\Psi(t, s) \in H^0_p$. We shall show that $\mathrm{R}_{x,y}z$ can be expressed in terms of derivatives of $\Psi(\eta)(z)$ for $\Psi(\eta) = \Psi(\sqrt{\eta}, \sqrt{\eta})$. We claim that

$$\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}} Z|_{(0,0)} = \lim_{t,s\to 0} \frac{\Psi(t, s)(z) - z}{ts} = \left.\frac{\partial^2}{\partial t\partial s}\right|_{(0,0)} \Psi(t, s)(z). \tag{2.2}$$

Noting that the left-hand side is $\mathrm{R}_{x,y}z \doteqdot -\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z\big|_p$, letting $t = \sqrt{\eta}, s = \sqrt{\eta}$, by claim (2.2), we have that $\mathrm{R}_{x,y}z = \lim_{\eta\to 0} \frac{\Psi(\eta)(z)-z}{\eta}$. For the proof of claim (2.2), note that

$$\nabla_{\frac{\partial}{\partial s}}\nabla_{\frac{\partial}{\partial t}} Z|_{(0,0)} = \lim_{s\to 0} \frac{P^{s,0}_{\beta_p}((\nabla_{\frac{\partial}{\partial t}} Z)(0, s)) - (\nabla_{\frac{\partial}{\partial t}} Z)(0, 0)}{s},$$

$$(\nabla_{\frac{\partial}{\partial t}} Z)(0, s) = \lim_{t\to 0} \frac{P^{t,0}_{\alpha_{\beta_p(s)}}(Z(t, s)) - Z(0, s)}{t}, \quad \text{and} \quad P^{s,0}_{\beta_p}(Z(0, s)) = z.$$

Claim (2.2) follows by putting the above three identities together.

The general case follows from the observation that $\gamma H_q^0 \gamma^{-1} = H_p^0$, where γ is a path joining q to p. Hence $\gamma(\mathrm{R}^q)_{x,y} = \gamma \cdot \mathrm{R}_{\gamma^{-1}(x),\gamma^{-1}(y)} \cdot \gamma^{-1}$ lies in the Lie algebra of H_p^0 since $\mathrm{R}_{\gamma^{-1}(x),\gamma^{-1}(y)}$ is in the Lie algebra of H_q^0. More precisely, if $\Psi(\eta)$ is the element in H_q^0 corresponding to $\gamma^{-1}(x), \gamma^{-1}(y)$, recall from (2.1) that $\gamma(\mathrm{R}^q)_{x,y}z = \gamma(\mathrm{R}^q_{\gamma^{-1}(x),\gamma^{-1}(y)}\gamma^{-1}(z))$, thus

$$\gamma(\mathrm{R}^q)_{x,y}z = \gamma\left(\lim_{\eta\to 0} \frac{\Psi(\eta)(\gamma^{-1}(z)) - \gamma^{-1}(z)}{\eta}\right) = \lim_{\eta\to 0} \frac{\gamma \cdot \Psi(\eta) \cdot \gamma^{-1}(z) - z}{\eta}.$$

The result follows since $\gamma \cdot \Psi \cdot \gamma^{-1} \in H_p^0$. □

Lemma 2.2 is the easy part of Ambrose–Singer's theorem [1], perhaps known to Cartan. The second part of Ambrose–Singer's theorem asserts that $\gamma(\mathrm{R}^q)_{x,y}$ is all that is needed to generate $\mathfrak{h}$ if γ runs through all possible paths. This part, however, is not needed/used for our discussion. Note that the argument above proves for a Riemannian connection on any Riemannian vector bundle E that $\gamma(\mathrm{R}^q)_{x,y} \in \mathfrak{h}$ with $\mathfrak{h}$ being the Lie algebra of $H_p^0(E)$. Ambrose–Singer's theorem also asserts that $\{\gamma(\mathrm{R}^q)_{x,y}\}$, $\forall x, y \in M_p$, with γ exhausting all possible paths, generates the Lie algebra of $H_p^0(E)$.

Given $S = \{V, \mathrm{R}, G\}$, $g(\mathrm{R})$ can be defined algebraically. First recall the action of $\mathsf{SO}(n)$ (hence G) on $\wedge^2(V)$. Let $x \otimes y(z) \doteqdot \langle y, z\rangle x$. Then $g(x\otimes y) \doteqdot gx \otimes gy$. Direct calculation then shows that $g(x \wedge y) = g \cdot x \wedge y \cdot g^{-1}$ ($g \in \mathsf{SO}(n)$ is used). (Note that $x \wedge y$ can be identified with an element in $\mathfrak{so}(n)$, and we have identified $\wedge^2 V$, $\wedge_2 V$, and $\mathrm{Hom}(V, V)$ using the metric on V.) Hence $g(x\wedge y) = Ad_g(x \wedge y)$. Since $g^{tr} = g^{-1}$, note that Ad_g acts on $\wedge^2(V)$ isometrically with respect to the metric on $\mathfrak{gl}(V)$:

$$\langle A, B\rangle \doteqdot \frac{1}{2}\sum_i \langle A(e_i), B(e_i)\rangle = \frac{1}{2}\,\mathrm{trace}(B^{tr}A)$$

since $(Ad_g)^{tr} = Ad_{g^{\mathrm{tr}}}$, which is $Ad_{g^{-1}}$ for $g \in \mathsf{O}(n)$. It is easy to see for $A \in \mathfrak{so}(n)$ that

$$\begin{aligned} 2\langle A, x\wedge y\rangle &= \sum \langle A(e_i), (x\wedge y)(e_i)\rangle = \sum \big(\langle A(e_i), \langle y, e_i\rangle x\rangle - \langle A(e_i), \langle x, e_i\rangle y\rangle\big) \\ &= -\sum \langle e_i, A(x)\rangle \cdot \langle y, e_i\rangle + \sum \langle e_i, A(y)\rangle \cdot \langle x, e_i\rangle \\ &= -\langle y, A(x)\rangle + \langle x, A(y)\rangle = -2\langle y, A(x)\rangle = 2\langle x, A(y)\rangle. \end{aligned} \tag{2.3}$$

Here $\{e_i\}$ is an orthonormal basis of V. With this convention $\mathrm{R}(x\wedge y)$ is identified with $-\mathrm{R}_{x,y}$. Recall that R can be viewed as a symmetric tensor of $\wedge^2(V)$ with

$$\langle \mathrm{R}(x\wedge y), z\wedge w\rangle = \langle \mathrm{R}_{x,y}z, w\rangle = \langle -\mathrm{R}_{x,y}w, z\rangle = \mathrm{R}(x, y, z, w).$$

To be compatible with (2.1) when $V = M_p$, $G = H_p^0$, we define

$$g(\mathrm{R})_{x,y} \doteqdot g \cdot \mathrm{R}_{g^{-1}(x), g^{-1}(y)} \cdot g^{-1}.$$

