On orthogonal Ricci curvature

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ABSTRACT. In this paper we discuss some recent progresses in the study of compact Kähler manifolds with positive orthogonal Ricci curvature, a curvature condition defined as the difference between Ricci curvature and holomorphic sectional curvature. In the recent works by authors and the joint work of authors with Q. Wang the comparison theorems, vanishing theorems, and structural theorems for such manifolds have been proved. We also constructed examples of this type of manifolds, and give some classification results in low dimensions.

1. Orthogonal Ricci curvature

Let (M^n, g) be a Kähler manifold of complex dimension n. Its *orthogonal Ricci curvature* $Ric^{\perp}$ is defined by (cf. [**21**]):

$$Ric^{\perp}_{X\overline{X}} = Ric(X, \overline{X}) - R(X, \overline{X}, X, \overline{X})/|X|^2,$$

where X is a non-zero type $(1,0)$ tangent vector at a point $x \in M^n$. This curvature arises in the study of the comparison theorem for Kähler manifolds and the previous study of manifolds with so-called *nonnegative quadratic orthogonal bisectional curvature* (cf. [**4**], [**26**], [**16**], [**5**]). We refer the readers to [**21**] for a more detailed account on this topic. Clearly this curvature is closely related to Ricci curvature Ric and holomorphic sectional curvature H. It is natural to ask, what is the relationship between $Ric^{\perp}$ and Ric or H (other than the obvious one that $Ric^{\perp} + H = Ric$ for unit length tangent vectors), and what kind of compact complex manifolds M^n can admit Kähler metrics with $Ric^{\perp} > 0$ (or ≥ 0, or ≤ 0, or < 0, or $\equiv 0$) everywhere?

In this paper, we will focus on the curvature condition $Ric^{\perp}$ and pay particular attention to the class of compact Kähler manifolds with $Ric^{\perp} > 0$ everywhere, except in Section 2 where complete noncompact Kähler manifolds are also considered. Throughout this paper, we will assume that the complex dimension $n \geq 2$ unless stated otherwise, since $Ric^{\perp} \equiv 0$ when $n = 1$.

We start with the following observation. At a point $x \in M^n$, let us denote by $\mathbb{S}^{2n-1}_x$ the unit sphere of all type $(1,0)$ tangent vector at x of unit length. By a classic result of Berger,

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the average value of Ric or H over $\mathbb{S}^{2n-1}_x$ is $\frac{S(x)}{2n}$ or $\frac{S(x)}{n(n+1)}$, respectively, where $S(x)$ is the scalar curvature. Based on that, we get the following

LEMMA 1.1. *For a Kähler manifold (M^n, g) with $n \geq 2$, the average value of $Ric^\perp$ over the unit tangent sphere $\mathbb{S}^{2n-1}_x$ at any $x \in M^n$ is $\frac{(n-1)S(x)}{2n(n+1)}$. In particular, $Ric^\perp > 0$ (or ≥ 0, or < 0, or ≤ 0, or $\equiv 0$) implies $S(x) > 0$ (or ≥ 0, or < 0, or ≤ 0, or $\equiv 0$).*

So just like Ric or H, $Ric^\perp$ also dominates the scalar curvature S, in the sense that the sign of $Ric^\perp$ determines the sign of S. On the other hand, $Ric^\perp$ is clearly dominated by the bisectional curvature $B = R_{X\overline{X}Y\overline{Y}}$, where $|X| = |Y| = 1$, just like Ric or H. It is also dominated by the weaker curvature conditions *orthogonal bisectional curvature* $B^\perp$, which is defined by $R_{X\overline{X}Y\overline{Y}}$ for $|X| = |Y| = 1$ and $X \perp Y$, and the *quadratic orthogonal bisectional curvature* QB, which is defined in the following way:

The Kähler manifold (M^n, g) is said to have $QB > 0$ at $x \in M^n$, if for any unitary tangent frame $\{e_1, \ldots, e_n\}$ at x and any real numbers $a_1, \ldots, a_n$, not all equal to each other, it holds that $\sum_{i,j=1}^n R_{i\bar{i}j\bar{j}}(a_i - a_j)^2 > 0$.

This is a weaker curvature condition than $B^\perp > 0$, and yet by taking all but one of these a_i to be zero, we get

LEMMA 1.2. *A Kähler manifold (M^n, g) with $QB > 0$ (or ≥ 0, or < 0, or ≤ 0, or $\equiv 0$) will have $Ric^\perp > 0$ (or ≥ 0, or < 0, or ≤ 0, or $\equiv 0$).*

Some more elementary facts about orthogonal Ricci curvature. In complex dimension $n = 1$, one always have $Ric^\perp = 0$. For $n \geq 2$, the complex space forms $\mathbb{P}^n$, $\mathbb{C}^n$, and $\mathbb{H}^n$ respectively satisfies $Ric^\perp > 0$, $= 0$, or < 0. For product manifolds, we have the following:

LEMMA 1.3. *If both of the Kähler manifolds (M, g) and (N, h) satisfy $Ric^\perp > 0$ and $Ric \geq 0$, then the product manifold $(M \times N,\, g \times h)$ will have $Ric^\perp > 0$.*

