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VANISHING THEOREMS ON COMPLETE KÄHLER MANIFOLDS AND THEIR APPLICATIONS

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1. Introduction

Semi-positive line bundles over compact Kähler manifolds have been the focus of studies for decades. Among them, there are several straddling vanishing theorems such as the Kodaira-Nakano Vanishing Theorem, Vesentini-Gigante-Girbau Vanishing Theorems and Kawamata-Viehweg Vanishing Theorem. As a corollary of the above mentioned vanishing theorems one can easily show that a line bundle over compact Kähler manifolds with negative degree has no non-trivial holomorphic sections. The high cohomology vanishing theorems for non-compact complex manifolds were also studied by several authors. Among them, there are the Nakano's vanishing theorem for Nakano-positive vector bundle over weakly 1-complete manifolds, and Andreotti-Vesentini's vanishing theorem for the q-complete manifolds. In the case where M is a non-compact manifold there are also many works on the finite dimensionness of cohomology group. One of these results proved by N. Mok in [16] gave the finite dimensional estimate for the space of L^2-sections in the case where M is a complete noncompact Kähler manifold with finite volume.

In this paper we first show some vanishing theorems for the L^2-sections of the holomorphic vector bundles over complete nonparabolic Kähler manifolds. By applying the vanishing results and the L^2-estimate of $\bar{\partial}$ of Andreotti-Vesentini, we show, among other things, that if M is a non-parabolic Kähler manifold with nonnegative Ricci curvature and

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E is a quasi-positive line bundle over M, then the degree of E has to be infinity.

On the other hand, a uniformization type theorem for simply-connected complete Kähler manifold M^m of complex dimension m, which was proved twenty years ago by [22], says that if M has non-positive sectional curvature and the sectional curvature decays faster than quadratic, then M is isometric-biholomorphic to $\mathbf{C}^m$. This can be interpreted as a gap phenomenon of the sectional curvature on Kähler manifolds (A more general theorem in Riemannian category was proved later by Greene and Wu in [8]). As a corollary of the result we mentioned in the preceding paragraph we show that there is also a similar gap phenomenon for the Ricci curvature over complete Kähler manifolds when the manifold M is non-parabolic. More precisely, we show that

Suppose M^m is a complete non-parabolic Kähler manifold with non-negative Ricci curvature. If the Ricci curvature is quasi-positive (i.e., the Ricci curvature is semipositive and at least positive at one point) then the total scalar curvature $\int_M S(x)\; dv = \infty$. Similarly if M^m has quasi-negative Ricci curvature (i.e., Ricci curvature is seminegative and strictly negative at least at one point) then $\int_M S(x)\; dv = -\infty$.

Since a manifold of dimension n (real dimension) with nonnegative Ricci curvature has at most polynomial volume growth of order n we can conclude from the above statement that if a nonparabolic Kähler manifold has nonnegative Ricci curvature which is positive at least at one point then the Ricci curvature can not decay very fast. This can be interpreted as an analogy of the gap theorem of Siu-Yau for Ricci curvature.

The above result is also a natural generalization of one of Huber's theorems (cf. [11]), which says that a complete Riemann surface with integrable curvature is parabolic. From our vanishing theorem one can conclude that

If M^m has quasi-positive (quasi-negative) Ricci curvature and integrable scalar curvature, then M is parabolic.

The generalization of above mentioned Huber's theorem along another direction was proved by Peter Li and S. T. Yau in [15], where they proved the Liouville type theorem for bounded pluriharmonic functions instead of harmonic functions. But their assumption on the Ricci curvature is more flexible than our case.

When M is parabolic we show that the finiteness of the degree of a semi-positive line bundle E implies the finite dimensionness of the space of L^2 holomorphic sections of that line bundle. This together with our vanishing theorem generalizes a previous result of Mok in [16].

Suppose (M^m, h) *is a complete Kähler manifold, and* (E, g) *is a Hermitian holomorphic line bundle over* M. *Let* $C(E, g)$ *be the curvature form of* (E, g), *and* $S(x)$ *be the trace of* $C(E, g)$ *with respect to* h. *If* $\int_M S_+(x)\, dv < \infty$, *and* $S(x)$ *is bounded from above, we have*

$$\dim(H^0_{L^2}(M, E^p)) \le Cp^m,$$

for some constant $C = C(M, E)$.

Applying the above result to the canonical bundle of M we have an upper bound for the L^2 plurigenus in terms of the integral of the negative part of the scalar curvature.

By the technique of deforming Hermitian metric on the line bundle we are able to prove similar vanishing result when $S(x)$, the trace of the curvature form, belongs to $L^p(M)$. Using the L^2 $\bar\partial$-method, in the case where M has quasi-positive (quasi-negative) Ricci curvature, we show similar results on the relation between the growth of scalar curvature and the volume growth of the manifold. More precisely we show the following result.

Let M^m *be a complete Kähler manifold with quasi-positive Ricci curvature of complex dimension* m. *If the scalar curvature* $S(x)$ *belongs to* $L^p(M)$ *for some* $p \ge 1$, *then*

$$\int_1^\infty \frac{1}{\left(V_x(\sqrt{t})\right)^{\frac{1}{p}}} dt = \infty \quad \text{for any point } x \in M.$$

This result also can be thought of as a generalized Huber's theorem since the parabolicity is equivalent to $\int_1^\infty \frac{1}{V_x(\sqrt{t})} dt = \infty$ in the case where M has nonnegative Ricci curvature.

In the second part of this paper, by using the vanishing theorems, the results we proved about the quasi-strictly plurisubharmonic functions and the technique of solving Poincaré-Lelong equation as in [17], we prove a general version of Mok-Siu-Yau's gap theorem. More precisely, we have:

Theorem 1.1. *Suppose* M^m *is a complete noncompact Kähler manifold of complex dimension* $m \ge 2$ *with bounded nonnegative holomorphic bisectional curvature. Suppose* M *is a Stein manifold and satisfies* *for some $p \geq 1$*

$$\int_M S^p(x)\ dv_x < \infty, \tag{1.1}$$

and for every $\delta > 0$ there is a positive number $B(\delta)$ such that for any x_0

$$\int_\delta^\infty \frac{1}{\left(V_{x_0}(\sqrt{t})\right)^{\frac{1}{p}}}\ dt \leq B < \infty, \tag{1.2}$$

where $S(x)$ is the scalar curvature and $V_{x_0}(r)$ is the volume of the ball centered at x_0 with radius r. Then M is isometrically biholomorphic to a flat complete Kähler manifold. In particular, if either

(i) M is simply-connected or

(ii) M has maximum volume growth,

then M is isometrically biholomorphic to $\mathbf{C}^m$.

Even in the case where M has maximum volume growth the above result still provides a generalization of Mok-Siu-Yau's theorem since (1.2) is a consequence of volume comparison and (1.1) is weaker than the pointwise decay assumption in [17]. If we replace the condition that M is a Stein manifold by that M is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ we can relax (1.2) by

$$\int_1^\infty \frac{1}{\left(V_{x_0}(\sqrt{t})\right)^{\frac{1}{p}}}\, dt\ <\ \infty \ \text{ for some point } x_0 \in M. \tag{1.3}$$

In this case we will include complex cylinders $T^k \times \mathbf{C}^{m-k}$, where T^k is an Abelian variety. These cases are excluded by the Steinness assumption in Theorem 1.1.

Corollary 1.2. *Suppose M^m is a complete noncompact Kähler manifolds of complex dimension $m \geq 2$ with bounded nonnegative holomorphic bisectional curvature. Suppose M is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ and satisfies (1.1) and (1.3). Then M is isometrically biholomorphic to a flat complete Kähler manifold. In particular, if either*

(i) M is simply-connected or

(ii) M has maximum volume growth,

then M is isometrically biholomorphic to $\mathbf{C}^m$.

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2. Vanishing Theorems

Let (M,h) be a complete Kähler manifold, (E,g) be a Hermitian vector bundle on M. For simplicity we only prove the theorem for the case that E is a line bundle. The proof for general case can follow verbatim from this special case. Let $c_1(E,g) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log g$ be the first Chern class of E. As in the compact case (see for example [9]) we define the degree of a vector bundle to be

$$deg(E) \;=\; \int_M c_1(E)\wedge(\omega_h)^{n-1}.$$

It is well-known that the degree is a holomorphic invariant of E and independent of the choice of Hermitian metrics on E.

To prove our vanishing theorem we first need Bochner type formulae just as other vanishing theorems

Proposition 2.1 (cf. [12]). *Let (E,g) be a Hermitian vector bundle over a Kähler manifold (M,h). Let D be the Hermitian connection of E and $\Theta(E,g)$ be its curvature. Let $\hat{K}$ be the mean curvature of E. If ξ is a holomorphic section of E, then we have*

$$\partial\bar{\partial}\|\xi\|^2 = <g, D\xi\wedge\bar{D}\bar{\xi}> - <\Theta(\xi),\xi>, \tag{2.1}$$

$$\frac{1}{4}\Delta\|\xi\|^2 = Tr_h(<g, D\xi\wedge\bar{D}\bar{\xi}>) - \hat{K}(\xi,\xi), \tag{2.2}$$

where

$$<g, D\xi\wedge\bar{D}\bar{\xi}> = g_{i\bar{j}}D\xi^i\wedge\bar{D}\bar{\xi}^j$$

and

$$\hat{K}(\xi,\xi) = h^{\alpha\bar{\beta}} <\Theta_{\alpha\bar{\beta}}(\xi),\xi> (Tr_h(<\Theta(\xi),\xi>)).$$

$\hat{K}$ is called the mean curvature sometimes.