Lemma 2.2 asserts that $\{M_p, \mathrm{R}, H_p^0\}$ is a holonomy system. If $S = \{V, \mathrm{R}, G\}$ is a holonomy system, $g(\mathrm{R})_{x,y} \in \mathfrak{g}$ for all $g \in G$ since $g \cdot A \cdot g^{-1} = Ad_g(A) \in \mathfrak{g}$ if $A = \mathrm{R}_{g^{-1}(x),g^{-1}(y)} \in \mathfrak{g}$. In the mean time,

$$\begin{aligned}\langle g(\mathrm{R})(x \wedge y), z \wedge w\rangle = \langle g(\mathrm{R})_{x,y} z, w\rangle &= \langle \mathrm{R}_{g^{-1}(x),g^{-1}(y)} g^{-1}(z), g^{-1}(w)\rangle \\ &= \langle \mathrm{R}(Ad_{g^{-1}}(x \wedge y)), Ad_{g^{-1}}(z \wedge w)\rangle \\ &= \langle Ad_g \cdot \mathrm{R} \cdot Ad_{g^{-1}}(x \wedge y), z \wedge w\rangle.\end{aligned}$$

Namely, $g(\mathrm{R}) = Ad_g \cdot \mathrm{R} \cdot Ad_{g^{-1}}$ in $S_B^2(\wedge^2 V)$. Viewing it as a $(4, 0)$ tensor, one can check directly that $g(\mathrm{R})(x, y, z, w) = \mathrm{R}(g^{-1}x, g^{-1}y, g^{-1}z, g^{-1}w)$. It is easy to see that $g(\mathrm{R})$ satisfies the first Bianchi identity. Similarly, $\forall \mathrm{R}_1, \mathrm{R}_2 \in S_B^2(\wedge^2 V)$, we define the inner product by $\langle \mathrm{R}_1, \mathrm{R}_2\rangle \doteqdot \sum_\alpha \langle \mathrm{R}_1(b_\alpha), \mathrm{R}_2(b_\alpha)\rangle$ with $\{b_\alpha\}$ being an orthonormal basis of $\wedge^2(V)$. It is easy to see that $\langle g(\mathrm{R}_1), g(\mathrm{R}_2)\rangle = \langle \mathrm{R}_1, \mathrm{R}_2\rangle$, namely the action is an isometry. The Ricci curvature of R is defined as $\mathrm{Ric}_\mathrm{R}(x, y) \doteqdot \sum \langle \mathrm{R}(e_i, x, e_i, y)$, where $\{e_i\}$ is an orthonormal basis of V.

For $A \in \mathfrak{g}$ (or $\mathfrak{so}(n)$), let $g_t = \exp(tA)$. Define $A(\mathrm{R}) \doteqdot \lim_{t\to 0} \frac{g_t(\mathrm{R})-\mathrm{R}}{t}$. Since $g_t(\mathrm{R}) \in S_B^2(\wedge^2 V), \forall g_t \in \mathsf{SO}(n)$, $A(\mathrm{R}) \in S_B^2(\wedge^2 V), \forall A \in \mathfrak{so}(n)$. Direct calculation shows that $A(x \wedge y) \doteqdot \lim_{t\to 0} \frac{g_t(x\wedge y)-x\wedge y}{t} = [A, x \wedge y]$. Alternatively,

$$A(x \wedge y) \doteqdot \lim_{t\to 0} \frac{g_t(x \wedge y) - x \wedge y}{t} = A(x) \wedge y + x \wedge A(y) \doteqdot 2(A \wedge \mathrm{id})^1(x \wedge y).$$

Denote $[A, x \wedge y]$ also as $\mathrm{ad}_A(x \wedge y)$. Then $A(\mathrm{R}) = \mathrm{ad}_A \cdot \mathrm{R} - \mathrm{R} \cdot \mathrm{ad}_A$ and

$$A(\mathrm{R}) = 2(A \wedge \mathrm{id} \cdot \mathrm{R} - \mathrm{R} \cdot A \wedge \mathrm{id}) = -2\big((A \wedge \mathrm{id})^{tr} \cdot \mathrm{R} + \mathrm{R} \cdot A \wedge \mathrm{id}\big),$$

noting that $(A \wedge B)^{tr} = A^{tr} \wedge B^{tr}$. Recalling $\mathrm{R}(x \wedge y) = -\mathrm{R}_{x,y}$, we also have that

$$\begin{aligned}A(\mathrm{R})_{x,y} &= -A(\mathrm{R})(x \wedge y) = (A \cdot \mathrm{R}_{x,y} - \mathrm{R}_{x,y} \cdot A - \mathrm{R}_{Ax,y} - \mathrm{R}_{x,Ay}), \\ A(\mathrm{R})(x, y, z, w) &= -\big(\mathrm{R}(Ax, y, z, w) + \mathrm{R}(x, Ay, z, w) \\ &\quad + \mathrm{R}(x, y, Az, w) + \mathrm{R}(x, y, z, Aw)\big), \quad \text{viewing as } (4, 0) \text{ tensors.}\end{aligned}$$

From this it is easy to confirm again that $A(\mathrm{R})$ satisfies the first Bianchi identity. For a holonomy system, it is easy to see that $A(\mathrm{R})_{x,y} \in \mathfrak{g}, \forall g \in G, A \in \mathfrak{g}$. The conclusion of Theorem 1.2, namely S being symmetric (i. e., $g(\mathrm{R}) = \mathrm{R}, \forall g \in G$), is equivalent to $A(\mathrm{R}) = 0$ for any $A \in \mathfrak{g}$.

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1 Note here that $A \wedge \mathrm{id}$ only satisfies the first Bianchi identity if A is symmetric. Hence $A \wedge \mathrm{id} \notin S_B^2$ if $A \in \mathfrak{g}$ (which is skew-symmetric).

3 A derivation of Theorem 1.1

Our derivation of Theorem 1.1 from Theorem 1.2 here follows the argument of [7]. The main difference is that no result from the theory of symmetric spaces, or the full Ambrose–Singer's theorem, is needed. As in [7], it starts with a result of Kostant. First, let P be the projection from $\wedge^2(V)$ onto $\mathfrak{g}$, and $T : \mathfrak{g} \to \mathfrak{g}$ be the symmetric isomorphism corresponding to the negative definite bilinear form on $\mathfrak{g}$,

$$B(A, A') \doteqdot K(A, A') - 2\langle A, A' \rangle, \tag{3.1}$$

with K being the Killing form of $\mathfrak{g}$ (defined as $K(A, A') = \mathrm{trace}(\mathrm{ad}_A \cdot \mathrm{ad}_{A'})$). Namely, T is defined by $B(A, A') = \langle T(A), A' \rangle$.

Theorem 3.1 (Kostant). *Assume that $S = \{V, \mathrm{R}, G\}$ is an irreducible symmetric holonomy system. Then there exists a constant λ such that $\mathrm{R}_{z,w} = -\lambda(T^{-1} \cdot P)(z \wedge w)$. Moreover, $\mathrm{R}_{x,y} = 0$ if and only if $\mathrm{R}(x, y, x, y) = 0$, and if $\mathrm{R} \neq 0$ (hence $\lambda \neq 0$), $\mathrm{Ric}_{\mathrm{R}}(x, x) = \sum \frac{1}{\lambda} B([x, e_i], [x, e_i])$.*

Proof. Assume that $\mathrm{R} \neq 0$ (otherwise the conclusion is obvious). A construction of a Lie algebra J (due to Cartan) is the key: Let $J = \mathfrak{g} \oplus V$ (orthogonal sum with the inner product of V and $\langle A, B \rangle$ on $\mathfrak{g}$ as elements in $\mathfrak{so}(n)$) and define a Lie algebra structure of J by letting

$$[A, A'] \doteqdot [A, A'], \quad [x, y] \doteqdot \mathrm{R}_{x,y}, \quad [A, x] \doteqdot A(x), \forall A, A' \in \mathfrak{g},\ x, y \in V.$$

Since $A(\mathrm{R}) = 0, \forall A \in \mathfrak{g}$, it is easy to check that the bracket so defined satisfies the Jacobi identity, namely J is a Lie algebra.[2] That R satisfies the first Bianchi identity is also needed in checking the Jacobi identity for J.