This is because any tangent vector X of type $(1,0)$ on $M \times N$ can be uniquely written as $U + V$ where U is tangent to M and V is tangent to N, and

$$\begin{aligned} |X|^2 R_{X\overline{X}} - R_{X\overline{X}X\overline{X}} &= |X|^2(R_{U\overline{U}} + R_{V\overline{V}}) - (R_{U\overline{U}U\overline{U}} + R_{V\overline{V}V\overline{V}}) \\ &\geq \big(|U|^2 R_{U\overline{U}} - R_{U\overline{U}U\overline{U}}\big) + \big(|V|^2 R_{V\overline{V}} - R_{V\overline{V}V\overline{V}}\big). \end{aligned}$$

Here we used $R_{X\overline{Y}}$ to denote the Ricci tensor. In particular, $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ has $Ric^\perp > 0$ whenever all $n_1, \ldots, n_r \geq 2$.

There also exists an algebraic consideration viewing $Ric^\perp$ as the *holomorphic sectional curvature* of an algebraic curvature operator risen from the one acting on the two-forms via the Bochner formula. Recall the notations from the appendix of [**19**] and define an algebraic (Kähler) curvature operator

$$R_{Ric} = Ric \bar{\wedge}\, \mathrm{id},$$

where for any $A, B : T'_x M \to T'_x M$ Hermitian symmetric ($\bar{A}(X) = A(\bar{X}) = 0$)

$$\begin{aligned} \langle A \bar{\wedge} B (X \wedge \bar{Y}), \overline{Z \wedge \bar{W}} \rangle \ &\doteqdot\ \frac{1}{2}\Big(\langle \big(A \wedge \bar{B} + B \wedge \bar{A}\big)(X \wedge \bar{Y}), \overline{Z \wedge \bar{W}} \rangle \\ &\qquad + \langle \big(A \wedge \bar{B} + B \wedge \bar{A}\big)(W \wedge \bar{Y}), \overline{Z \wedge \bar{X}} \rangle \Big). \end{aligned}$$

It is easy to check that $Ric^\perp(X, \overline{X}) = H_{R_{Ric} - R}(X)/|X|^2$. Here $H_{\widetilde{R}}(X)$ is the holomorphic sectional curvature of $\widetilde{R} = R_{Ric} - R$. From this it is easy to see that $Ric^\perp \equiv 0$ implies that

$\widetilde{R} \equiv 0$. Hence $Ric^{\perp} \equiv 0$, via the decomposition of the curvature operators, induces that either $n = 1$, or $n = 2$ R is conformally flat, or $n \geq 3$ and R is flat. If $Ric^{\perp}(X, \overline{X}) = c|X|^2$ for a constant $c \neq 0$, similarly one can conclude that either $n = 2$, R is conformally flat or R is a multiple if identity. Hence $Ric^{\perp}$-Einstein is a very special condition.

2. Comparison theorems

The Laplacian comparison theorem is a cornerstone in Riemannian geometry and global analysis. In the Kähler case, the Laplacian of the distance function decomposes as the sum of the so-called *holomorphic Hessian* and *orthogonal Laplacian* in a natural manner [**21**].

Let (M^n, g) be a Kähler manifold and let us fix a point $p \in M^n$. Denote by ρ the function on M which is the distance from p. Let $Z = \frac{1}{\sqrt{2}}(\nabla\rho - \sqrt{-1}J\nabla\rho)$ be the type $(1,0)$ unit tangent vector in the radial direction, then the orthogonal Laplacian is defined by

$$\Delta^{\perp}\rho = \Delta\rho - \nabla^2(Z, \overline{Z}),$$

and the second term on the right hand side is the holomorphic Hessian. As observed in [**21**], the comparison of orthogonal Ricci curvature will lead to comparison on orthogonal Laplacians:

Theorem 2.1. *Let (M^n, g) be a complete Kähler manifold with $Ric^{\perp} \geq (n-1)\lambda$, where λ is a constant. Let $(\tilde{M}, \tilde{g})$ be a complex space form of the same dimension with constant holomorphic sectional curvature 2λ. Fix $p \in M$ and $\tilde{p} \in \tilde{M}$, and denote by ρ, $\tilde{\rho}$ the distance function from p or $\tilde{p}$, respectively. Then for any $x \in M$ not in the cut locus of p, it holds*

$$\Delta^{\perp}\rho(x) \;\leq\; \Delta^{\perp}\tilde{\rho}|_{\tilde{\rho}=\rho(x)} \;=\; (n-1)\cot_{\frac{\lambda}{2}}(\rho).$$

Similarly, the comparison of holomorphic sectional curvature H leads to comparison on holomorphic Hessians. For distance function to points, this was proved by G. Liu in [**17**], using the argument of [**15**]. In [**21**], we generalized it to distance functions to complex submanifolds:

Theorem 2.2. *Let (M^n, g) be a complete Kähler manifold with $H \geq 2\lambda$, and let $(\tilde{M}, \tilde{g})$ be a complex space form of the same dimension with constant holomorphic sectional curvature 2λ. If $P \subset M$ and $\tilde{P} \subset \tilde{M}$ are complex submanifolds, and denote by ρ, $\tilde{\rho}$ the distance function from P or $\tilde{P}$, respectively. Then for any $x \in M$ not in the focal locus of P, it holds*

$$\nabla^2\rho(Z, \overline{Z})|_x \;\leq\; \nabla^2\tilde{\rho}(\tilde{Z}, \overline{\tilde{Z}})|_{\tilde{\rho}=\rho(x)}.$$