Remark. We also denote $Tr_h(< g, D\xi \wedge \bar{D}\bar{\xi} >)$ by $\|D\xi\|^2 = \Sigma_{i,j,\alpha,\beta} h^{\alpha\bar{\beta}} D_\alpha \xi^i D_{\bar{\beta}} \bar{\xi^j} g_{i\bar{j}}$.

Proof. See [12] for a proof.

As a corollary of the above proposition we can have the following differential inequality. It also follows as a corollary of the Poincaré-Lelong equation if E is a line bundle.

Corollary 2.2. *Let (E, g) be a Hermitian vector bundle over a Kähler manifold (M, h). Let D be the Hermitian connection of E and $\Theta(E, g)$ its curvature. Let $\hat{K}$ be the mean curvature of E. If ξ is a holomorphic section of E and $f = \|\xi\|$, then we have*

$$f\Delta f - |\nabla f|^2 \geq -2\hat{K}(\xi, \xi). \tag{2.3}$$

In particular, if E is a line bundle we will have

$$f\Delta f - |\nabla f|^2 \geq -2S(x)f^2, \tag{2.4}$$

where $\hat{K}(\xi, \xi) = (Tr_h(< \Theta(\xi), \xi >)) = \Sigma_{i,j,\alpha,\beta} h^{\alpha\bar{\beta}} R^i_{j\alpha\bar{\beta}} \xi^i \bar{\xi^j}$ *and* $S(x) = \Sigma_{\alpha\beta} h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$.

Proof. For a real function f we have

$$|\nabla f|^2 = 2(|D'f|^2 + |D''f|^2) = 4|D'f|^2.$$

On the other hand, direct calculation shows that

$$|\nabla f|^2 = \left|\frac{1}{2}\frac{\nabla f^2}{f}\right|^2 = \frac{1}{4}\frac{|\nabla f^2|^2}{f^2},$$

and

$$|\nabla f^2|^2 = 4|D'f^2|^2 = 4|D' < \xi, \xi > |^2 = 4| < D\xi, \xi > |^2.$$

Using Cauchy-Schwarz inequality we have

$$|\nabla f|^2 \leq \|D\xi\|^2 = \Sigma_{i,j,\alpha,\beta} h^{\alpha\bar{\beta}} D_\alpha \xi^i D_{\bar{\beta}} \bar{\xi^j} g_{i\bar{j}}.$$

Now (2.3) and (2.4) follow easily from the above inequality and (2.2).

Now we can prove our first vanishing theorem. The key here is a theorem of Li-Yau (cf. [15]).

Theorem 2.3. *Let (M, h) be a complete Kähler manifold, and let (E, g) be a Hermitian holomorphic line bundle over M. Suppose that M is nonparabolic and $S_+(x)$, the positive part of $S(x) = Tr_h(\frac{2\pi}{\sqrt{-1}}c_1(E))$, is integrable. Then $H^0_{L^2}(M, E) = \{0\}$.*

Proof. Let $f = \|\xi\|$. By above corollary we will have f satisfies the following differential inequality

$$f\Delta f - |\nabla f|^2 \geq -2qS_+(x)f^2,$$

and $f \in L^2(M)$. On the other hand, the following theorem of [15] implies that if f satisfies the above differential inequality and

$$\int_M S_+(x)\ dv_h < \infty, S_+(x) \geq 0,$$

then $f \equiv 0$, which is contradictory to the fact that ξ is nontrivial. Here is the statement of Li-Yau's theorem, for the convenience of the reader.

Theorem (Li-Yau). *Let M be a complete nonparabolic Riemannian manifold. Assume u is a nonnegative function on M and satisfies*

$$\Delta u \ - \ \frac{|\nabla u|^2}{u} \ \geq \ Ku^{q+1} \ - \ Su,$$

for some $q \geq 0$ and for some function $K \geq 0$ and S on M. If we assume that S_+ is integrable and

$$\int_{B_{q_0}(r)} u^p \ = \ o(r^2),$$

for some positive constant p and fixed point $q_0 \in M$, then u must be identically zero.

In the case where (M, h) is parabolic we can have the following finite dimensionality result which generalizes one of Mok's previous theorems. We can also regard this as a generalization of Li-Yau's vanishing result.

Theorem 2.4. *Let (M^m, h) be a complete Kähler manifold of complex dimension m, and let (E, g) be a Hermitian holomorphic line bundle over M. Suppose that $S_+(x)$, the positive part of*

$$S(x)\ (=\ Tr_h(\frac{2\pi}{\sqrt{-1}}c_1(E))),$$

is integrable and $S(x)$ is bounded from above, then there exists a constant

$$C \ = \ C(M, \|S_+(x)\|_{L^1}) > 0$$

such that $\dim(H^0_{L^2}(M, E^q)) \ \leq \ Cq^m$.

Remarks. Just as before, we can state the similar result for vector bundles. When E in Theorem 2.4 is a quasi-positive line bundle we know that the above upper bound is the sharpest.

Proof. By the so-called Siegel-Poincaré argument we know that in order to prove our dimension estimate we only need a multiplicity estimate for the zero divisor of any L^2-holomorphic section.

Let $\xi \in H^0_{L^2}(M, E^q)$ be any L^2 holomorphic section of E^q. First we claim that $\|D\xi\| \in L^2(M)$. The proof of claim follows from the Bochner formula directly. By Proposition 2.1 we know

$$\frac{1}{4}\Delta\|\xi\|^2 \ = \ \|D\xi\|^2 \ - \ qS(x)\|\xi\|^2.$$

Let φ be a cut-off function supported in $B(x_0, 2r)$ and be equal to 1 in $B(x_0, r)$. Multiplying φ^2 on both sides and integrating by parts we have

$$-\frac{1}{2}\int_M < \nabla\|\xi\|^2, \nabla\varphi > \varphi \, dv + q\int_M S(x)\|\xi\|^2\varphi^2 \, dv \ = \ \int_M \|D\xi\|^2\varphi^2 \, dv.$$

By assumption there exists a constant $A > 0$ such that

$$S(x) \leq A.$$

Similar calculation as in Corollary 2.2 yields

$$|\nabla\|\xi\|^2|^2 \ = \ 4| < D\xi, \xi > |^2 \ \leq \ 4\|D\xi\|^2\|\xi\|^2.$$

Combining above inequalities gives

$$A\int_M \|\xi\|^2\varphi^2 \, dv \ + \ \int_M \|D\xi\| \, \|\xi\| \, |\nabla\varphi|\varphi \, dv \ \geq \ \int_M \|D\xi\|^2\varphi^2 \, dv.$$

Schwarz inequality implies that

$$2A\int_M \|\xi\|^2\varphi^2 \, dv \ + \ \int_M \|\xi\|^2|\nabla\varphi|^2 \, dv \geq \int_M \|D\xi\|^2\varphi^2 \, dv.$$

Letting $r \to \infty$ we have

$$2A\int_M \|\xi\|^2 \, dv \ \geq \int_M \|D\xi\|^2 \, dv.$$

Now we can do the multiplicity estimate. By the Poincaré-Lelong equation we obtain

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|\xi\|^2 \ = \ [\xi] \ - \ qc_1(L).$$

Using the definition of the multiplicity of analytic variety and integrating above equality it is sufficient to show that

$$\int_M \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|\xi\|^2 \wedge (\omega_h)^{n-1} \;=\; \frac{1}{4\pi}\int_M \Delta\log\|\xi\|^2 \; dv \;\leq\; 0.$$

The existence of the above integral will be explained in the following argument. First, for any $\epsilon > 0$, we can show that

$$\Delta\log(\|\xi\|^2+\epsilon) \geq -4qS_+(x).$$

Then the integrals $\int_M \Delta\log(\|\xi\|^2+\epsilon)$ and $\int_M \Delta\log\|\xi\|^2$ make sense (see Royden's Real Analysis page 93, exercise.13). Furthermore, we have

$$\int_M \Delta\log\|\xi\|^2 \;\leq\; \lim_{\epsilon\to 0}\int_M \Delta\log(\|\xi\|^2+\epsilon).$$

In order to show that $\Delta\log(\|\xi\|^2+\epsilon) \geq -4qS_+(x)$, around any point x, we choose local holomorphic coordinates $(z_1, z_2, \ldots, z_n)$ with x to be the origin such that the Kähler metric $h_{\alpha\bar{\beta}}(x) = \delta_{\alpha\bar{\beta}}$ and $dh_{\alpha\bar{\beta}}(x) = 0$. We also choose the holomorphic basis e for E^q such that $\xi = \xi_0 e$ and $g(x) = 1$ and $dg(x) = 0$. Direct calculation as in [16] shows that

$$\frac{1}{4}\Delta\log(\|\xi\|^2+\epsilon) \geq -qS(x)\frac{\|\xi\|^2}{\|\xi\|^2+\epsilon}(x) + \frac{\epsilon\partial\xi_0\wedge\bar{\partial}\bar{\xi_0}}{(\|\xi\|^2+\epsilon)^2}(x) \;\geq\; -qS_+(x).$$

Now the only thing left is to show that

$$\int_M \Delta\log(\|\xi\|^2+\epsilon) \;\leq\; 0.$$

This follows directly from the integrability of $\|D\xi\|^2$.