Let B' be the Killing form of J. It is a basic result of Lie algebra that B' is ad_J-invariant (see, for example, page 180 of [6]). The proof now follows from the following claims: *(i) $B'|_{\mathfrak{g}}$ is given by B defined by* (3.1), *hence is negative definite*; *(ii) $B'(A, x) = 0$*; (the proofs of (i) and (ii) are computational and shall be given at the very end), and *(iii) $B'|_V$ is $\mathrm{ad}_{\mathfrak{g}}$-invariant*, *hence G-invariant*, which, together with the irreducibility of G-action on V, implies that $B'(x, y) = \lambda\langle x, y \rangle$ *for some λ. Moreover, $\lambda \neq 0$*. Otherwise, $B'([x, y], [x, y]) = B'(x, [y, [x, y]]) = 0$ since $[y, [x, y]] \in V$. On the other hand, by (i), which implies $B'|_{\mathfrak{g}}$ is negative definite, we have that $[x, y] = \mathrm{R}_{x,y} = 0, \forall x, y \in V$. This contradicts $\mathrm{R} \neq 0$.

Now observe that (a) $\langle [[x, y], z], w \rangle = -\langle [x, y], (z \wedge w) \rangle$ (using (2.3), namely $\langle A, z \wedge w \rangle = -\langle A(z), w \rangle, \forall A \in \mathfrak{so}(\mathrm{n})$) and (b) $\langle [[x, y], z], w \rangle = \frac{1}{\lambda} B'([[x, y], z], w) = \frac{1}{\lambda} B'([x, y], [z, w])$ which equals to $\frac{1}{\lambda}\langle [x, y], T([z, w]) \rangle$. Theorem 3.1 now follows from (a)

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2 A result of Borel, whose proof is also a by-product of the proof of Theorem 3.1, asserts that J is semisimple.

and (b) above, together with *claim (c):* $\mathrm{span}\{\mathrm{R}_{x,y}\} \doteqdot \mathfrak{g}^{\mathrm{R}} = \mathfrak{g}$, since (a)–(c) together imply that

$$-\lambda\big\langle \mathrm{R}_{x,y}, P(z\wedge w)\big\rangle = \big\langle \mathrm{R}_{x,y}, T(\mathrm{R}_{z,w})\big\rangle, \quad \forall \mathrm{R}_{x,y} \implies T(\mathrm{R}_{z,w}) = -\lambda P(z\wedge w).$$

For claim (c), note that $\mathfrak{g}^{\mathrm{R}}$ is an ideal (cf. Lemma 4.1), let $\mathfrak{a}$ be its orthogonal complement (w. r. t. B') in $\mathfrak{g}$. It is easy to see that $\forall A \in \mathfrak{a}, y \in V, B'([A,y],[A,y]) = B'(A,[y,[A,y]]) = 0$ since $[y,[A,y]] = \mathrm{R}_{y,A(y)} \in \mathfrak{g}^{\mathrm{R}}$. Hence due to $B'|_V = \lambda\langle\cdot,\cdot\rangle$ this implies that $[A,y] = A(y) = 0, \forall y \in V$. Thus $A = 0, \forall A \in \mathfrak{a}$, namely $\mathfrak{a} = 0$.

Finally, we prove (i) and (ii). By the definition $B'(A,B) = \mathrm{trace}(\mathrm{ad}_A \cdot \mathrm{ad}_B) = \sum_{i=1}^n \langle \mathrm{ad}_A \cdot \mathrm{ad}_B(e_i), e_i\rangle + \sum_\alpha \mathrm{ad}_A \cdot \mathrm{ad}_B(A_\alpha), A_\alpha\rangle$ where $\{e_i\}$ ($\{A_\alpha\}$) is an orthonormal frame of V ($\mathfrak{g}$ respectively). The second summand is $K(A,B)$. By the definition of the Lie bracket, the first summand is $-\langle B(e_i), A(e_i)\rangle = -2\langle A,B\rangle$. This proves (i). The proof of (ii) is by a similar straightforward computation.

Note that (b) above implies that $\langle \mathrm{R}_{x,y}z, y\rangle = \frac{1}{\lambda}B(\mathrm{R}_{x,y}, \mathrm{R}_{x,y})$. Hence the sectional curvature $K(x,y) = 0 \iff \mathrm{R}_{x,y} = 0$. The formula for the Ricci curvature is via direct computations. In fact, $\mathrm{Ric}(x,x) = -\frac{1}{2}B'(x,x)$ (cf. page 182 of [6]). □

The following argument deriving Berger's theorem, namely M has $\nabla \mathrm{R} = 0$, using Theorem 1.2 is the same as in [7].

Proof. We assume $n \geq 3$ since $n = 2$ case is obvious. The goal is to show that M is a locally symmetric space. Assume that H_p^0 acts on $\mathbb{S}^{n-1}$ nontransitively. By Lemma 2.2, $S = \{M_p, \mathrm{R}, H_p^0\}$ is a holonomy system. By the assumption of Theorem 1.1, S is irreducible. It is easy to see that $\mathrm{Ric}_{g(\mathrm{R})}(x,y) = \mathrm{Ric}_{\mathrm{R}}(g^{-1}(x), g^{-1}(y))$. Also use $\mathrm{Ric}_{\mathrm{R}}$ to denote the corresponding symmetric automorphism of V. Namely, $\langle \mathrm{Ric}_{\mathrm{R}}(x), y\rangle \doteqdot \mathrm{Ric}_{\mathrm{R}}(x,y)$. Then $\mathrm{Ric}_{g(\mathrm{R})} = g\cdot \mathrm{Ric}_{\mathrm{R}} \cdot g^{-1}$. By the irreducibility of the system S, I. Schur's lemma implies that $\mathrm{Ric}_{\mathrm{R}} = f(p)\,\mathrm{id}$ (i. e., R is Einstein at M_p). Now for any q, pick a path γ from q to p. Consider the system $S_\gamma = \{M_p, \gamma(\mathrm{R}^q), H_p^0\}$. By Lemma 2.2 again, S_γ is an irreducible Riemannian holonomy system. By Theorems 1.2 and 3.1, we conclude that $\gamma(\mathrm{R}^q) = c\mathrm{R}^p$ for some constant c. This implies that $\mathrm{Ric}_{\gamma(\mathrm{R}^q)} = c\mathrm{Ric}_{\mathrm{R}^p} = cf(p)\,\mathrm{id}$. On the other hand, by F. Schur's lemma $f(x) = \beta$ for a constant β. Namely, $\mathrm{Ric}_{\mathrm{R}^{p'}} = \beta\,\mathrm{id}$ for all $p' \in M$. Thus $\mathrm{Ric}_{\gamma(\mathrm{R}^q)} = \gamma \mathrm{Ric}_{\mathrm{R}^q}\gamma^{-1} = \beta\,\mathrm{id}$, since $\mathrm{Ric}_{\mathrm{R}^q} = \beta\,\mathrm{id}$. This implies $c = 1$, hence $\nabla\mathrm{R} = 0$. The claim rank ≥ 2 follows from Proposition 4.2 below. □

4 Simons' constructions of flats and totally geodesic subspaces

Let S be an irreducible Riemannian holonomy system. The induction assumption is that Theorem 1.2 holds for S with degree smaller than n and $\dim(G(\mathrm{R})) \leq k$. The goal is to prove it for $\dim(V) = n$ and $\dim(G(\mathrm{R})) \leq k+1$. The constructions in this section are almost the same as those of [7]. The only difference is the alternate argument for Proposition 4.2 which was shown to us by Nolan Wallach. The original proof in [7] was due to I. Singer. We start with a lemma on $\mathfrak{g}^{\mathrm{R}}$ of [7].