In particular, when $\lambda = 0$ and $\tilde{P}$ is a point, it holds that

$$\nabla^2\rho(Z, \overline{Z})|_x \leq \frac{1}{2\rho(x)} \Longleftrightarrow \nabla^2 \log\rho(Z, \overline{Z}) \leq 0.$$

When the curvature assumptions in the above two theorems are both valid, then as in [**15**] one has the volume comparison theorem

Corollary 2.3. *Let (M^n, g) be a complete Kähler manifold with $Ric^{\perp} \geq (n-1)\lambda$ and $H \geq 2\lambda$, and $(\tilde{M}, \tilde{g})$ the complex space form of the same dimension with constant holomorphic* *sectional curvature* 2λ*. Then for any* $x \in M$ *and* $\tilde{x} \in \tilde{M}$*, it holds that* $\Delta\rho(x) \leq \Delta\tilde{\rho}|_{\rho(x)}$*, and for any* $0 < r \leq R$*, it holds*

$$\frac{Vol(B(x,R))}{Vol(B(x,r))} \leq \frac{Vol(\tilde{B}(\tilde{x},R))}{Vol(\tilde{B}(\tilde{x},r))},$$

where B *and* $\tilde{B}$ *are geodesic balls in* M *and* $\tilde{M}$*, respectively. The equality holds if and only if* $B(x,R)$ *is holomorphically isometric to* $\tilde{B}(\tilde{x},R)$*.*

Note that the lower bounds on $Ric^{\perp}$ and H gives the condition $Ric \geq (n+1)\lambda$, so there is volume comparison in the Riemannian setting. However, the above comparison in the Kähler setting is sharper.

Theorem 2.1 can be generalized to the case of complex hypersurfaces, which can be viewed as the Kähler version of Heintze-Karcher Theorem [**10**] with the assumption on Ricci curvature being replaced by $Ric^{\perp}$:

Theorem 2.4. *With* M*,* $\tilde{M}$ *in Theorem 2.1, if* $P \subset M$ *and* $\tilde{P} \subset \tilde{M}$ *are complex hypersurfaces and* ρ*,* $\tilde{\rho}$ *are distance functions to* P*,* $\tilde{P}$*, respectively, then for any* x *not in the focal locus of* P*,*

$$\Delta^{\perp}\rho(x) \;\leq\; \Delta^{\perp}\tilde{\rho}|_{\tilde{\rho}=\rho(x)} \;=\; (n-1)\tan_{\frac{\lambda}{2}}(\rho).$$

Similarly, one can generalize Theorem 2.2 to the orthogonal Hessian of the distance function. For any real value u on M, we will denote by $\nabla^{2\perp}u(X,\overline{X})$ the restriction of $\nabla^2 u(X,\overline{X})$ on the spaces of all type $(1,0)$ vectors X perpendicular to both ∇u and $J\nabla u$. We have the following:

Theorem 2.5. *Let* (M^n, g) *be a complete Kähler manifold with orthogonal bisectional curvature* $B^{\perp} \geq \lambda$*, and let* $(\tilde{M}, \tilde{g})$ *be a complex space form of the same dimension with constant holomorphic sectional curvature* 2λ*. Fix* $p \in M$ *and* $\tilde{p} \in \tilde{M}$ *and denote by* ρ*,* $\tilde{\rho}$ *the distance function from* p *or* $\tilde{p}$*, respectively. Then for any* $x \in M$ *not in the cut locus of* p*, it holds*

$$\nabla^{2\perp}\rho(x) \;\leq\; \nabla^{2\perp}\tilde{\rho}|_{\tilde{\rho}=\rho(x)}.$$

A similar argument as in the classical Bonnet-Myers Theorem case would imply that, for any complete Kähler manifold (M^n, g) with $Ric^{\perp}$ bounded from below by a positive constant, the diameter of M^n is bounded from above, hence M^n must be compact. As a consequence, we get the following:

Corollary 2.6. *Let* (M^n, g) *be a compact Kähler manifold with* $Ric^{\perp} > 0$ *everywhere. Then the fundamental group* $\pi_1(M)$ *is finite.*

It was conjectured [**21**] that such manifolds are all simply-connected, in fact, they all should be rationally-connected. But so far, we have only been successful in proving this for $n \leq 4$.

3. Vanishing theorems

In [**27**], it was shown that any compact Kähler manifold M^n with positive holomorphic sectional curvature must be projective, answering affirmatively a question raised in [**28**]. The proof was done by showing that the Hodge number $h^{2,0}$ vanishes, namely, there are no non-trivial global holomorphic 2-forms on M^n. It was actually proved that all the Hodge number $h^{p,0}$ vanishes for any $1 \leq p \leq n$, that is, any global holomorphic p-form on M^n must be identically zero.