Let φ be a cut-off function supported in $B_p(r)$, where p is a fixed point in M, and r is the distance function to p. Then

$$\begin{aligned}\int_M \Delta\log(\|\xi\|^2+\epsilon)\varphi^2 \; dv \;&=\; -2\int_M < \frac{\nabla\|\xi\|^2}{\|\xi\|^2+\epsilon}, \nabla\varphi > \varphi \; dv \\ &\leq \frac{8}{\epsilon}\int_M \|D\xi\| \; \|\xi\| \; |\nabla\varphi|\varphi \; dv \\ &\leq \frac{8}{\epsilon}(\int_M \|D\xi\|^2\varphi^2 \; dv)^{\frac{1}{2}}(\int_M \|\xi\|^2 \; |\nabla\varphi|^2 \; dv)^{\frac{1}{2}}.\end{aligned}$$

Letting $r \to \infty$ we have our claim.

As a conclusion we arrive at the following theorem.

Theorem 2.5. *Let (M^m, h) be a complete Kähler manifold and (E, g) be a semi-positive Hermitian line bundle over M. Let $C(E, g)$ be the curvature form of (E, g) and $S(x)$ be the trace of $C(E, g)$ with respect to h. If $\int_M S_+(x)\, dv < \infty$ we have the following:*

(i) If $S(x)$ is bounded from above, then $\dim(H^0_{L^2}(M, E^p)) \leq Cp^m$, for some constant $C = C(M, E)$.

(ii) If M is nonparabolic, then $H^0_{L^2}(M, E^p) = \{0\}$.

As a corollary we obtain

Corollary 2.6. *Let (M^m, h) be a complete Kähler manifold. Suppose that the integral of the negative part of the scalar curvature is finite, and $p_{q,L^2} = \dim(H^0_{L^2}(M, K^q_M))$ is the L^2 q-th plurigenus of M. Then if the scalar curvature is bounded from below, then there exists a constant $C = C(M)$ such that*

$$p_{q,L^2} \leq Cq^n.$$

If M is nonparabolic, then

$$p_{q,L^2} = 0, \text{ for all } q.$$

In the next we are going to establish the vanishing theorem for the case where $S_+(x) \in L^p(M)$ for some $p > 1$. The method of proving these types of vanishing theorems is to deform the metric on the line bundle along the opposite direction of the positive part of $S(x)$, and then to apply Theorem 2.3. The idea of deforming metrics as above was used first by Donaldson [6] to prove the existence of Hermitian-Einstein metric on stable vector bundles. But in our case since our bundle is of rank one we have an easy situation. In [20] we treated the vector bundle case and proved the existence of Hermitian-Einstein metrics for vector bundles over a class of complete Kähler manifolds.

Let g_0 be the Hermitian metric on E at the starting time, and $S_0(x) = 4(-tr_h(\partial\bar{\partial}\log g_0))_+$ and $u(t, x)$ be the solution of the following heat-equation

$$\frac{\partial u}{\partial t} = \Delta u - S_0(x) \qquad u(0, x) = 0. \tag{2.5}$$

We deform the metric by $g_t = g_0 \exp(u)$. Direct calculation shows that

$$S(t, x) = -4tr_h(\partial\bar{\partial}\log g_t)$$

$$= -\Delta u + (-4tr_h(\partial\bar\partial \log g_0))$$
$$= -u_t - (-4tr_h(\partial\bar\partial \log g_0))_-.$$

If we can solve the above equation for $[0,\infty)\times M$ and show that $u_\infty(x) = \lim_{t\to\infty} u(t,x)$ exists with $u_\infty(x) \le 0$ and $\lim_{t\to\infty} u_t(t,x) = 0$, we will have, for the $g_\infty = g_0 \exp(u_\infty)$,

$$S_\infty(x) = -(-4tr_h(\partial\bar\partial \log g_0))_- \le 0.$$

Since $u_\infty \le 0$ we know $\|\xi\|_{g_0} \ge \|\xi\|_{g_\infty}$. Then the vanishing of L^2-sections of (E, g_∞) will imply the vanishing of L^2-sections of (E, g_0). Once we also know that M is nonparabolic we will be able to apply Theorem 2.3 to prove the vanishing theorem for L^2-sections of (E, g_0).

Theorem 2.7. *Let M be a complete Kähler manifold with nonnegative Ricci curvature, and let (E, g) be a Hermitian line bundle on M. If $S_+(x) = (-tr_h(\partial\bar\partial \log g_0))_+$ belongs to $L^p(M)$ for some $p > 1$ and*

$$\int_1^\infty \frac{1}{\left(V_{x_0}(\sqrt{t})\right)^{\frac{1}{p}}} dt < \infty, \quad \textit{for some point } x_0 \in M,$$

then $H^0_{L^2}(M, E) = \{0\}$.

Proof. Let $H(x,y,t)$ be the heat-kernel of M. One can easily verify that

$$u(t,x) = -\int_0^t ds \int_M H(x,y,t-s) S_0(y)\, dv_y, \tag{2.6}$$

provides a solution of (2.5) and

$$u_t(t,x) = -\int_M H(x,y,t) S_0(y) dv_y. \tag{2.7}$$

Clearly $u(t,x) \le 0$, and $v(t,x) = u_t(t,x)$ satisfies the heat equation

$$v_t = \Delta v, \tag{2.8}$$
$$v(0,x) = S_0(x).$$

By the reasoning in the paragraph before Theorem 2.7 we only need to verify that

$$u(\infty,x) = -\int_0^\infty ds \int_M H(x,y,s) S_0(y)\, dv_y$$

exists and

$$\lim_{t\to\infty} u_t(t,x) = \lim_{t\to\infty} -\int_M H(x,y,t)S_0(y)dv_y = 0.$$

But this follows from the well-known heat-kernel estimate of Li-Yau. More precisely,

$$\begin{aligned}
-u_t(t,x) &= \int_M H(x,y,t)S_0(y)dv_y \\
&\le \left(\int_M (H(x,y,t))^q\, dv_y\right)^{\frac{1}{q}} \left(\int_M (S_0(y))^p\, dv_y\right)^{\frac{1}{p}} \\
&\le \left(\sup_{y\in M} (H(x,y,t))^{\frac{q-1}{q}} \left(\int_M H(x,y,t)dv_y\right)^{\frac{1}{q}}\right) \|S_0(y)\|_p \\
&\le \sup_{y\in M} (H(x,y,t))^{\frac{q-1}{q}} \|S_0(y)\|_p.
\end{aligned}$$

Here we have used the fact that

$$\int_M H(x,y,t)dv_y = 1.$$

By the heat-kernel estimate of [15] one has that

$$H(x,y,t) \le c(n)V_x^{-1}(\sqrt{t}).$$

Combining the above estimates gives

$$u_t(t,x) \le C(n)\frac{1}{\left(V_x(\sqrt{t})\right)^{\frac{q-1}{q}}}\|S_0(y)\|_p.$$

By the assumption that M has nonnegative Ricci curvature and

$$\int_1^\infty \frac{1}{\left(V_x(\sqrt{t})\right)^{\frac{q-1}{q}}}dt = \int_1^\infty \frac{1}{\left(V_x(\sqrt{t})\right)^{\frac{1}{p}}}dt < \infty,$$

one can easily see that

$$u_\infty(x) = -\int_0^\infty\ ds \int_M H(x,y,s)S_0(y)\ dv_y$$

exists and

$$\lim_{t\to\infty} u_t(t,x) = \lim_{t\to\infty} -\int_M H(x,y,t)S_0(y)dv_y = 0.$$

Thus we complete our proof by noting the fact that under our assumption M is a nonparabolic manifold.

Remark. When M has nonnegative Ricci curvature one can think the above theorem as a general version of Theorem 2.3 since when $p \to 1$, the condition on the volume growth becomes $\int_1^\infty \frac{1}{V_{x_0}(\sqrt{t})} dt < \infty$, which is equivalent to the assumption that M is nonparabolic.

From the proof of the above theorem one can see that the only thing needed to prove the vanishing theorem is a good enough estimate of the heat kernel. Because of that, using an upper bound for the heat kernel proved by A. Grigor'yan, one can have the following theorem:

Theorem 2.8. *Let M^m be a complete Kähler manifold with complex dimension m, and E be a Hermitian line bundle on M. Suppose $S_+(x) = (-tr_h(\partial\bar{\partial} \log g_0))_+$ belongs to $L^p(M)$ for some $p \geq 1$, and M satisfies one of the following conditions;*

a) M covers a compact manifold with superpolynomial growth deck transformation group , , or

b) M has positive $\lambda_1(M)$.
Then $H^0_{L^2}(M, E) = \{0\}$.