Lemma 4.1. *(i) $\mathfrak{g}^{\mathrm{R}}$ is an ideal in $\mathfrak{g}$. (ii) If $\mathfrak{g} = \mathfrak{g}^{\mathrm{R}} \oplus \mathfrak{g}_{\mathrm{R}}$ is an orthogonal decomposition then $A(Q) = 0$ for any $A \in \mathfrak{g}_{\mathrm{R}}$ and $Q \in G(\mathrm{R})$.*

Proof. By the definition, $\forall A \in \mathfrak{g}$, $[A, \sum_{i,\alpha} g_i(\mathrm{R})(b_\alpha)] = \sum \mathrm{ad}_A \cdot g_i(\mathrm{R})(b_\alpha)$ with $g_i \in G$, $b_\alpha \in \wedge^2(V)$. But $A(g_i(\mathrm{R})) = \mathrm{ad}_A \cdot g_i(\mathrm{R}) - g_i(\mathrm{R}) \cdot \mathrm{ad}_A$ with $A(g_i(\mathrm{R}))(b_\alpha) \in \mathfrak{g}^{\mathrm{R}}$ since $A(G(\mathrm{R})) \subset G(\mathrm{R})$. Part (i) follows from the facts that $g_i(\mathrm{R})(\mathrm{ad}_A(b_\alpha)) \in \mathfrak{g}^{\mathrm{R}}$ and $A(\mathrm{R})(b_\alpha) \in \mathfrak{g}^{\mathrm{R}}$.

For (ii), observe that for any x, y, z, w, $A \in \mathfrak{g}_{\mathrm{R}}$, $Q \in G(\mathrm{R})$, $\langle A(Q)(x \wedge y), z \wedge w\rangle = \langle \mathrm{ad}_A \cdot Q(x \wedge y), z \wedge w\rangle - \langle Q \cdot \mathrm{ad}_A(x \wedge y), z \wedge w\rangle$. The first term vanishes since $[\mathfrak{g}_{\mathrm{R}}, \mathfrak{g}^{\mathrm{R}}] = 0$, in particular $[A, Q_{x,y}] = 0$. The second term is

$$\big\langle Q \cdot \mathrm{ad}_A(x \wedge y), z \wedge w \big\rangle = \big\langle Q(z \wedge w), \mathrm{ad}_A(x \wedge y)\big\rangle = -\big\langle \mathrm{ad}_A \cdot Q(z \wedge w), x \wedge y \big\rangle = 0,$$

by reducing to the first. Putting them together, the lemma is proved. □

A subspace $W \subset V$ (with $\dim(W) \geq 2$) is called a *flat* if $Q(x \wedge y) = 0$ for any $x, y \in W$, for any $Q \in G(\mathrm{R})$, or equivalently, $Ad_{g^{-1}}(\wedge^2 W) \in \ker(\mathrm{R})$ for any $g \in G$. Clearly, W being a flat implies that $g(W)$ is a flat. For a flat W, we have that $\forall x, y \in W$, $\forall z, w \in V$, $\langle Q_{x,y} z, w\rangle = \langle Q_{z,w} x, y\rangle = 0$. Hence W is a flat if and only if $\mathfrak{g}^{\mathrm{R}}(W) \subset W^{\perp}$ and W is maximal if and only if W is a maximal subspace such that $\mathfrak{g}^{\mathrm{R}}(W) \subset W^{\perp}$.

The main ingredients of Simons' proof are the construction of flats out of the nontransitivity, and of total geodesic subspaces out of the maximal flats. A subspace $E \subset V$ is called *totally geodesic* if for any $Q \in G(\mathrm{R})$, any $x, y, z \in E$, $Q_{x,y} z \in E$. Note that if E is totally geodesic, one then can view R as a curvature operator on the space E, which provides a possible reduction on $\dim(V)$.

Proposition 4.2. *If G^{R} is nontransitive on $\mathbb{S}^{n-1}$ then there exists a flat W ($\dim(W) \geq 2$). In fact, $\forall u \in V$, there exists a flat W with $u \in W$.*

Proof. Assume $\mathrm{R} \neq 0$, otherwise the claim is true. The nontransitivity implies that there exist $u, v \in \mathbb{S}^{n-1}$ such that $u \neq g(v), \forall g \in G^{\mathrm{R}}$. Now consider the function $f(g) = \langle u, g(v)\rangle$. Note $f(g) \in [-1, 1)$. Since G^{R} is compact, $f(g)$ attains its maximum somewhere, say at $g_0 \in G^{\mathrm{R}}$. We then have that for any $A \in \mathfrak{g}$, $\langle u, A g_0(v)\rangle = 0$. In particular, we have that $\forall Q \in G(\mathrm{R})$,

$$\big\langle u, Q_{x,y} g_0(v)\big\rangle = \langle Q_{g_0(v),u} x, y\rangle = 0, \quad \forall x, y \in V.$$

This implies that $Q_{u, g_0(v)} = 0$. Since $g_0(v) \neq u$ nor $g_0(v) = -u$, we conclude that $W = \mathrm{span}\{u, g_0(v)\}$ is a flat. The part $g_0(v) \neq -u$ is due to that $f(g_0)$ attains its maximum which cannot be -1, unless $v = -u$ and $g(-u) = -u, \forall g \in G^{\mathrm{R}}$, which then implies that the eigenspace $E(1)$ (with eigenvalue 1) of G^{R} is nonempty and invariant under the action of G (since G^{R} is a normal subgroup of G by, say Theorem 2.13.4 of Varadarajan's book [8]), thus either $G^{\mathrm{R}} = \{\mathrm{id}\}$ or $E(1) \neq V$, contradicting the assumption that V is irreducible. The argument effectively shows that for any $u \in \mathbb{S}^{n-1}$, there exists a flat W such that $u \in W$ since the nontransitivity assumption implies that $\forall u$, there exists v such that $u \neq g(v)$ for any $g \in G^{\mathrm{R}}$. □

Note that Proposition 4.2 implies Theorem 1.2 for $n = 2$, for it implies that V is a flat, hence $\mathrm{R} = 0$. Clearly, any flat is totally geodesic.[3]