The proof of this vanishing theorem used the form version of the Bochner identity

$$\partial\overline{\partial}|s|^2 = \langle \nabla s, \overline{\nabla s} \rangle - \tilde{R}(s, \overline{s}, \cdot, \cdot),$$

where s is any holomorphic p-form on M^n, i.e., any holomorphic section of $\bigwedge^p \Omega$, where Ω is the holomorphic cotangent bundle of M^n, and $\tilde{R}$ is the curvature of $\bigwedge^p \Omega$. Following the same approach, we were able to show that

THEOREM 3.1. *Let (M^n, g) be a compact Kähler manifold with $Ric^{\perp} > 0$ everywhere. Then $h^{p,0} = 0$ for $p = 1, 2, n-1, n$. In particular, M^n is always projective.*

Of course it is believed that such a manifold will have $h^{p,0} = 0$ for any $1 \leq p \leq n$, in fact, the manifold should be rationally connected. But so far we are not able to show that, as we don't know how to deal with the vanishing of holomorphic p-forms for $p \geq 3$. At present, we also don't know how to prove that any compact Kähler manifold with $Ric^{\perp} > 0$ must be simply-connected, except when the dimension is at most 4 which is a consequence of the above vanishing theorem.

Note that if we already know that $Ric^{\perp} > 0$ implies that $h^{p,0} = 0$ for all $1 \leq p \leq n$, then all such M^n must be simply-connected, by the following well-known argument. Note that the Euler characteristic of the structure sheaf $\mathcal{O}_M$ of M is given by

$$\chi(\mathcal{O}_M) = 1 - h^{1,0} + h^{2,0} - \cdots + (-1)^n h^{n,0},$$

so the vanishing of the Hodge numbers implies that $\chi(\mathcal{O}_M) = 1$. Let $\pi : \widetilde{M} \to M$ be the universal covering space. π is finite of degree d since $\pi_1(M)$ is finite. So we have $\chi(\mathcal{O}_{\widetilde{M}}) = d \cdot \chi(\mathcal{O}_M) = d$. On the other hand, since $\widetilde{M}$ with the pull back metric also has $Ric^{\perp} > 0$, thus $\chi(\mathcal{O}_{\widetilde{M}}) = 1$, so we must have $d = 1$, namely, M is simply-connected.

In a related recent work [**22**], we examined the vanishing theorems for a new set of curvature conditions, where we were able to achieve optimal results. Given a Kähler manifold (M^n, g), if Σ is a k-dimensional complex subspace of the tangent space $T^{1,0}_x M$ of M at $x \in M$, then we will denote by $S_k(x, \Sigma)$ the average value of the holomorphic sectional curvature function H, integrated over the unit sphere $\Sigma \cap \mathbb{S}^{2n-1}_x$.

S_k will be called the k-scalar curvature, which interpolates between $S_1 = H$ and $S_n = S$, the usual scalar curvature. We will say that M^n has positive k-scalar curvature, denoted as $S_k > 0$, if $S_k(x, \Sigma) > 0$ for any $x \in M$ and any k-dimensional subspace Σ at x. Clearly, $S_k > 0$ implies $S_l > 0$ for any $l > k$. So the strength of S_k deceases as k increases. In [**22**], it was proved that

THEOREM 3.2. *Let M^n be a compact Kähler manifold with $S_2 > 0$. Then $h^{p,0} = 0$ for any $2 \leq p \leq n$. In particular, M^n is always projective.*

Note that there are complex 2-tori that are not projective. By taking the product of such a torus with a complex projective space, we see that $S_3 > 0$ does not guarantee projectiveness. So the above result is sharp in some sense. For general k, we also have the following

THEOREM 3.3. *Let M^n be a compact Kähler manifold with $S_k > 0$, where k is an integer between 2 and n. Then $h^{p,0} = 0$ for any $k \leq p \leq n$.*

Note that the statement is also true when $k = 1$, which is exactly the vanishing theorem of [**27**] for compact Kähler manifolds with positive holomorphic sectional curvature.

The curvature S_k is related to $Ric^\perp$ in the following way. For any k-subspace $\Sigma \subset T^{1,0}_x M$, we define

$$S_k^\perp(x,\Sigma) = k \cdot \fint_{Z\in\Sigma, |Z|=1} Ric^\perp(Z,\overline{Z})\, d\theta(Z)$$

where $\fint f(Z)\, d\theta(Z)$ denotes $\frac{1}{Vol(\mathbb{S}^{2k-1})}\int_{\mathbb{S}^{2k-1}} f(Z)\, d\theta(Z)$. We say $S_k^\perp(x) > 0$ if for any k-subspace $\Sigma \subset T^{1,0}_x M$, $S_k^\perp(x,\Sigma) > 0$. The following generalization of Theorem 3.1 can also be obtained.

Theorem 3.4. *Let (M^n, g) be a compact Kähler manifolds such that $S_2^\perp(x) > 0$ for any $x \in M$. Then $h^{2,0} = 0$. In particular, M is projective.*