Proof. As in the proof of the last theorem we only need to estimate

$$u_t(t,x) = -\int_M H(x,y,t) S_0(y) dv_y.$$

As before we have that

$$\begin{aligned} -u_t(t,x) &\leq \sup_{y \in M} \left(H(x,y,t)\right)^{\frac{q-1}{q}} \|S_0(y)\|_p \\ &= \sup_{y \in M} \left(H(x,y,t)\right)^{\frac{1}{p}} \|S_0(y)\|_p. \end{aligned}$$

While Grigor'yan's estimate says that $H(x,y,t)$ has exponential decay (cf. [10]). More precisely,

$$H(x,y,t) \leq \text{Const} \exp(-ct^{\frac{\alpha}{\alpha+2}}), \quad \text{for some } c > 0,\ 1 \geq \alpha > 0$$

under the assumption a) and

$$H(x,y,t) \leq \text{Const} \exp(-\lambda_1 t), \quad \text{for } t \geq 1$$

under the assumption b). In both cases,

$$u_\infty(x) = -\int_0^\infty ds \int_M H(x,y,s) S_0(y)\ dv_y$$

exists and

$$\lim_{t\to\infty} u_t(t,x) = \lim_{t\to\infty} -\int_M H(x,y,t)S_0(y)dv_y = 0.$$

Moreover,

$$\int_0^\infty H(x,y,t)\ dt < \infty,$$

which implies that M is a non-parabolic manifold. Thus we complete our proof by the same reasoning as in the proof of the last theorem.

Applying another heat kernel estimate due to Nash [18] we can state the following result.

Theorem 2.9. *Let M^m be a complete Kähler manifold with complex dimension m(real dimension $n = 2m$) such that L^2-Sobolev inequality holds on M, i.e.,*

$$\int_M |\nabla\phi|^2 \geq C_S\|\phi\|^2_{\frac{2n}{n-2}}, \quad \textit{for any } \phi \in C_c^\infty(M),$$

and let E be a Hermitian line bundle on M. If

$$S_+(x) = (-tr_h(\partial\bar\partial \log g_0))_+$$

belongs to $L^p(M)$ for some $m > p \geq 1$, then $H^0_{L^2}(M,E) = \{0\}$.

Proof. As in the proof of the last theorem we only need to estimate

$$u_t(t,x) = -\int_M H(x,y,t)S_0(y)dv_y.$$

As before we have that

$$\begin{aligned} -u_t(t,x) &\leq \sup_{y\in M} (H(x,y,t))^{\frac{q-1}{q}} \|S_0(y)\|_p \\ &= \sup_{y\in M} (H(x,y,t))^{\frac{1}{p}} \|S_0(y)\|_p. \end{aligned}$$

Under our assumption $H(x,y,t)$ can be estimated by $\frac{1}{t^m}$, thanks to an argument of Nash (cf. [18]). More precisely

$$H(x,y,t) \leq C(M)\frac{1}{t^m}.$$

This estimate implies that M is nonparabolic since integrating $H(x,y,t)$ along the direction of time gives a positive Green's function. Under our condition that $m > p$, our argument of the proof of the last theorem completes the proof.

When at the critical case, i.e., $S_+(x) \in L^m(M)$ we still have the following vanishing theorem. This vanishing theorem can be thought of as a gap theorem.

Theorem 2.10. *Let M^m be a complete Kähler manifold with complex dimension m (real dimension $n = 2m$) such that L^2-Sobolev inequality holds on M, i.e.,*

$$\int_M |\nabla \phi|^2 \geq C_S \|\phi\|^2_{\frac{2n}{n-2}}, \quad \text{for any } \phi \in C_c^\infty(M),$$

and let E be a Hermitian line bundle on M. If $S_+(x) = (-tr_h(\partial\bar{\partial}\log g_0))_+$ is bounded and belongs to $L^m(M)$ with

$$\left(\int_M S_+^m(x)\right)^{\frac{1}{m}} < C_S,$$

then $H^0_{L^2}(M, E) = \{0\}$.

Proof. We argue by contradiction. Let ξ be a non-trivial L^2 holomorphic sections. By (2.4), for $f(x) = \|\xi\|(x)$,

$$f\Delta f - |\nabla f|^2 \geq -2S_+(x)f^2.$$

Let $r(x)$ be the distance function to a fixed point $p \in M$, and $\psi(t)$ be a function satisfying that

$$\psi(t) = \begin{cases} 1 & \text{for } t \leq 1 \\ 0 & \text{for } t \geq 2, \end{cases}$$

$$-C_1 \leq \psi'(t) \leq 0.$$

Futher, let $\phi(x) = \psi(\frac{r(x)}{R})$ satisfy

$$\phi(x) = \begin{cases} 1 & \text{for } x \in B_p(R) \\ 0 & \text{for } x \in M \setminus B_p(2R), \end{cases}$$

$$|\nabla\phi|^2 \leq C_1 R^{-2}.$$

Multiplying ϕ^2 on both side of the basic differential inequality and integrating over M we have

$$\int_M (\Delta f) f\phi^2 - \int_M |\nabla f|^2\phi^2 \geq -2\int_M S_+ f^2\phi^2.$$

Integrating by parts yields

$$\begin{aligned}2\int_M S_+ f^2\phi^2 &\geq 2\int_M |\nabla f|^2\phi^2 + 2\int_M <\nabla f, \nabla\phi> f\phi \\ &\geq (2-\epsilon)\int_M |\nabla(f\phi)|^2 - 2(2-\epsilon)\int_M <\nabla f, \nabla\phi> f\phi \\ &\quad - (2-\epsilon)\int_M |\nabla\phi|^2 f^2 \\ &\quad + 2\int_M <\nabla f, \nabla\phi> f\phi + \epsilon\int_M |\nabla f|^2\phi^2.\end{aligned}$$

Thus

$$\begin{aligned}&2\int_M S_+ f^2\phi^2 + (2-\epsilon)\int_M |\nabla\phi|^2 f^2 \\ &\geq (2-\epsilon)\int_M |\nabla(f\phi)|^2 \\ &\quad - 2(1-\epsilon)\int_M <\nabla f, \nabla\phi> f\phi + \epsilon\int_M |\nabla f|^2\phi^2 \\ &\geq (2-\epsilon)\int_M |\nabla(f\phi)|^2 - \epsilon\int_M |\nabla f|^2\phi^2 \\ &\quad - \frac{(1-\epsilon)^2}{\epsilon}\int_M |\nabla\phi|^2 f^2 + \epsilon\int_M |\nabla f|^2\phi^2 \\ &\geq (2-\epsilon)\int_M |\nabla(f\phi)|^2 - \frac{(1-\epsilon)^2}{\epsilon}\int_M |\nabla\phi|^2 f^2.\end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned}\left(\int_M S_+^m\right)^{\frac{1}{m}}\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} + \left((2-\epsilon) + \frac{(1-\epsilon)^2}{\epsilon}\right)\int_M |\nabla\phi|^2 f^2 &\geq (2-\epsilon)\int_M |\nabla(f\phi)|^2 \\ &\geq (2-\epsilon)C_S\left(\int_M (f\phi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.\end{aligned}$$

Let $R \to \infty$. Then

$$\left(\int_M S_+^m\right)^{\frac{1}{m}}\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \geq (2-\epsilon)C_S\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}.$$

Using the assumption of the theorem and choosing a small enough ϵ, one can conclude that

$$\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \equiv 0,$$

which implies that ξ is a trivial section. In the proof we have used the fact that

$$\left(\int_M f^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} < \infty,$$

which follows from the fact $|\nabla f| \in L^2(M)$ (cf. Theorem 2.4) and the Sobolev inequality.

3. Applications of the vanishing theorems

As before, we only state and prove our results for the line bundle case. We can easily see from the proof that they are also true for the vector bundle case.

Theorem 3.1. *Suppose M^m is a complete nonparabolic Kähler manifold of complex dimension m with non-negative Ricci curvature, and E is a quasi-positive (quasi-negative) holomorphic hermitian vector bundle over M. Then* $\deg(E) = \infty$.

The proof of the theorem is based on the L^2-estimate of $\bar{\partial}$ of Andreotti-Vesentini [2].