The existence of flats leads to some totally geodesic subspaces via the *Jacobi curvatures* (named after the curvature term in the equation $J'' + \mathrm{R}_{\gamma',J}\gamma' = 0$ defining a Jacobi field J along a geodesic γ). Given any $Q \in G(\mathrm{R})$ and $x, y \in W$, consider the linear transformation $T_Q^{x,y} : V \to V$ (the Jacobi curvature) as $\langle T_Q^{x,y}(z), w\rangle \doteqdot Q(x,z,y,w)$. By the first Bianchi identity and since $Q_{x,y} = 0$,

$$\langle T_Q^{x,y}(z), w\rangle = Q(y,z,x,w) = \langle T_Q^{y,x}(z), w\rangle = Q(x,w,y,z) = \langle T^{x,y}(w), z\rangle. \tag{4.1}$$

In particular, $T_Q^{x,y} : V \to V$ is symmetric. More importantly, $\forall x, y, s, t \in W$,

$$T_Q^{x,y} \cdot T_Q^{s,t} = T_Q^{s,t} \cdot T_Q^{x,y}, \quad \text{more generally } T_Q^{x,y} \cdot T_P^{s,t} = T_Q^{s,t} \cdot T_P^{x,y}, \forall P, Q \in G(\mathrm{R}). \tag{4.2}$$

Since $s, t, x, y \in W$, we have that $Q_{Ax,y} = -Q_{x,Ay}$ for any $A \in \mathfrak{g}$. Hence we have

$$\begin{aligned} T_Q^{x,y} \cdot T_P^{s,t}(z) &= Q_{x,P_{s,z}t}y = -Q_{P_{s,z}x,t}y = -Q_{P_{x,z}s,t}y \\ &= Q_{t,P_{x,z}s}y = Q_{y,P_{x,z}s}t = -Q_{P_{x,z}y,s}t = T_Q^{s,t} \cdot T_P^{x,y}(z). \end{aligned}$$

When $Q = P$, we have (4.2), namely $\{T_Q^{x,y}\}_{x,y \in W}$ forms a family of commutative symmetric operators on V, which hence can be diagonalized simultaneously (cf. [9], Section 5 of Chapter 1 for an illuminating proof). Thus there exist unit $X_1^Q, \dots, X_n^Q$ (when there is no confusion we omit the superscript) such that they are eigenvectors of $T_Q^{x,y}$ (for any $x, y \in W$). Since clearly $T_Q^{x,y}(W) = 0$, we may assume that $\{X_i\}_{1 \le i \le \mu}$ forms an orthonormal basis of W ($\mu = \dim(W)$). We denote the corresponding eigenvalues by $\Lambda_Q^k(x,y)$ with $1 \le k \le n$. Namely, $T_Q^{x,y}X_k = \Lambda_Q^k(x,y)X_k$. Clearly, $\Lambda_Q^k(x,y)$ is a bilinear form of W. Equation (4.1) also implies that $\Lambda_Q^k(x,y)$ is symmetric, which can also be viewed as a symmetric linear map of W via $\langle \Lambda_Q^k(x), y\rangle = \Lambda_Q^k(x,y)$. For $1 \le k \le \mu$, $\Lambda_Q^k = 0$. For $\mu + 1 \le k \le n$, we shall show that either $\Lambda_Q^k \equiv 0$ (write as $\Lambda_Q^k = 0$) or it is of rank one.

First we observe that if $\Lambda_Q^k(x,y) = 0, \forall x, y \in W$ and for all $1 \le k \le n$, we then have that $Q_{x,z}y = 0, \forall z \in V, x, y \in W$. In particular, $Q(x,z,x,z) = 0, \forall x \in W, z \in V$. On the other hand, Proposition 4.2 asserts that for any $x \in V$ there exists a flat W with $x \in W$. Hence if $\Lambda_Q^k = 0$ for all $1 \le k \le n$ and for all flats, $Q(x,z,x,z) = 0$, for any $x, z \in V$, hence $Q = 0$, and $\mathrm{R} = 0$. Thus $\mathrm{R} \neq 0$ implies that for any Q, there exists at least one flat W and one k with $\Lambda_Q^k \neq 0$ on W.

3 The flats and total geodesic subspaces are Lie algebra analogues of the maximum toruses and the centralizers of the torus subgroup of the Lie group of isometries.

Lemma 4.3. *Let W be a flat. Assume for some k $\Lambda_Q^k \neq 0$. (i) The rank of $\Lambda_Q^k : W \to W$ is one; (ii)*

$$P_{s,X_k} = 0, \quad \textit{for any } s \in U_{k,Q}, P \in G(\mathrm{R}), \textit{ with } U_{k,Q} = \ker(\Lambda_Q^k). \tag{4.3}$$

Proof. Pick $x \in W$ with $\Lambda_Q^k(x,x) \neq 0$. Then $T_Q^{x,x}X_k = \Lambda_Q^k(x,x)X_k \neq 0$. Let $U_{k,Q} \doteqdot \{s \in W \mid \Lambda_Q^k(x,s) = 0\}$, which defines a hypersurface in W. Since $P_{x,Ay} = -P_{Ax,y}$, for any $A \in \mathfrak{g}$, $x, y \in W$, we have that

$$\Lambda_Q^k(x,x)P_{s,X_k} = P_{s,Q_{x,X_k}x} = -P_{Q_{x,X_k}s,x} = -\Lambda_Q^k(x,s)P_{X_k,x} = 0.$$

This proves (4.3). In particular, for any $s \in U_{k,Q}, t \in W$,

$$\Lambda_Q^k(s,t)X_k = T_Q^{s,t}X_k = Q_{s,X_k}t = 0, \implies \Lambda_Q^k(s,t) = 0, \quad \forall s \in U_{k,Q}, t \in W.$$

Thus $\Lambda_Q^k|_{U_{k,Q}} \equiv 0$, which implies the rank one assertion. □

The above shows that Jacobi curvatures associated with vectors from a flat are special, and $U_{k,Q} = \ker(\Lambda_Q^k)$ is independent of the choice of $x \in W$. Note that for $X_k \in W^\perp$, $\Lambda_Q^k(x,x) = K_\Sigma^Q$, the sectional curvature of $\Sigma = \mathrm{span}\{x, X_k\}$ for $x \in W, |x| = 1$. For $\Lambda_Q^k \neq 0$, let $\lambda_k^Q \neq 0$ and $x_k \in W$ be the nonzero eigenvalue and an eigenvector of $\Lambda_Q^k : W \to W$ (with $\Lambda_Q^k(x_k) = \lambda_k^Q x_k$). If $\Lambda_Q^k = 0$, let $\lambda_k^Q = 0$ and pick any unit $x_k \in W$. Order k by $|\lambda_k^Q|$,

$$|\lambda_{\mu+1}^Q| \le |\lambda_{\mu+2}^Q| \le \cdots \le |\lambda_{n-1}^Q| \le |\lambda_n^Q|, \quad \text{with } \lambda_j^Q \doteqdot Q(x_j, X_j, x_j, X_j).$$

The first construction of totally geodesic subspaces: For a maximal flat W, $\Lambda_Q^k \neq 0$, let

$$E_{k,Q} \doteqdot \{m \in V \mid P_{s,m} = 0, \text{ for all } P \in G(\mathrm{R}), s \in U_{k,Q}\}.$$

By (4.3), we have that $X_k \in E_{k,Q}$ and $U_{k,Q} \subsetneq W \subsetneq E_{k,Q}$. We show that $E_{k,Q} \subsetneq V$.