Observe that, if $\Sigma = \text{span}\{E_1, E_2, \cdots, E_k\}$, then we have

$$\begin{aligned}
\frac{1}{k} S_k^\perp(x,\Sigma) &= \fint_{Z\in\Sigma,\, |Z|=1} Ric^\perp(Z,\overline{Z})\, d\theta(Z) = \fint_{Z\in\Sigma,\, |Z|=1} \left(Ric(Z,\overline{Z}) - H(Z)\right)\, d\theta(Z) \\
&= \fint \frac{1}{Vol(\mathbb{S}^{2n-1})}\left(\int_{\mathbb{S}^{2n-1}} \left(nR(Z,\overline{Z},W,\overline{W}) - H(Z)\right)\, d\theta(W)\right) d\theta(Z) \\
&= \frac{1}{Vol(\mathbb{S}^{2n-1})}\int_{\mathbb{S}^{2n-1}} \left(\fint \left(nR(Z,\overline{Z},W,\overline{W}) - H(Z)\right)\, d\theta(Z)\right) d\theta(W) \\
&= \frac{1}{k}\left(Ric(E_1,\overline{E}_1) + Ric(E_2,\overline{E}_2) + \cdots + Ric(E_k,\overline{E}_k)\right) - \frac{2}{k(k+1)} S_k(x,\Sigma)
\end{aligned}$$

where $S_k(x,\Sigma)$ is the scalar curvature of R restricted to Σ defined in the above. The positivity of the partial sum $Ric(E_1,\overline{E}_1) + Ric(E_2,\overline{E}_2) + \cdots + Ric(E_k,\overline{E}_k)$ for any unitary frame is called the k-positivity of Ricci.

Given the results on the projectivity for compact Kähler manifolds with $Ric^\perp > 0$ and $S_2 > 0$, naturally questions could be asked for compact Kähler manifolds with $Ric^\perp < 0$, or $S_2 < 0$. For example, *are all compact Kähler manifolds with $Ric^\perp < 0$ projective? When (or if) the K_M of such a manifold is ample? Similarly one can ask when a compact Kähler manifold with $S_2 < 0$ is projective, and when K_M is ample.* Regarding the question for $S_2 < 0$ manifolds, there have been some recent progresses ([**25**], [**24**]) for $S_1 = H < 0$ case.

4. Examples: classical Kähler C-spaces with $b_2 = 1$

It is well known that any compact Hermitian symmetric space M^n will have bisectional curvature nonnegative everywhere, while its Ricci curvature and holomorphic sectional curvature are positive everywhere. It is verified in [**20**] that such M^n will have $Ric^\perp > 0$ if and only if M^n does not contain $\mathbb{P}^1$ as a factor:

Theorem 4.1. *A compact Hermitian symmetric space M^n has $Ric^\perp > 0$ if and only if it does not have a $\mathbb{P}^1$ factor.*

More generally, the set of all compact Hermitian symmetric spaces is contained in the larger set of all Kähler C-spaces, which are the orbit spaces of the adjoint representations of compact simple Lie groups. Note that not all irreducible Kähler C-spaces of dimension at least 2 satisfy $Ric^{\perp} > 0$, for instance, we will see later that the flag threefold

$$M^3 = \{([z],[w]) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid \sum_{i=0}^{2} z_i w_i = 0\}$$

cannot admit any Kähler metric with $Ric^{\perp} > 0$. However, this may be caused by the fact that its second betti number is bigger than one. We propose the following:

Conjecture 4.2. *Any Kähler C-spaces with $b_2 = 1$ and $n \geq 2$ will satisfy $Ric^{\perp} > 0$.*

Note that Kähler C-spaces with $b_2 = 1$ consist of the four classical sequences plus finitely many exceptional ones, and by using the computations by Itoh [**13**] and by Chau and Tam [**5**], we verified in [**20**] the following

Theorem 4.3. *Any classical Kähler C-space with $b_2 = 1$ and $n \geq 2$ satisfies $Ric^{\perp} > 0$.*

To describe the story about the exceptional ones, let us recall that Kähler C-space with $b_2 = 1$ are characterized as $(\mathfrak{g}, \alpha_i)$ (see [**13**], [**5**], or [**16**] for more details). Here $\mathfrak{g}$ is a simple complex Lie algebra, and $\{\alpha_1, \ldots, \alpha_r\}$ is a fundamental root system with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, with r the rank and $1 \leq i \leq r$. Simple complex Lie algebras are fully classified as the four classical sequences $A_r = \mathfrak{sl}_{r+1}$ $(r \geq 1)$, $B_r = \mathfrak{so}_{2r+1}$ $(r \geq 2)$, $C_r = \mathfrak{sp}_{2r}$ $(r \geq 3)$, $D_r = \mathfrak{so}_{2r}$ $((r \geq 4)$, plus the exceptional ones E_6, E_7, E_8, F_4 and G_2.

For the exceptional ones, $(E_6, \alpha_1) = (E_6, \alpha_6)$ is the compact Hermitian symmetric space of type V, which has dimension 16 and rank 2. (E_7, α_7) is the compact Hermitian symmetric space of type VI, which has dimension 27 and rank 3. Theses two form the only exceptional irreducible compact Hermitian symmetric spaces. Also, $(G_2, \alpha_1) = \mathbb{Q}^5$ is the quadratic hypersurface in $\mathbb{P}^6$ which is a type IV Hermitian symmetric space. In [**5**], it was proven that

Theorem 4.4 (Chau-Tam [**5**]). *The following exceptional Kähler C-spaces with $b_2 = 1$ all have $QB > 0$, hence have $Ric^{\perp} > 0$:*

$$(G_2, \alpha_2),\ (F_4, \alpha_i)_{i=1,2,4},\ (E_6, \alpha_i)_{2,3,5},\ (E_7, \alpha_i)_{i=1,2,5},\ (E_8, \alpha_i)_{i=1,2,8}.$$

For the remaining exceptional Kähler C-spaces with $b_2 = 1$, namely, the following list

$$\mathcal{E}_0 = \{(F_4, \alpha_3),\ (E_6, \alpha_4),\ (E_7, \alpha_i)_{i=3,4,6},\ (E_8, \alpha_i)_{i=3,4,5,6,7}\},$$

Chau and Tam proved in [**5**] that each of them does not satisfy $QB \geq 0$. However, we do believe that each of them satisfy $Ric^{\perp} > 0$, which is a much weaker condition than $QB > 0$.