Theorem 3.2 cf. [2], [5]). *Let (E,g) be a Hermitian line bundle with semi-positive curvature on complete Kähler manifold (M,h) of dimension n. Suppose $\varphi : M \to [-\infty, 0]$ is a function of class C^∞ outside a discrete subsets s of M and, near each point $p \in S$, $\varphi(z) = A_p \log|z|^2$ where A_p is a positive constant and $z = (z_1, z_2, \ldots, z_n)$ are local holomorphic coordinates centered at p. Assume that*

$$\Theta(E, g\exp(-\varphi)) \;=\; \Theta(E,g) \;+\; \partial\bar{\partial}\varphi \geq 0$$

on $M \setminus S$, and $\epsilon : M \to [0,1]$ be a continuous function such that $\Theta(E,g) + \partial\bar{\partial}\varphi \geq \epsilon\omega_h$ on $M \setminus S$. Then, for every C^∞ form θ of type (n,1) with values in L on M which satisfies

$$\bar{\partial}\theta \;=\; 0 \text{ and } \int_M \epsilon^{-1}|\theta|^2 e^{-\varphi}\, dv_h < \infty,$$

there exists a C^∞ form η of type (n,0) with values in L on M such that

$$\bar{\partial}\eta \;=\; \theta \textit{ and } \int_M |\eta|^2 e^{-\varphi}\, dv_h \le \int_M \epsilon^{-1}|\theta|^2 e^{-\varphi}\, dv_h < \infty.$$

From this existence theorem we can easily show the following corollaries.

Corollary 3.3. *Let (E, g) be a Hermitian line bundle with quasi-positive curvature on a complete Kähler manifold M with nonnegative Ricci curvature. Then there exists a positive number q_0 such that there exists nontrivial L^2 holomorphic sections on E^q for $q \ge q_0$. Moreover, there even exists a constant $C = C(m, M, E)$ such that*

$$\dim(H^0_{L^2}(M, E^q)) \ge Cq^m.$$

Remark. The above corollary was first proved in [1]. The dimension estimate was given in [19] for covering spaces. The construction was also used by Siu-Yau in [23] earlier.

Proof. Let $\bar{E}^q = E^q \otimes K_M^{-1}$. It is easy to see $\bar{E}^q$ is semi-positive since M has nonnegative Ricci curvature. Thus we can apply Demailly's theorem to $\bar{E}^q$. The only thing left is to construct the weighted function φ. It is quite standard that for any point $o \in M$ we can construct φ as follows: Let U_0 be a coordinates neighborhood around o, U_i be two nested neighborhoods of o satisfying that $\overline{U_2} \subset U_1, \overline{U_1} \subset U_0$, $\rho(x)$ be a cut-off function which equals zero outside U_0 and equals 1 inside U_1, and $\varphi_p \;=\; \rho(x)\log(|z(x) - z(p)|^2)$. If o is one of the points where $\Theta(E, g)$ is positive, we can arrange U_0 to be the neighborhood such that $\Theta(E, g)$ is positive for any $x \in U_0$. Then we can find a positive number q_0 such that $q_0\Theta(E, g) + \partial\bar{\partial}\varphi_p \ge 0$ on M and $q_0\Theta(E, g) + \partial\bar{\partial}\varphi_p \;>\; 0$ for any $p \in U_2$ and $x \in U_0$. Since

$$\Theta(\bar{E}^q) \;+\; \partial\bar{\partial}\varphi_p = \; q\Theta(E) \;+\; \textit{Ricci}(M) \;+ \partial\bar{\partial}\varphi_p,$$

applying Demailly's theorem we can now use the singularity of φ_p to construct holomorphic sections on $\Omega^0(M, K_M \otimes \bar{E}^q) \;=\; \Omega^0(M, E^q)$ with prescribed local vanishing behavior around o. One can refer to [23] or [19] for detailed estimate of the lower bounds.

Corollary 3.4. *Suppose that M is a complete Kähler manifold with quasi-negative Ricci curvature. Then there exists a positive number q_0 such that there exist nontrivial L^2 holomorphic sections on K_M^q, where*

K_M is the canonical bundle of M. Moreover, there exists a constant $C = C(m, M)$ such that

$$\dim(H^0_{L^2}(M, K^q_M)) \geq Cq^m.$$

Proof. One only needs to notice that in this case K_M is quasi-positive line bundle and $\bar{E}^q = K_M^{q-1}$. Then

$$\Theta(\bar{E}^q) + \partial\bar{\partial}\varphi_p = (q-1)\Theta(E) + \partial\bar{\partial}\varphi_p \geq 0$$

on M and $\Theta(\bar{E}^q) + \partial\bar{\partial}\varphi_p > 0$ on U_0.

The proof of Theorem 3.1 now easily follows from above Corollary 3.3 and the vanishing Theorem 2.3. By the same reasoning and applying Corollaries 3.3 and 3.4, we can have the following result which was mentioned before in the introduction.

Corollary 3.5. *Suppose M^m is a complete non-parabolic Kähler manifold with nonnegative Ricci curvature. If the Ricci curvature is quasi-positive, then the total scalar curvature $\int_M S(x)\ dv = \infty$. Similarly if M^n has quasi-negative Ricci curvature (i.e., Ricci curvature is nonpositive and strictly negative at one point), then $\int_M S(x)\ dv = -\infty$.*

The following generalization of Huber's theorem (cf. [11]) follows verbatim;

Theorem 3.6. *Suppose M^m is a complete Kähler manifold with nonnegative(nonpositive) Ricci curvature. If the Ricci curvature is quasi-positive(quasi-negative) and the total scalar curvature $\int_M S(x)\ dv < \infty$ ($\int_M S(x)dv > -\infty$), then M is parabolic, i.e., there is no non-trivial bounded subharmonic functions on M.*

Remark. One certainly needs the assumption that M has quasi-positive Ricci curvature since the examples constructed by [24] provide Ricci flat nonparabolic complete Kähler manifolds.

The above result can be thought as an upper bound on the volume growth for the Kähler manifolds with quasi-positive Ricci curvature and integrable scalar curvature. By applying the vanishing Theorem 2.7 and Corollaries 3.3 and 3.4, we can get more information on the relation between scalar curvature and the volume growth. This kind of relation between the integrability of certain curvature quantity and the volume growth was studied by [15] and [7] earlier. But the known results are on the integrability of the lower bound of the Ricci curvature in the case of Riemannian manifolds. Our result is totally a Kähler phenomenon and new.

Theorem 3.7. *Let M be a complete Kähler manifold with quasi-positive Ricci curvature. If the scalar curvature $S(x)$ belongs to $L^p(M)$ for some $p \geq 1$, then*

$$\int_1^{\infty} \frac{1}{\left(V_x(\sqrt{t})\right)^{\frac{1}{p}}} dt = \infty, \quad \textit{for any point } x \in M.$$

Recall the order of a Riemannian manifold M is defined as (cf. [4])

$$O(M) = \inf\{\alpha : \lim_{r \to \infty} \inf \frac{V(r)}{r^{\alpha}} < \infty\}.$$

As a corollary of Theorem 3.7 we have

Corollary 3.8. *Let M be a complete Kähler manifold with quasi-positive Ricci curvature. If the scalar curvature $S(x)$ belongs to $L^p(M)$ for some $p \geq 1$, then $O(M) \leq 2p$.*

Similarly, by applying the vanishing Theorems 2.8, 2.9 together with Corollaries 3.3 and 3.4, we can have some restrictions on the complete Kähler manifolds with quasi-negative Ricci curvature.

Theorem 3.9. *Let M be a complete Kähler manifold with quasi-negative Ricci curvature. If the scalar curvature $S(x)$ belongs to $L^p(M)$ for some $p > 1$, then $\lambda_1(M) = 0$.*

Proof. The proof follows easily from Theorem 2.8 and Corollary 3.4.

In the case where $p \geq m$, this is a consequence of the volume growth estimate of [15], and it is true for Riemannian manifolds. But for $p < m$ it is a new result even though we do not know whether it is only true for Kähler manifolds or not.

Theorem 3.10. *Let M^m be a complete Kähler manifold of complex dimension m with nonpositive sectional curvature and quasi-negative Ricci curvature. If the scalar curvature $S(x)$ belongs to $L^p(M)$ for some $1 \leq p < m$, then the fundamental group $\pi_1(M)$ of M must be an infinite group.*

Proof. If M is simply-connected, it is well-know that the Sobolev inequality holds on M. By applying Theorem 2.9 we have vanishing result which will be contradictory to Corollary 3.4. For the case where M has finite fundamental group, we can lift everything to the universal cover and apply the preceding argument.

As a corollary we have the following restriction on the possible Kähler metrics defined on $\mathbf{C}^m$ and $\mathbf{B}^m$.

Corollary 3.11. *There is no such Kähler metric ω_h on either $\mathbf{C}^m$ or $\mathbf{B}^m$ that h has nonpositive sectional curvature, Ricc(h) is quasi-negative and the scalar curvature $S(x)$ belongs to $L^p(M)$ for some $p < m$.*

There are examples showing that for $p > m$ there do exist complete Kähler metrics on both $\mathbf{C}^m$ and $\mathbf{B}^m$ satisfying all the described properties of the above corollary. For the critical case when $p = m$ one can has a similar result by using Theorem 2.10 and Corollary 3.4. We leave this to the interested reader.

As a final remark, it might be interesting to understand when one can have a complex splitting theorem under any one of the assumptions in Theorem 3.1 to Theorem 3.6. In the last section of this paper we will address this question under the assumption that M has nonnegative holomorphic bisectional curvature.