Proposition 4.4. *Consider W, Q with $\Lambda_Q^k \neq 0$. Then: (i) $E_{k,Q}$ is totally geodesic; (ii) $E_{k,Q} \neq V$ unless $\mathrm{R} = 0$; (iii) For a maximal W,*

$$E_{k,Q} \cap E_{k',Q'} = W, \quad \textit{if } E_{k,Q} \neq E_{k',Q'}. \tag{4.4}$$

Proof. For any $A \in \mathfrak{g}, P \in G(\mathrm{R})$, since $g(P)_{s,m} = 0$ for any $g \in G, s \in U_{k,Q}, m \in E_{k,Q}$,

$$0 = \frac{d}{dt}\bigg|_{t=0} P_{\exp(-tA)s,\exp(-tA)m} = -P_{As,m} - P_{s,Am} = -(P \cdot \mathrm{ad}_A)_{s,m}. \tag{4.5}$$

(i) Let $u, v, m \in E_{k,Q}$ and $s \in U_{k,Q}$. For any $P, P' \in G(\mathrm{R})$, $P'_{s,P_{u,v}m} = -P'_{P_{u,v}s,m} = P'_{P_{v,s}u+P_{s,u}v,m} = 0$. Hence $P_{u,v}m \in E_{k,Q}$.

(ii) Note that $V' = \{x \mid P_{x,y} = 0, \text{ for all } y \in V, P \in G(\mathrm{R})\}$ is a G-invariant subspace. If $E_{k,Q} = V$ then $U_{k,Q} \subset V' \implies V' = V$. It then implies that V is a flat and $\mathrm{R} = 0$.

(iii) For any $m \in E_{k,Q} \cap E_{k',Q'} \doteqdot I_{k,k'}$, we have that for any $P \in G(\mathrm{R})$, $P_{m,v} = 0$ for $v \in U_{k,Q} \cup U_{k',Q'}$. If $E_{k,Q} \neq E_{k',Q'}$, then $U_{k,Q} \neq U_{k',Q'}$. Hence $P_{m,v} = 0, \forall v \in W$. This implies $m \in W$ (hence $W = I_{k,k'}$) by the maximality of W, otherwise $W' = \mathrm{span}\{W, m\}$ is a flat, contradicting that W is maximal. □

The second construction of totally geodesic subspaces: For a fixed flat W, clearly $W \subset Z_Q \doteqdot \mathrm{span}\{X_k, \Lambda_Q^k = 0\}$. Define $Z(W) \doteqdot \bigcap_{Q \in G(\mathrm{R})} Z_Q$. Since $W \subset Z_Q, \forall Q$, $W \subset Z(W)$. Hence $V = Z(W) \oplus N$, where N is spanned by the eigenvectors X_k^Q with $\Lambda_Q^k(x,x) \neq 0$ for some $Q \in G(\mathrm{R}), x \in W$.

Proposition 4.5. *$Z(W)$ is totally geodesic, namely for any $Q, P \in G(\mathrm{R})$, $P_{w,Q_{x,y}z}w = 0$, $\forall w \in W, x, y, z \in Z(W)$.*

Proof. Note that $x \in Z(W)$ if and only if $Q_{w,x}v = 0, \forall Q \in G(\mathrm{R}), \forall w, v \in W$. Now note that if $x \in Z(W)$, then $\forall P \in G(\mathrm{R}), A \in \mathfrak{g}$, $g_t = \exp(tA)$,

$$P_{w,x}v = 0 \implies g_t(P)_{w,x}v = 0, \quad \forall w, v \in W \implies -P_{w,x} \cdot Av - P_{Aw,x}v - P_{w,Ax}v = 0.$$

Hence $P_{w,Q_{x,y}z}v = -P_{w,z}Q_{x,y}v - P_{Q_{x,y}w,z}v$. If $x \in W$, $Q_{x,y}v = \Lambda_Q^k(x,v)y = 0 = Q_{x,y}w = \Lambda_Q^k(x,w)y$, hence $P_{w,Q_{x,y}z}v = 0$, which proves $Q_{x,y}z \in Z(W)$. For the general case, since $Q_{y,v}x, Q_{x,v}y \in Z(W)$, the first Bianchi identity implies that $Q_{x,y}v = -Q_{y,v}x + Q_{x,v}y \in Z(W)$. Similarly, $Q_{x,y}w \in Z(W)$. Therefore applying the above special case, $P_{w,z}Q_{x,y}v, P_{Q_{x,y}w,z}v \in Z(W)$. The first Bianchi identity again implies $P_{w,Q_{x,y}z}v \in Z(W)$. Now for any $z' \in Z(W)$, we have

$$\langle P_{w,Q_{x,y}z}v, z' \rangle = \langle P_{v,z'}w, Q_{x,y}z \rangle = 0.$$

This implies that $P_{w,Q_{x,y}z}v = 0$, hence $Q_{x,y}z \in Z(W)$. □

Proposition 4.6. *Assume that $Z(W) \neq V$.*[4] *Then $\forall w \in W, x \in Z(W)$, $Q_{w,x}|_{Z(W)} = 0, \forall Q \in G(\mathrm{R})$.*

Proof. This is the place where the induction on $\dim(V)$ is applied by letting $Q' = Q|_{Z(W)}$. Note that $\mathfrak{g}^{Z(W)} \doteqdot \mathrm{span}\{P_{x,y}, x, y \in Z(W), P \in G(\mathrm{R})\}$ is a subalgebra of $\mathfrak{g}^{\mathrm{R}}$ by the proof of Lemma 4.1. Let $G^{Z(W)} \subset G$ be the closed subgroup generated by $\mathfrak{g}^{Z(W)}$. Denote it by G'. Clearly, $G'(Z(W)) \subset Z(W)$. Consider the holonomy system $S' = (Z(W), Q', G')$. By the definition, $\forall h \in G'$, $h(Q')_{w,x}w = 0, \forall w \in W, x \in Z(W)$. If S' is irreducible, by induction we can assert that either it is symmetric or the action is transitive. In either case, we show that $Q'_{w,x} = 0$.

(i) If $S' = (Z(W), Q', G')$ is symmetric, by Theorem 3.1, $\langle Q'_{x,w}x, w \rangle = 0 \iff Q'_{x,w} = 0$. Hence we have $Q'_{x,w} = 0$ from $w \in W, x \in Z(W)$.

4 A priori for a fixed W it is possible that all $\lambda_j^Q = 0$ (namely $\Lambda_Q^k = 0, \forall k \in \{\mu+1, \dots, n\}$). If it is the case for all Q, then $Z(W) = V$. Since this cannot be precluded, we assume $Z(W) \neq V$. However, Theorem 1.2 eventually implies that $\lambda_j^Q \neq 0$ for $\mu + 1 \leq j \leq n$.

(ii) If S' is transitive (on the unit sphere of $Z(W)$), then for a fixed $w \in W$, $|w| = 1$, for any $z \in Z(W), |z| = 1$, there exists $h \in G'$ such that $h(w) = z$. Then from $\langle h^{-1}(Q')_{w,x}w, x\rangle = 0, \forall x \in Z(W)$ we have that $\langle (h^{-1}(Q'))_{h^{-1}(z),x}h^{-1}(z), x\rangle = 0 \Longleftrightarrow \langle Q'_{z,h(x)}z, h(x)\rangle = 0, \forall z, x \in Z(W), |z| = 1$. Since G' acts transitively on $Z(W)$, this implies that the curvature tensor $Q'|_{Z(W)} = 0$.