For Kähler C-spaces with $b_2 > 1$, it would be a very interesting question to determine the subset which satisfies $Ric^{\perp} > 0$.

5. Examples: projectivized vector bundles

We have seen that $\mathbb{P}^n \times \mathbb{P}^m$ has $Ric^{\perp} > 0$ for any $n, m \geq 2$. Now we would like to generalize that to projectivized vector bundles. Let (M^n, g) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle of rank r over M, equipped with a Hermitian metric h. Let $\pi : P = \mathbb{P}(E^*) \to M$ be the projectivized bundle associated with E, that is, for any

$x \in M$, the fiber $\pi^{-1}(x) = \mathbb{P}(E_x)$ is the projective space of all complex lines in E_x through the origin.

Denote by L be the holomorphic line bundle on P dual to the tautological subbundle. L is determined by the short exact sequence on P:

$$0 \to \mathcal{O}_P \to \pi^*E^* \otimes L \to T_{P|M} \to 0,$$

where $T_{P|M} = \ker(d\pi : T_P \to \pi^* T_M)$ is the relative tangent bundle. The metric h on E induces naturally a Hermitian metric $\hat{h}$ on L, whose curvature form is

$$C_1(L, \hat{h}) = \omega_{\rm FS} - \frac{\sqrt{-1}}{|v|^2}\Theta^h_{v\overline{v}}$$

at any point $(x, [v]) \in P$, where $x \in M$ and $0 \neq v \in E_x$. Here $\omega_{\rm FS}$ is the Kähler form of the Fubini-Study metric on the fiber of π. For a positive constant λ, consider the closed $(1,1)$ form on P:

$$\omega_G = \lambda \pi^* \omega_g + C_1(L, \hat{h}).$$

Then for λ sufficiently large, G is a Kähler metric on P.

In [**11**], Hitchin showed that any Hirzebruch surface $\mathbb{F}_k$ admits Kähler metrics with positive holomorphic sectional curvature. Here $\mathbb{F}_k = \mathbb{P}(E^*)$ for $E = \mathcal{O} \oplus \mathcal{O}(k)$ over $\mathbb{P}^1$, where k is any nonnegative integer. In [**1**], this was generalized to any projectivized vector bundle over any compact Kähler manifold with positive holomorphic sectional curvature, namely, it was shown that when the base manifold (M^n, g) has positive holomorphic sectional curvature and when λ is sufficiently large, the above metric G always has positive holomorphic sectional curvature. Following this computation, in [**20**], we obtained the following:

Theorem 5.1. *Let (M^n, g) be a compact Kähler manifold with $Ric^{\perp} > 0$, and (E, h) be a Hermitian vector bundle over M of rank $r \geq 3$ such that for any $x \in M$ and any $0 \neq v \in E_x$,*

$$Ric^{g\perp}_{X\overline{X}} + R(\det E)_{X\overline{X}} - \frac{r}{|v|^2} R^h_{v\overline{v}X\overline{X}} \ > \ 0 \tag{5.1}$$

for any tangent vector $0 \neq X \in T^{1,0}_x M$. Here $R(\det E)$ is the curvature of the determinant line bundle $\det E = \bigwedge^r E$ equipped with the metric induced by h. Then on the projectivized bundle $P = \mathbb{P}(E^*)$, the Kähler metric G with $\omega_G = \lambda\, \pi^* \omega_g + C_1(L, \hat{h})$ will have $Ric^{\perp} > 0$ everywhere when λ is sufficiently large.

We remark that the rank requirement $r \geq 3$ here is necessary, as we shall see later that any $\mathbb{P}^1$-bundle over any space can never admit a Kähler metric with $Ric^{\perp} > 0$. We also remark that the curvature condition (5.1) is independent of the scaling of metrics g or h, as well as tensoring of E by a line bundle.

When the dimension of the base manifold is 3 or higher, the above theorem gives non-trivial examples of manifolds with $Ric^{\perp} > 0$, including those which are not Kähler C-spaces. For instance, we have the following

Corollary 5.2. *Consider $E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ on $\mathbb{P}^n$ where $a_1 \geq a_2 \geq \cdots \geq a_r$ are integers. If $r \geq 3$, and $n - 1 > (a_1 - a_2) + \cdots + (a_1 - a_r)$, then $P = \mathbb{P}(E^*)$ will admit a Kähler metric with $Ric^{\perp} > 0$.*

For instance, for $E = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(-1)$ over $\mathbb{P}^3$, the Fano fivefold $P^5 = \mathbb{P}(E^*)$ has $Ric^{\perp} > 0$. Note that it is not a Kähler C-space, as it contains a section with negative normal bundle.