4. Quasi-strictly plurisubharmonic functions

In this section we first study the plurisubharmonic functions on complete nonparabolic Kähler manifolds with nonnegative Ricci curvature. The simplest model of this type of Kähler manifolds is C^n with the standard flat metric. Other nontrivial examples are the quasi-projective Kähler manifolds constructed in [24], which has flat Ricci form and maximum volume growth. Over C^n one can show easily that any nonconstant plurisubharmonic function have at least logarithmical growth. In this section we first show that the same property holds for the quasi-strictly pluriharmonic functions (see the following definition) on Ricci nonnegative Kähler manifolds.

Definition. A plurisubharmonic function f defined on a complex manifold M is called quasi-strictly pluriharmonic if there is a point $p \in M$ such that

$$\partial\bar{\partial} f\ (p)\ > 0.$$

Proposition 4.1. *Let M be a nonparabolic complete Kähler manifold with nonnegative Ricci curvature. If f is a quasi-strictly plurisubharmonic function on M, then*

$$\limsup_{r\to\infty} \frac{f(z)}{\log(r(z))} > 0, \tag{4.1}$$

where r is the distance function to a fixed point $p_o \in M$.

Remarks. i) For parabolic Riemannian manifolds Peter Li and L. F. Tam showed in [13] that for the nonconstant subharmonic functions similar inequality as (4.1) holds. More precisely they showed that if f is a non-constant subharmonic function (by parabolicity we know that f cannot be bounded from above), then there exists a constant $C > 0$ such that

$$C\int_1^{r(z)} \frac{t}{V(t)}\, dt \;\le\; f(z).$$

ii) For a nonparabolic manifold (4.1) will not hold for subharmonic functions anymore since one can have bounded subharmonic functions by definition. Proposition 4.1 can be thought as a generalization of Li-Tam's result on Kähler category. On the other hand it is not clear whether the quasi-strict pluriharmonicity assumption is necessary or not.

Proof. The proof of Proposition 4.1 follows the same line as in the proof of theorems in the last section.

Without loss of generality we assume that $\partial\bar{\partial} f \;>\; 0$ around point p_0. Now we apply Theorem 3.2 to construct L^2 holomorphic functions with respect to certain weighted norm. Just as in Corollary 3.3 we let E be the trivial line bundle, correspondingly $\bar{E} \;=\; K_M^{-1}$. Now we apply Demailly's theorem to $\bar{E}$. In our case we use $\varphi \;=\; (n+1)\rho(x)\varphi_{p_0} + Cf$, where φ_{p_0} is the function with singularity at p_0 as in Corollary 3.2, ρ is a cut-off function, and C is a positive constant. We can always choose C large enough to have

$$\Theta(\bar{E}, g\exp(-\varphi)) \;=\; Ricci(M) \;+\; \partial\bar{\partial}\varphi \;>\; 0$$

around point p_0 and nonnegative outside that neighbourhood. Similarly Demailly's theorem implies that there exists a nonconstant holomorphic function u such that

$$\int_M |u|^2\, e^{-\varphi}\, dv \;<\; \infty. \tag{4.2}$$

Now the proof of theorem follows from an argument by contradiction. Assume that the theorem is not true. We can find $R >> 1$ such that

$$f \le \frac{1}{C}\log(r(z)), \tag{4.3}$$

where C is the positive constant we used in above paragraph. Since ρ is a cut-off function, from (4.2) one can find a compact set $K \subset B(R)$

such that

$$\int_{M\setminus K} |u|^2 \ e^{-Cf} \ < \ \infty.$$

Combining (4.2) and (4.3) we have that

$$\begin{aligned}\int_{B_{p_0}(R_1)} |u|^2 \ dv \ &\leq \int_{B_{p_0}(R)} |u|^2 \ dv \ + \int_{B_{p_0}(R_1)\setminus B_{p_0}(R)} r\frac{|u|^2}{r} \ dv \\ &\leq \int_{B_{p_0}(R)} |u|^2 \ dv \ + \ R_1 \int_{B_{p_0}(R_1)\setminus B_{p_0}(R)} |u|^2 \ e^{-Cf}.\end{aligned}$$

This implies that

$$\int_{B_{p_0}(R_1)} |u|^2 \ dv \ = \ O(R_1).$$

On the other hand, it is easy to see that $|u|$ is a subharmonic function. Therefore if we denote $g = |u|$, then $\Delta g^2 \geq 2|\nabla g|^2$ and for cut-off function φ,

$$\begin{aligned}2\int_M |\nabla g|^2 \varphi^2 \ dv \ &\leq \ \int_M (\Delta g^2)\varphi^2 \ dv \\ &= \ -4\int_M < \nabla g, \nabla\varphi > g\varphi \ dv \\ &\leq \ \int_M |\nabla g|^2 \varphi^2 \ dv + 4\int_M |\nabla\varphi|^2 g^2 \ dv.\end{aligned}$$

By choosing suitable cut-off function, the above inequality leads to that $g = 0$, so that we have u is a constant, which is a contradiction.

Corollary 4.2. *Let M be as in Proposition 4.1. Then there is no quasi-strictly plurisubharmonic function bounded from above on M.*

When M is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ we do not need to assume the quasi-strict pluriharmonicity. In fact we can show:

Proposition 4.3. *Suppose (M, ω_1), a complete Kähler manifold, is a Zarisiki open subset of some compact Kähler manifold $(\overline{M}, \omega_2)$, where, restricted to M, ω_1 and ω_2 are two different metrics. Then there is no non-constant bounded plurisubharmonic function on M.*

Proof. First we should point out that when we refer to metric property in the proof of this proposition we always mean ω_2. Assume f is a plurisubharmonic function on M. Then with respect to ω_2, f is a subharmonic function. The proposition will be proved if we can show that f can be extended to be a subharmonic function on $\overline{M}$. Since f is bounded this can be done. The following argument is adapted from [21].

As we observed we only need to show that as a L^1 function on $\overline{M}$ (the integrability follows from the fact f is bounded and plurisubharmonic) f satisfies

$$\int_{\overline{M}} f\,\Delta\varphi\,dv \geq 0, \qquad \text{for all } \varphi \in C_c^\infty(\overline{M}).$$

By induction we can assume $Y = \overline{M} - M$ is smooth and of codimention $k \leq m$ (here m is the complex dimension of M). Let $G_Y(x)$ be the Green's function of Y. By defintion we have that

$$\int_{\overline{M}} G_Y\,\Delta\varphi\,dv = -\int_Y \varphi\,dv_Y.$$

It can be constructed from the regular Green's function. By the construction of the $G_Y(x)$ we also have the following asymptotic expansion:

$$G_Y(x) \sim \begin{cases} d(x,Y)^{-(2k-2)} & \text{for } k \geq 2, \\ -\log(x,Y) & \text{for } k = 1. \end{cases}$$

As in [21], we can approximate $G_Y(x)$ by a sequence of smooth functions $\phi_N(x)$ with the properties

$$\phi_N(x) \;=\; N, \text{ for } x \text{ close to } Y,$$

$$\int_{\overline{M}} |\Delta\phi_N|\,dv \leq C$$

and

$$\int_{\overline{M}} |\nabla\,\phi|\,dv \leq C.$$

Then

$$\begin{aligned}
\int_{\overline{M}} f\Delta\varphi\,dv &= \lim_{N\to\infty}\int_{\overline{M}} f(1-\frac{1}{N}\phi_N)\Delta\varphi\,dv \\
&= \lim_{N\to\infty}\left(\int_{\overline{M}} f\Delta\left((1-\frac{1}{N}\phi_N)\varphi\right)\,dv + \int_{\overline{M}} f\frac{\Delta\phi_N}{N}\,dv \right.\\
&\qquad \left. +2\frac{1}{N}\int_{\overline{M}} f < \nabla\phi_N, \nabla\varphi > \,dv\right) \\
&\geq \lim_{N\to\infty}\left(\int_{\overline{M}} f\frac{\Delta\phi_N}{N}\,dv + 2\int_{\overline{M}}\frac{1}{N} f < \nabla\phi_N, \nabla\varphi > \,dv\right) \\
&\geq 0.
\end{aligned}$$

The value of f at a point $p \in Y$ can be assigned to be the sub-limit of the average over small balls centered at p. Since any subharmonic function on a compact Riemannian manifold is a constant function, we conclude that f is a constant function. This completes the proof of the proposition.

5. Gap theorem

In this section we apply the results from the previous sections to study the complete Kähler manifolds with nonnegative holomorphic bisectional curvature. Much work have been done on the following conjecture of Yau and Green-Wu.