When S' is reducible we split $Z(W) = \oplus_{i=1}^{\ell} Z_i$ into orthogonal subspaces with $\{Z_i\}$ being irreducible G'-invariant subspaces. Observe that $Q'_{w,x}(Z_i) \subset Z_i$. It is easy to see that $Q'_{w,x} = \sum_{i=1}^{\ell} Q'_{w_i,x_i}$. Moreover, for $w' \in W$, $0 = Q'_{w,x}w' = \sum Q'_{w_i,x_i}w'_i$. Hence $Q'_{w_i,x_i}w'_i = 0$ on Z_i with w_i, w'_i, x_i being the orthogonal projections of w, w', x into Z_i. The arguments (i) and (ii) above show that $Q'_{w_i,x_i}|_{Z_i} = 0$. This proves $Q'_{w,x} = 0$ for the reducible case. □

5 An alternate proof via the induction on dim(G(R))

We start with extending a result of [7] on the totally geodesic subspaces.

Theorem 5.1. *Assume that $E \subsetneq V$ is totally geodesic w. r. t. $S = \{V, \mathrm{R}, G\}$. Define $\mathfrak{J} \subset \mathfrak{g}$, a subspace of $\mathfrak{g}$, and $K \subset G(\mathrm{R})$, a subspace of $G(\mathrm{R})$ as*

$$\mathfrak{J} \doteqdot \{A \in \mathfrak{g} \mid A(Q)|_E = 0, \forall Q \in G(\mathrm{R})\};\ K \doteqdot \{Q \in G(\mathrm{R}) \mid Q_{x,y}z = 0, \forall x, y, z \in E\}.$$

Then $A \in \mathfrak{J}$ if and only if $A(G(\mathrm{R})) \subset K$ and the following statements hold:

(i) For all $A \in \mathfrak{g}$, $A(K) \subset K$, hence $G(K) \subset K$; (ii) $\mathfrak{J}$ is an ideal; (iii) $\mathfrak{g}_{\mathrm{R}} \subset \mathfrak{J}$;

(iv) $A(E) \subset E^{\perp}$ implies that $A \in \mathfrak{J}$; (v) $Q_{x,y} \in \mathfrak{J}$, if $x \in E$ and $y \in E^{\perp}$.

Proof. Since E is totally geodesic, $\forall x, y, z \in E$, $A(Q)_{x,y}z \in E$. Hence $\langle A(Q)_{x,y}z, w\rangle = 0$, $\forall w \in E \Longleftrightarrow A(Q)|_E = 0 \Longleftrightarrow A(Q)_{x,y}z = 0, \forall x, y, z \in E$. This shows that $A \in \mathfrak{J}$ if and only if $A(G(\mathrm{R})) \subset K$.

Let $Q \in K$, we claim that $A(Q)_{x,y}z = 0$ for any $x, y, z \in E$. This implies (i). The claim follows from noting $\mathrm{ad}_A(z \wedge w) = A(z) \wedge w + z \wedge A(w)$ and $\forall x, y, z, w \in E$,

$$\begin{aligned} -\langle A(Q)_{x,y}z, w\rangle &= \langle A(Q)(x \wedge y), z \wedge w\rangle \\ &= \langle \mathrm{ad}_A Q(x \wedge y), z \wedge w\rangle - \langle Q\mathrm{ad}_A(x \wedge y), z \wedge w\rangle \\ &= -\langle Q(x \wedge y), \mathrm{ad}_A(z \wedge w)\rangle - \langle \mathrm{ad}_A(x \wedge y), Q(z \wedge w)\rangle = 0. \end{aligned} \tag{5.1}$$

For (ii), observe that for $A \in \mathfrak{J}, B \in \mathfrak{g}$, $A(B(Q)) \in K$ by the equivalent definition of $\mathfrak{J}$, and $A(Q) \in K$ implies that $B(A(Q)) \in K$ by (i). Hence $[A, B](Q) \in K$. Claim (iii) follows from Lemma 4.1. Equation (5.1) also proves (iv) since if $A(E) \subset E^{\perp}$, $\forall Q \in G(\mathrm{R})$, $\langle Q(x \wedge y), \mathrm{ad}_A(z \wedge w)\rangle = \langle Q_{x,y}w, A(z)\rangle - \langle Q_{x,y}z, A(w)\rangle = 0$. Similarly, $\langle \mathrm{ad}_A(x \wedge y), Q(z \wedge w)\rangle = 0$. Hence $\langle A(Q)_{x,y}z, w\rangle = 0$, $\forall x, y, z, w \in E$. Claim (v) follows from (iv) and that $Q_{x,y}(E) \subset E^{\perp}$, since $\forall x \in E, y \in E^{\perp}$ $\langle Q_{x,y}z, w\rangle = \langle Q_{z,w}x, y\rangle = 0, \forall z, w \in E$. □

Corollary 5.2. *For $A = Q_{x,y}$ with $x \in E, y \in E^{\perp}$, $Ad_g(A) \in \mathfrak{J}$, and $G(A(\mathrm{R})) \subset K$. In particular, it holds for $E = E_{l,Q}$ with $\Lambda_Q^l \neq 0$.*

Proof. Integrating part (ii) of the above theorem $Ad_g(A) \in \mathfrak{J}$ follows from $A \in \mathfrak{J}$. By (v) of the above theorem, we have that $A \in \mathfrak{J}$, which implies that $A(\mathrm{R}) \in K$. By part (i) of Theorem 5.1, $G(A(\mathrm{R})) \in K$. □

The alternate induction on $\dim(G(\mathrm{R}))$ below seems a more efficient way of applying the theorem and its corollary. The following identity (5.2) (Lemma 10 of [7]) is the key step.

Proposition 5.3. [5] *Let W be a fixed maximal flat with $Z(W) \neq V$. Then (i) there are nonzero Λ^k_Q and $\Lambda^{k'}_{Q'}$ such that $E_{k,Q} \neq E_{k',Q'}$; (ii) In particular,*

$$W^{\perp} = \sum_{\Lambda^k_Q \neq 0, Q \in G(\mathrm{R})} E^{\perp}_{k,Q}, \quad \textit{or equivalently,} \quad W = \bigcap_{\Lambda^k_Q \neq 0, Q \in G(\mathrm{R})} E_{k,Q}. \tag{5.2}$$

Proof. By Proposition 4.4 (particularly (4.4)), (5.2) follows from (i), namely the existence of two totally geodesic $E_{k,Q}, E_{k',Q'}$ with $E_{k,Q} \neq E_{k',Q'}$. We prove (i) by contradiction. Assume that all the totally geodesic subspaces $E_{k,Q}$ (with $\Lambda^k_Q \neq 0$) are the same. We denote it by E, which is totally geodesic by Proposition 4.4. Hence the respective kernels of Λ^k_Q, $U_{k,Q}$, are also the same by the duality. We denote the common $U_{k,Q}$ by U. By Proposition 4.4 and the discussion after it, we deduce that $E^{\perp} \neq \emptyset$ consists of vectors which are in the null space of $T^{x,y}_Q, \forall x, y \in W, \forall Q \in G(\mathrm{R})$. Namely, $E^{\perp} \subsetneq Z(W)$, $V = E + Z(W)$. (Note that $W \subset E \cap Z(W)$.) Pick $x \in E^{\perp}$. Clearly, $x \notin W$. Below we show that $P_{w,x} = 0, \forall w \in W, \forall P \in G(\mathrm{R})$. This implies (i) since it is a contradiction to the maximality of W.