Similarly, for any $n \geq 3$, one can check that the curvature condition (5.1) is satisfied for the holomorphic cotangent bundle $E = \Omega_{\mathbb{P}^n}$ over $\mathbb{P}^n$, so we have:

COROLLARY 5.3. *For any $n \geq 3$, the $(2n-1)$-dimensional manifold $\mathbb{P}(T_{\mathbb{P}^n})$ has $Ric^{\perp} > 0$.*

In contrast, when the base manifold is 2-dimensional, the theorem does not give much information. In fact, we have the following result which is in sharp contrast with the higher base dimensional cases:

THEOREM 5.4. *Let P be a holomorphic fiber bundle over a compact complex surface S with fiber $\mathbb{P}^m$, where $m \geq 2$. If P admits a Kähler metric with $Ric^{\perp} > 0$ everywhere, then S is biholomorphic to $\mathbb{P}^2$ and P is biholomorphic to $\mathbb{P}^2 \times \mathbb{P}^m$.*

So if we take any non-trivial $\mathbb{P}^m$-bundle over $\mathbb{P}^2$, for instance, $\mathbb{P}(T_{\mathbb{P}^3}|_{\mathbb{P}^2})$ or $\mathbb{P}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$ over $\mathbb{P}^2$, we know by the above theorem that the total spaces do not admit any Kähler metric with $Ric^{\perp} > 0$. So the total space of a holomorphic fiber bundle may not admit any Kähler metric with $Ric^{\perp} > 0$ even when both the fiber and the base do.

6. Structural results

In the two previous sections, we have seen some existence results. Now let us turn our attention to the obstruction or non-existence side, and use them to obtain some structural results. Our goal is to obtain differential and algebraic geometric consequences from the curvature condition $Ric^{\perp} > 0$.

In [**20**], a generalization of a theorem of Frankel [**7**] was obtained:

THEOREM 6.1. *Let M^n be a compact Kähler manifold with $Ric^{\perp} > 0$. If Y_1 and Y_2 are smooth compact complex hypersurfaces in M, then $Y_1 \cap Y_2 \neq \phi$.*

Note that when the hypersurfaces Y_1 and Y_2 in the above theorem are singular, the same conclusion holds. This can be proved by a slight modification of the proof given in [**20**].

As an immediate corollary, we know that manifolds with $Ric^{\perp} > 0$ cannot be the blowing up of a (smooth or singular) point, or a fiberation over a curve:

COROLLARY 6.2. *Let M^n be a compact Kähler manifold with $Ric^{\perp} > 0$. Then there exists no surjective holomorphic map from M^n onto a complex curve, and there exists no birational morphism $f : M \to Z$ onto a normal variety Z, where a smooth hypersurface in M is mapped to a (smooth or singular) point.*

A Lefschetz type theorem can also be proved for compact Kähler manifolds with $Ric^{\perp} > 0$, namely, for a pair of complex hypersurfaces (Y_1, Y_2) in M, or for a hypersurface Y in M. The key here is that for any pair of hypersurfaces Y_1, Y_2, one can consider the space Ω of all paths in M originating from Y_1 and ending in Y_2. The energy $\mathcal{E}(\gamma)$ of a path $\gamma \in \Omega$ is defined in the usual way, and it is well known that the critical points of the energy functional are normal geodesics, namely, geodesics which intersects Y_i orthogonally. The same argument as in the proof of the above theorem implies the following index estimate, which includes the theorem as a special case since the minimizers can be identified with $Y_1 \cap Y_2$ (cf. [**23**]).

Corollary 6.3. *Let γ be a nontrivial critical point (namely a nonconstant normal geodesic after* [**21**]*). Then the index $ind(\gamma) \geq 1$. In particular,*

$$\pi_0(\Omega, Y_1 \cap Y_2) = \{0\}, \quad \iota_* : \pi_1(Y_1, Y_1 \cap Y_2) \to \pi_1(M, Y_2) \text{ is surjective.} \tag{6.1}$$

When $Y_1 = Y_2 = Y$, this implies that $\pi_1(M, Y) = \{0\}$.

Note that in [**21**] it was conjectured that $\pi_1(M) = \{0\}$. The last statement of the corollary is clearly a consequence of an affirmative answer to the conjecture.

An important geometric property for manifolds with $Ric^{\perp} > 0$ is the following:

Theorem 6.4. *Let (M^n, g) be a compact Kähler manifold with $Ric^{\perp} > 0$. Let C be an irreducible curve in M and $f : \tilde{C} \to C \subset M$ be its normalization. If we denote by K_M the canonical line bundle of M and let g be the genus of $\tilde{C}$, then we have*

$$K_M^{-1} C \geq 3 - 2g.$$

In particular, $K_M^{-1} C \geq 3$ for any rational curve C in M.

For a smooth rational curve $C \subset M$, we have the short exact sequence of vector bundles on C

$$0 \to T_C \to T_M|_C \to N_C \to 0$$

where N_C is the normal bundle of C in M. By taking their first Chern classes, we get

$$c_1(N_C) = c_1(T_M|_C) - c_1(T_C) = K_M^{-1} C - 2 > 0$$

by the above theorem. In other words, for any smooth rational curve in M^n with $Ric^{\perp} > 0$, the normal bundle must have positive first Chern class.