Conjecture. *Suppose M is a complete noncompact Kähler manifold with positive holomorphic bisectional curvature. Then M is biholomorphic to $\mathbf{C}^m$.*

In [17] N. Mok, Y. T. Siu and S. T. Yau proved above conjecture under some extra conditions on the volume growth and scalar curvature. More precisely, they proved the following gap theorem

Theorem (Mok-Siu-Yau). *Suppose M is a complete noncompact Kähler manifold of complex dimension $m \geq 2$ with bounded nonnegative holomorphic bisectional curvature. Suppose M is a Stein manifold and there exist constants $0 < \epsilon, C_0, C_1 < +\infty$ such that*

$$Vol(B(x_0, r)) \geq C_0 r^{2m}, \quad 0 \leq r + \infty, \tag{5.1}$$

and

$$0 \leq S(x) \leq \frac{C_1}{r(x, x_0)^{2+\epsilon}}, \quad x \in M, \tag{5.2}$$

where $S(x)$ is the scalar curvature and $r(x, x_0)$ is the distance between x and x_0. Then M is isometrically biholomorphic to $\mathbf{C}^m$ with the flat metric.

In this section we will prove an improved version of above theorem.

Theorem 5.1. *Suppose M is a complete noncompact Kähler manifold of complex dimension $m \geq 2$ with bounded nonnegative holomorphic bisectional curvature. Suppose M is a Stein manifold and satisfies for* *some* $p \geq 1$

$$\int_M S^p(x)\, dv_x < \infty, \tag{5.3}$$

and for any $\delta > 0$ *there is a positive number* $B(\delta)$ *such that for any* x_0

$$\int_\delta^\infty \frac{1}{\left(V_{x_0}(\sqrt{t})\right)^{\frac{1}{p}}}\, dt \leq B < \infty, \tag{5.4}$$

where $S(x)$ *is the scalar curvature, and* $V_{x_0}(r)$ *is the volume of the ball centered at* x_0 *with radius* r*. Then* M *is isometrically biholomorphic to a flat complete Kähler manifold. In particular, if either*

(i) M *is simply-connected or*

(ii) M *has maximum volume growth,*

then M *is isometrically biholomorphic to* $\mathbf{C}^m$*.*

Remarks. (i) As before one can see that the assumption (5.4) only makes sense when $p < m$, the complex dimension of the manifold.

(ii) It is not hard to check that if M satisfies (5.1) and (5.2), then there exists a positive number p with $1 \leq p < m$ such that (5.3) and (5.4) hold. Hence the above theorem does generalize Mok-Siu-Yau's theorem.

We first state and prove the following important step in the proof of our theorem.

Theorem 5.2. *Suppose* M *is a complete noncompact Kähler manifold of complex dimension* $m \geq 2$ *with bounded nonnegative holomorphic bisectional curvature. Suppose* M *satisfies (5.3) and (5.4). Then there exists a solution* u *of* $\Delta u = S(x)$ *such that* u *is bounded from above with* $\lim_{x\to\infty} u(x) = 0$ *and satisfies automatically* $\partial\bar{\partial} u = -\partial\bar{\partial}\log h$*, where* $h = \det(h_{\alpha\bar{\beta}})$*.*

Proof. Before we start the proof we fix our notation. Here we denote $Ricci(h) = -\sqrt{-1}\partial\bar{\partial}\log h$ and the scalar curvature $S(x) = -4tr_h(\partial\bar{\partial}\log h)$. And u solves the Poincaré-Lelong equation in the sense that $\sqrt{-1}\partial\bar{\partial} u = Ricci(h)$. In the following we will divide the proof of Theorem 5.2 into several steps.

The first step of the proof is the following proposition which reduces the Poincaré-Lelong equation to a scalar equation.

Proposition 5.3 ([17], [3]). *Let* M *be a complete Kähler manifold of nonnegative holomorphic bisectional curvature. Suppose* ρ *is a d-closed (1,1) form on* M*, and* f *is the trace* ρ *with respect to the Kähler* *metric. Let u be a solution of $\frac{1}{4}\Delta u = f$. Then $\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2$ is subharmonic, where $\|\cdot\|$ denotes the norms measured in terms of the Kähler metric.*

Because of Proposition 5.3, if we can solve the scalar equation $\Delta u = S(x)$ and show that $\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2 \to 0$ as $x \to \infty$, where ρ is the Ricci form, we can conclude that u solves the Poincaré-Lelong equation by the subharmonicity of $\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2$ and the maximum principle.

From the proof of Theorem 2.7 one can easily see that, under our assumption,

$$u(x) = -\int_0^\infty dt \int_M H(x,y,t)S(y)\ dv_y$$

provides a solution of the scalar equation $\Delta u = S(x)$ with $u(x) \le 0$. As the second step of the proof we want to show that $\lim_{x\to\infty} u(x) = 0$, which implies that u is bounded also.

By the explicit expression of $u(x)$ we know that

$$u(x) = \int_0^\infty v\ dt = (\int_0^\delta + \int_\delta^\infty) v\ dt,$$

where as in the proof of Theorem 2.7, $v = \int_M H(x,y,t)S(y)\ dv_y$ satisfies the scalar heat equation and the following estimate

$$|v| \le \begin{cases} \|S\|_{L^\infty} & \text{for } t \le 1, \\ C(n)\|S\|_{L^p}\left(\frac{1}{V_x(\sqrt{t})}\right)^{\frac{1}{p}} & \text{for } t \ge 1. \end{cases}$$

Now we can show that $\lim_{x\to\infty} u(x) = 0$ as follows;

$$\begin{aligned}\lim_{x\to\infty} |u|(x) \le & \lim_{x\to\infty}\left(\int_0^\delta |v|\ dt + \int_\delta^\infty |v|\ dt\right) \\ \le & \delta\|S\|_{L^\infty} + \lim_{x\to\infty}\int_\delta^\infty |v|\ dt.\end{aligned}$$

On the other hand

$$\begin{aligned}\int_\delta^\infty |v|\ dt = & \int_\delta^\infty \left(\int_M H(x,y,t)S(y)dv_y\right)\ dt \\ = & \int_\delta^\infty \left(\int_{B_p(R)} H(x,y,t)S(y)dv_y\right)\ dt \\ & + \int_\delta^\infty \left(\int_{M\setminus B_p(R)} H(x,y,t)S(y)dv_y\right)\ dt\end{aligned}$$

$$\leq \|S\|_{L^\infty} \int_\delta^\infty \left(\int_{B_p(R)} H(x,y,t) dv_y \right) dt$$
$$+ \int_\delta^\infty \left(\int_{M \setminus B_p(R)} H^q \right)^{\frac{1}{q}} \|S\|_{L^p(M\setminus B_p(R))}$$
$$\leq \|S\|_{L^\infty} \int_{B_p(R)} G(x,y) dv_y + \|S\|_{L^p(M\setminus B_p(R))} \int_\delta^\infty \left(\frac{1}{V_x(\sqrt{t})} \right)^{\frac{1}{p}} dt.$$

For any $\epsilon > 0$ we can find a $R >> 1$ such that $\|S\|_{L^p(M\setminus B_p(R))} \leq \epsilon$. Thus we have

$$\lim_{x\to\infty} \int_\delta^\infty |v| \ dt \leq \|S\|_{L^\infty} \lim_{x\to\infty} \int_{B_p(R)} G(x,y) dv_y + \epsilon \int_\delta^\infty \left(\frac{1}{V_x(\sqrt{t})} \right)^{\frac{1}{p}} dt$$
$$= \epsilon \int_1^\infty \left(\frac{1}{V_x(\sqrt{t})} \right)^{\frac{1}{p}} dt \leq \epsilon B.$$

Since ϵ is any positive number, we have

$$\lim_{x\to\infty} \int_\delta^\infty |v| \ dt = 0$$

and

$$\lim_{x\to\infty} u(x) \leq \delta \|S\|_{L^\infty},$$

for any positive number δ. Therefore

$$\lim_{x\to\infty} u(x) = 0.$$

To prove our theorem the only thing we need to show is that

$$\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2 \to 0$$

as $x \to \infty$. By Proposition 5.3 we have $\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2$ is subharmonic. Therefore we can reduce the pointwise estimate to the L^2-estimate due to the mean-value property proved by Li-Schoen. Now we come to the third step of the proof;

Integral estimate of $\|\sqrt{-1}\partial\bar{\partial}u - \rho\|^2$. Let A be the upper-bound of $S(x)$. Then for $p \leq 2$,

$$\int_M \|\rho\|^2 dv \leq \int_M S^2(x) \ dv_x$$

$$\leq A^{2-p} \int_M S^p(x)\ dv_x. \tag{5.5}$$

And for $p > 2$,

$$\begin{aligned} \int_{B_x(R)} \|\rho\|^2\ dv &\leq \int_{B_x(R)} S^2(x)\ dv \\ &\leq \left(\int_{B_x(R)} S^p(x)\ dv \right)^{\frac{2}{p}} (V_x(R))^{\frac{p-2}{p}}. \end{aligned} \tag{5.6}$$

Similarly we have

$$\int_M |\Delta u|^2 \leq A^{2-p} \int_M S^p(x)\ dv_x, \quad \text{for } p \leq 2 \tag{5.7}$$

and

$$\int_{B_x(R)} |\Delta u|^2 \leq \left(\int_{B_x(R)} S^p(x)\ dv \right)^{\frac{2}{p}} (V_x(R))^{\frac{p-2}{p}}, \quad \text{for } p > 2. \tag{5.8}$$