Observe that $\forall z \in E, w \in W, x \in E^{\perp}$, $P_{w,x}z \in E$ (namely $P_{w,x}(E) \subset E$), since

$$\forall u \in U, \quad -Q_{u,P_{w,x}z} = Q_{P_{w,x}u,z} = \Lambda^k_P(w,u)Q_{x,z} = 0, \ \forall P, Q \in G(\mathrm{R}),$$

by (4.5) and $\Lambda^k_P(u,w) = 0$. Now for any $z' \in V$ write it as $z' = e + e^{\perp}$ with respect to $V = E \oplus E^{\perp}$. Then $P_{w,x}z \in E$ implies that $\langle P_{w,x}z, e^{\perp}\rangle = 0$. On the other hand,

$$\langle P_{w,x}z, e\rangle = \langle P_{z,e}w, x\rangle = 0, \text{ since } P_{z,e}w \in E, x \in E^{\perp}.$$

Thus $P_{w,x}|_E = 0$. On the other hand, since $x \in Z(W)$, Proposition 4.6 implies that $P_{w,x}|_{Z(W)} = 0$. For any $z \in V$, we may write $z = z_1 + z_2$ with $z_1 \in E$, $z_2 \in Z(W)$. We then have that $P_{w,x} = 0, \forall P \in G(\mathrm{R})$, since $P_{w,x}z = P_{w,x}z_1 + P_{w,x}z_2 = 0$. □

Below we assume $\mathrm{R} \neq 0$ (otherwise nothing needs to be proved). Then it implies $\mathrm{R}(x,y,x,y) \neq 0$ for some $x, y \in V$. Pick $\{x_i\}$ a basis of V in a neighborhood x with $\mathrm{R}(x_i,y,x_i,y) \neq 0$. Now choose maximal flats W, W_i such that $x \in W$ and $x_i \in W_i$. By Proposition 4.2 and for W (and W_i) so chosen, there exists some k with $\Lambda^k_{\mathrm{R}} \neq 0$. Hence $Z(W) \neq V$ ($Z(W_i) \neq V$).

5 The argument effectively implies that $V = \sum E_{k,Q}$ (Lemma 9 of [7]), since for any z, which belongs the orthogonal complement of the right-hand side, the proof implies $P_{w,z} = 0, \forall w \in W$.

Lemma 5.4. *Let W and $E_k = E_{k,Q}$ be as those of the last section. Let $x \in E_k$, $y \in E_k^\perp$. Then (i) for $A = Q_{x,y}$, $S' = \{V, A(\mathrm{R}), G\}$ is a holonomy system such that* $\dim(G(A(\mathrm{R}))) < \dim(G(\mathrm{R}))$*; (ii) There exists a basis $\{A_\ell\}$ of $\mathfrak{g}^{\mathrm{R}}$ such that $S_\ell = \{V, A_\ell(\mathrm{R}), G\}$ satisfies (i).*

Proof. For part (i), Corollary 5.2 implies that $G(A(\mathrm{R})) \subset K$. On the other hand, the definition of E_k ensures that there exists $X_k \in E_k$ and a unit eigenvector $e_k \in W$ of Λ_Q^k with $Q_{e_k,X_k} e_k = \Lambda_Q^k(e_k, e_k)X_k = \lambda_k^Q X_k \neq 0$. Hence $Q \notin K$, and we have that $\dim(G(A(\mathrm{R}))) < \dim(G(\mathrm{R}))$.

To obtain a basis of $\mathfrak{g}^{\mathrm{R}}$ with (i), we first pick g^α such that $\{g^\alpha(\mathrm{R})\}$ (with finitely many α) generates $G(\mathrm{R})$. Then let $\{x_i\}$ be a basis of V chosen as above with $\mathrm{R}(x_i, y, x_i, y) \neq 0$. Now it is clear that $\{A_{i,j}^\alpha\}$ with $A_{i,j}^\alpha = g^\alpha(\mathrm{R})_{x_i,x_j}$ generates $\mathfrak{g}^{\mathrm{R}}$, hence contains a subset as a basis of $\mathfrak{g}^{\mathrm{R}}$. Write $Q^\alpha = g^\alpha(\mathrm{R})$. Now for each x_i, there exists a flat (which can be made into a maximal one) W_i with $x_i \in W_i$. Since W_i is a flat $Q_{x_i,x_j}^\alpha = Q_{x_i,x_j^\perp}^\alpha$ with $x_j^\perp$ denotes the orthogonal projection of x_j into $W_i^\perp$. Now apply the part (i) to W_i. The way of choosing W_i ensures that $Z(W_i) \neq V$, hence, by equation (5.2), we have $x_j^\perp = \sum b_l y_l$ with $y_l \in (E_{l,Q'})^\perp$. This effectively expresses $A_{i,j}^\alpha$ as a linear combination of Q_{x_i,y_l}^α with $x_i \in W_i$ and $y_l \in (E_{l,Q'})^\perp$. Applying (i) with $A = Q_{x_i,y_l}^\alpha$, $E = E_{l,Q'}$ we have $\dim(G(Q_{x_i,y_l}^\alpha(\mathrm{R}))) < \dim(G(\mathrm{R}))$. By the way $\{Q_{x_i,y_l}^\alpha\}$ is constructed, it is easy to see that we can select a basis of $\mathfrak{g}^{\mathrm{R}}$ out of this family. □

Now we prove Theorem 1.2. If $\dim(G(\mathrm{R})) = 1$, clearly, $g(\mathrm{R}) = \mathrm{R}, \forall g \in G$. Assume that Theorem 1.3 holds for any holonomy system with $\dim(G(\mathrm{R})) \le k, k \in \mathbb{Z}, k > 0$. Now we prove it for the system with $\dim(G(\mathrm{R})) \le k+1$. Apply Theorem 1.3 to $S_{i,l}^\alpha = \{V, Q_{x_i,y_l}^\alpha(\mathrm{R}), G\}$ with Q^α, $x_i \in W_i$ and $y_l \in E_l$ as in Lemma 5.4. Then by Lemma 5.4, $\dim(G(Q_{x_i,y_l}^\alpha)) < \dim(G(\mathrm{R}))$. Thus $S_{i,l}^\alpha$ is symmetric. Namely, we have that $A(Q_{x_i,y_l}^\alpha(\mathrm{R})) = 0$ for any $A \in \mathfrak{g}$. Writing $A = A_1 + A_2, A_1 \in \mathfrak{g}^{\mathrm{R}}, A_2 \in \mathfrak{g}_{\mathrm{R}}$ with $A_1 = \sum a_{i,l}^\alpha Q_{x_i,y_l}^\alpha$, this and Lemma 4.1 imply that $A^2(\mathrm{R}) = 0$, hence $A(\mathrm{R}) = 0$. This proves Theorem 1.3 for $\dim(G(\mathrm{R})) \le k+1$.

Our approach avoids Lemmata 6, 9, 11, 12 of [7]. Since $\dim(G^{\mathrm{R}}(\mathrm{R})) = \dim(G(\mathrm{R}))$, the same argument proves Theorem 1.3. The rank of S could be defined as the maximum $\dim(W)$ for all maximal flat W. It is the same as the rank of the symmetric space corresponding to the Cartan algebra J.

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