The above results already put severe restrictions to the class $\mathcal{M}_n^{\perp}$ of all compact complex manifolds of complex dimension n which admit Kähler metrics with $Ric^{\perp} > 0$ everywhere. For instance, $\mathcal{M}_2^{\perp}$ consists of $\mathbb{P}^2$ alone by result in [**9**] on manifold with positive orthogonal bisectional curvature $B^{\perp}$, as when $n = 2$, $Ric^{\perp}$ coincides with $B^{\perp}$.

By Mori's theory on extremal rays and the cone-contraction theorems, one can use the above numerical restriction for $Ric^{\perp} > 0$ manifolds to draw conclusions on low dimensional cases (see for instance [**18**], [**14**], [**6**]):

Theorem 6.5. *Let (M^3, g) be a compact Kähler manifold of dimension 3 with $Ric^{\perp} > 0$. Then M^3 is isomorphic to either $\mathbb{P}^3$ or $\mathbb{Q}^3$.*

Here and below we will denote by $\mathbb{Q}^n$ the smooth quadric in $\mathbb{P}^{n+1}$. In dimension 4, one could use the results in Mori's program (see for instance [**3**], [**12**], [**2**]) to narrow things down to the following list:

Theorem 6.6. *Let (M^4, g) be a compact Kähler manifold of dimension 4 with $Ric^{\perp} > 0$. Then M^4 is isomorphic to either $\mathbb{P}^2 \times \mathbb{P}^2$ or a Fano fourfold with $b_2 = 1$ and pseudo index $i(M) \geq 3$.*

Recall that the *pseudo index* of a Fano manifold M is defined to be the minimum of the intersection number $K_M^{-1} C$ with the anti-canonical line bundle, where C runs through all rational curves in M.

A key intermediate step in deriving the above theorem is to rule out the possibility of a class of fourfolds, including for instance a $\mathbb{P}^2$-bundles over a Barlow's surface. The result was stated as Theorem 5.4 in the previous section, and we need a slight modification of the vanishing theorem on holomorphic 2-forms to achieve that goal.

Recall that a *del Pezzo manifold* M^n is defined to be a Fano manifold with index $n-1$, where the index is the largest integer r such that $K_M^{-1} = rA$ for an ample divisor A. For $n \geq 3$, such manifolds were completely classified by Fujita in [**8**], arranged by their degree $d = A^n$:

- $d = 1$: $X_6^n \subset \mathbb{P}(1^{n-1}, 2, 3)$, a degree 6 hypersurface in the weighted projective space.
- $d = 2$: $X_4^n \subset \mathbb{P}(1^n, 2)$, a degree 4 hypersurface in the weighted projective space.
- $d = 3$: $X_3^n \subset \mathbb{P}^{n+1}$, a cubic hypersurface.
- $d = 4$: $X_{2,2}^n \subset \mathbb{P}^{n+2}$, a complete intersection of two quadrics.
- $d = 5$: Y^n, a linear section of $\mathbb{G}r(2,5) \subset \mathbb{P}^9$.
- $d = 6$: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, or $\mathbb{P}^2 \times \mathbb{P}^2$, or the flag threefold $\mathbb{P}(T_{\mathbb{P}^2})$.
- $d = 7$: $\mathbb{P}^3 \# \overline{\mathbb{P}^3}$, the blowing up of $\mathbb{P}^3$ at a point.

We believe that by further application of the deep and rich results in algebraic geometry, one should be able narrow things down even more, and in particular in dimension 4, we would like to propose the following:

CONJECTURE 6.7. *A compact complex manifold M^4 of dimension 4 admits a Kähler metric with Ric$^\perp > 0$ if and only if M^4 is biholomorphic to $\mathbb{P}^4$, or $\mathbb{Q}^4$, or a del Pezzo fourfold: X_6^4, X_4^4, X_3^4, $X_{2,2}^4$, Y^4, or $\mathbb{P}^2 \times \mathbb{P}^2$.*

Based on the structural theorems we obtained so far, we see that in dimension $n \leq 4$, the $Ric^\perp > 0$ condition is quite restrictive, it means (assuming the above conjecture holds true) Fano manifolds with index 3 or higher. This of course is more restrictive than $Ric > 0$, which means Fano, or $H > 0$, which is known to be rationally connected but the exact subset is still quite unclear. In fact, even for $\mathbb{P}^2 \# 2\overline{\mathbb{P}^2}$, the blowing up of $\mathbb{P}^2$ at two points, it is still unknown whether it admits a Kähler metric with $H > 0$ or not.

For dimensions 5 or higher, however, there are more examples of $Ric^\perp > 0$ manifolds, for instance, there are examples of Fano manifold with index 1 that lies in the set $\mathcal{M}_n^\perp$ of compact Kähler n-manifolds with $Ric^\perp > 0$. It is unclear how the set $\mathcal{M}_n^\perp$ look like, or how is it related to the Fano or the $H > 0$ class. We don't know if all such manifolds are Fano, even though all the examples in $\mathcal{M}_n^\perp$ that we were able to construct so far are Fano, but we do believe that all manifolds in $\mathcal{M}_n^\perp$ are rationally connected. In any event, we think they should form an interesting class of projective manifolds, and perhaps worth some attention from algebraic geometers.

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