Let $\phi(r(x))$ be the cut-off function as in the proof of Theorem 2.10. Integrating by parts shows

$$\begin{aligned} \int_{B_x(R)} (\Delta u)^2 \phi^2 &= \int_{B_x(R)} u_{ii} u_{jj} \phi^2 \\ &= - \int_{B_x(R)} u_{iij} u_j \phi^2 - 2 \int_{B_x(R)} u_{ii} u_j \phi \phi_j. \end{aligned}$$

Ricci identity yields

$$\begin{aligned} u_{iij} - u_{iji} &= - u_m R_{miji}. \\ \int_{B_x(R)} (\Delta u)^2 \phi^2 &= - \int_{B_x(R)} u_{iji} u_j \phi^2 \\ &\quad + \int_{B_x(R)} Ric(\nabla u, \nabla u) \phi^2 - 2 \int_{B_x(R)} u_{ii} u_j \phi \phi_j \\ &= \int_{B_x(R)} u_{ij} u_{ij} \phi^2 + \int_{B_x(R)} u_{ij} u_j \phi \phi_i \\ &\quad + \int_{B_x(R)} Ric(\nabla u, \nabla u) \phi^2 - 2 \int_{B_x(R)} u_{ii} u_j \phi \phi_j \\ &\geq \frac{1}{2} \int_{B_x(R)} (u_{ij})^2 \phi^2 - 2 \int_{B_x(R)} (u_j)^2 (\phi_i)^2 \\ &\quad - 2 \int_{B_x(R)} u_{ii} u_j \phi \phi_j. \end{aligned}$$

By direct calculation we obtain

$$\int_{B_x(R)} (u_{ij})^2\phi^2 \le 6\int_{B_x(R)} |\nabla u|^2|\nabla\phi|^2 + 4\int_{B_x(R)} |\Delta u|^2\phi^2. \tag{5.9}$$

From Li-Scheon's mean-value inequality, volume doubling property for Ricci nonnegative manifolds and (5.5) - (5.9) for $p \le 2$ it follows that

$$\begin{aligned}
&\|\sqrt{-1}\partial\bar\partial u - \rho\|^2(x)\\
&\quad\le C(M)\frac{1}{V_x(\frac{R}{2})}\int_{B_x(\frac{R}{2})} \|\sqrt{-1}\partial\bar\partial u - \rho\|^2(y)dv_y\\
&\quad\le C(M)\frac{1}{V_x(R)}\left(\int_{B_x(R)} (u_{ij})^2\phi^2 + \int_{B_x(R)} \|\rho\|^2\right)\\
&\quad\le C(M)\left(\frac{1}{V_x(R)}\int_{B_x(R)} |\nabla u|^2|\nabla\phi|^2 + \frac{A^{2-p}\|S(x)\|^p_{L^p}}{V_x(R)}\right).
\end{aligned} \tag{5.10}$$

Moreover, for $p > 2$ we have

$$\begin{aligned}
&\|\sqrt{-1}\partial\bar\partial u - \rho\|^2(x)\\
&\le C(M)\frac{1}{V_x(\frac{R}{2})}\int_{B_x(\frac{R}{2})} \|\sqrt{-1}\partial\bar\partial u - \rho\|^2(y)dv_y\\
&\le C(M)\frac{1}{V_x(R)}\left(\int_{B_x(R)} (u_{ij})^2\phi^2 + \int_{B_x(R)} \|\rho\|^2\right)\\
&\le C(M)\left(\frac{1}{V_x(R)}\int_{B_x(R)} |\nabla u|^2|\nabla\phi|^2 + \frac{\|S(x)\|^2_{L^p}\,(V_x(R))^{\frac{p-2}{p}}}{V_x(R)}\right).
\end{aligned} \tag{5.11}$$

Once we can show that $\frac{1}{V_x(R)}\int_{B_x(R)} |\nabla u|^2(y)$ is bounded, we can finish our proof by taking $R \to \infty$.

Gradient estimate. From above we know since u is bounded from above and below and u is subharmonic, by replacing u with $u + A$ we can always assume that $u \ge 0$ in order to get the integral estimate for the gradient. But this is not hard to have. In fact with a little more effort one can show that

$$\lim_{R\to\infty} \frac{R^2\int_{B_x(R)} |\nabla u|^2}{V_x(R)} = 0.$$

For our use the standard reversed Poincaré inequality will be enough. For completeness we can sketch the proof here.

First let ϕ be the cut-off function satisfying

$$\phi(x) = \begin{cases} 1 & \text{for } x \in B_p(R) \\ 0 & \text{for } x \in M \setminus B_p(2R), \end{cases}$$

$$|\nabla \phi|^2 \le C_1 R^{-2}.$$

Then

$$\begin{aligned} 0 \le & \int_{B_x(2R)} \phi^2 u \Delta u \\ = & -\int_{B_x(2R)} \phi^2 |\nabla u|^2 - 2\int_{B_x(2R)} \phi u \nabla\phi \nabla u. \end{aligned}$$

$$\begin{aligned} \int_{B_x(2R)} \phi^2 |\nabla u|^2 \le & -2\int_{B_x(2R)} \phi u \nabla \phi \nabla u \\ \le & 2\left(\int_{B_x(2R)} \phi^2 |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_{B_x(2R)} u^2 |\nabla \phi|^2\right)^{\frac{1}{2}}, \end{aligned}$$

which implies that

$$\begin{aligned} \int_{B_x(R)} |\nabla u|^2 \le & \frac{C}{R^2}\int_{B_x(2R)} u^2 \\ \le & \frac{C}{R^2} V_x(2R) \|u\|^2_{L^\infty}. \end{aligned}$$

Hence by (5.10) and (5.11) we complete our proof of Theorem 5.2.

The proof of Theorem 5.1 follows the same line of reasoning as in [17]. We will argue by contradiction that the solution of the Poincaré-Lelong equation constructed in Theorem 5.2 is a trivial solution.

Since we assume that M is a Stein manifold, M can be embedded as a closed complex submanifold of some $\mathbf{C}^N$ with coordinates $(z_1, \ldots, z_N)$. Let $\varphi(x)$ be the restriction of $\sum_{i=1}^N |z_i|^2$ to M. Suppose u is not identically zero. Then $M_c = \{u < c\}$ is relatively compact for every $c < 0$ since $u(x) \to 0$ as $x \to \infty$. By the Sard's theorem we can always choose c such that ∇u does not vanish on the boundary of M_c. Let x be a point on ∂M_c such that φ is maximum (on M_c). Then x is a strictly pseudoconvex boundary point of M_c. Since u is a defining function for M_c, $\partial\bar{\partial} u$ is positive definite on the complex tangent space of ∂M_c at x. Thus $\partial\bar{\partial}\exp(u) = \exp(u)\partial\bar{\partial}u + \exp(u)\partial u \wedge \bar{\partial} u$ is then positive definite at x, and therefore we have constructed a quasi-strictly plurisubharmonic function on M, which is bounded from above. By Theorem 4.3 we know it has to be constant, then u is a constant function.

When M is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ we can relax the assumption of Theorem 5.1. More precisely we can have the following results;

Theorem 5.4. *Suppose M^m is a complete noncompact Kähler manifolds of complex dimension $m \geq 2$ with bounded nonnegative holomorphic bisectional curvature. Suppose M is a Zariski open subset of a smooth compact Kähler manifold $\overline{M}$ and satisfies (5.3) and*

$$\int_1^\infty \frac{1}{\left(V_{x_0}(\sqrt{t})\right)^{\frac{1}{p}}}\, dt \;<\; \infty \quad \text{for some point } x_0 \in M. \tag{5.12}$$

Then M is isometrically biholomorphic to a flat complete Kähler manifold. In particular, if either

(i) M is simply-connected or

(ii) M has maximum volume growth,

then M is isometrically biholomorphic to $\mathbf{C}^m$.

Proof. The proof follows the same line as of Theorem 5.1. Notice that if we replace (5.4) by (5.12), we can still solve the Possion equation $\Delta u = S(x)$ as in Theorem 2.7. The thing we lose here is that we can not say $\lim_{x\to\infty} u(x) = 0$. But under the boundedness assumption of $S(x)$ we do know that u is bounded by tracing the proof of Theorem 5.2. On the other hand, all estimates on the first and second derivatives in the proof of Theorem 5.2 remain valid under our assumption here. Therefore we know that u also solves the Poincaré-Lelong equation

$$\sqrt{-1}\partial\bar{\partial} u \;=\; Ricci(h).$$

Now we can use Proposition 4.3 to conclude that u is a constant function since it is a bounded plurisubharmonic function. We therefore conclude that M is flat.

In [17], they also considered the case where M has nonnegative Riemannian sectional curvature and maximum volume growth. We close this section by the following generalization. The proof is simply a combination of our construction of u in the proof of Theorem 5.1, Proposition 4.1, Lemma 2 of [17] applying to the Buseman function and the piecing arguement of [17].

Corollary 5.5. *Let M^m be a complete Kähler manifold with non-negative Riemannian sectional curvature and maximum volume growth. If $S(x) \in L^p(M)$ for some $p < m$, then M is isometrically biholomorphic to C^m.*

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