Ricci flow and manifolds with positive curvature

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Dedicated to Nolan Wallach on his 70th birthday

Abstract This is an expository article based on the author's lecture delivered at the conference *Lie Theory and Its Applications* in March 2011, UCSD. We discuss various notions of positivity and their relations with the study of the Ricci flow, including a proof of the assertion, due to Wolfson and the author, that the Ricci flow preserves the positivity of the complex sectional curvature. We discuss the examples of Wallach of the manifolds with positive pinched sectional curvature and the behavior of Ricci flow on some examples. Finally we discuss the recent joint work with Wilking on the manifolds with pinched flag curvature and some open problems.

Keywords: Positivity of the Curvature • Ricci flow • Flag curvature pinching

Mathematics Subject Classification: 53C44

1 Introduction

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Gauss curvature was defined for surfaces in three-dimensional Euclidean space $\mathbb{R}^3$ by the determinant of the second fundamental form of the embedding with respect to the first fundamental form, namely the induced metric. The *Theorem*

The research of the author was partially supported by NSF grant DMS-1105549.

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R. Howe et al. (eds.), *Symmetry: Representation Theory and Its Applications*,
Progress in Mathematics 257, DOI 10.1007/978-1-4939-1590-3_17

Egregium of Gauss [11] asserts that it is in fact an invariant depending only on the first fundamental form, namely the metric of the given surface. Let (M, g) be a Riemannian manifold with metric $g = g_{ij} dx^i \otimes dx^j$. For any given point $p \in M$, let T_pM be the tangent space at p and let $\exp_p : T_pM \to M$ be the exponential map at p. The concept of the sectional curvature was introduced by Riemann [26], which can be described via the Gauss curvature in the following way. For any two-dimensional subspace σ, say spanned by e_1, e_2 with $\{e_i\}$ being an orthonormal frame of T_pM, take an open neighborhood (of the origin) $U \subset \sigma$, the sectional curvature $K(\sigma)$ is defined by the Gauss curvature of the surface $\exp_p U$ at p. It is the same as $R(e_1, e_2, e_1, e_2)$, where $R(\cdot, \cdot, \cdot, \cdot)$ is the curvature tensor defined by

$$R(X, Y, Z, W) = \langle -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z, W \rangle,$$

which measures the commutability of the second-order covariant differentiations.

Understanding the topology/differential topology of manifolds with positive sectional curvature has been one of the central problems in the study of Riemannian geometry. In this article we shall illustrate how Hamilton's Ricci flow can be applied to study manifolds with positive sectional curvature. In this regard we shall focus on (1) Ricci flow and various notions of positivity; (2) Wilking's general result on the invariance of various positive cones; (3) examples of manifolds with positive sectional curvature, particularly by Wallach and Aloff–Wallach, and on which the Ricci flow does not preserve the positivity of the sectional curvature by the author and by Böhm and Wilking; (4) the most recent classification result by Wilking and the author on manifolds with so-called pinched flag curvature. The selection of the topics is of course completely subjective. One should consult the excellent survey articles [29, 32] on the subject of the manifolds with positive sectional curvature, particularly on more comprehensive overviews about recent progress via other techniques, e.g., the actions of the isometry groups. These articles also contain many more open problems, some of which ambitious readers may find interesting.

Acknowledgment. The author would like to thank Ann Kostant for the careful editing of the article.

2 Ricci flow and preserved positivities

Let $\mathrm{Ric} = r_{ij} dx^i \otimes dx^j$ be the Ricci curvature tensor which is defined as $r_{ij} = g^{kl} R_{ikjl}$. The Ricci flow is a parabolic PDE which deforms a Riemannian metric by its Ricci curvature:

$$\frac{\partial}{\partial t} g_{ij} = -2 r_{ij}. \tag{1}$$

It is parabolic since under a "good coordinate" (precisely the harmonic coordinate),

$$r_{ij} = -\frac{1}{2} g^{st} \frac{\partial^2}{\partial x^s \partial x^t} g_{ij} + o(1).$$

Here $o(1)$ means terms involving at most the first-order derivatives. This also explains the number "2" in the equation (1).

Since the "good coordinates" are not invariant under the flow, to prove the short time existence, the most economic approach is via the De-Turck trick:

First solve the Ricci–DeTurck equation

$$\begin{cases} \frac{\partial g_{ij}(x,t)}{\partial t} = -2\,\mathrm{Ric}_{ij}(g)(x,t) + \nabla_j W_i + \nabla_i W_j, \\ g(x,0) = g_0(x). \end{cases} \tag{2}$$

Here $W_i = g_{ir} g^{st} \left(\Gamma^r_{st} - \widetilde{\Gamma}^r_{st} \right)$ with $\Gamma^r_{st}, \widetilde{\Gamma}^r_{st}$ being the Christoffel symbols for the metric $g_{ij}(x,t)$ and a fixed background metric $\widetilde{g}_{ij}$ respectively. Computation under the local coordinates shows that the Ricci–DeTurck equation is a quasilinear strictly parabolic system, whose short time existence can be proved via, say, a modified standard implicit function theorem argument. Denote its solution by $\bar{g}(x,t)$. Now let W be the vector field given by $W = W^i \frac{\partial}{\partial x^i}$ where $W^j = \bar{g}^{st} \left(\bar{\Gamma}^j_{st} - \widetilde{\Gamma}^j_{st} \right)$. Let Φ_t be the diffeomorphism generated by the vector field $-W(x,t)$. Define $g(x,t) = \Phi_t^*(\bar{g}(x,t))$. Direct calculation shows that

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= \Phi_t^*(-2\,\mathrm{Ric}(\bar{g}) + \bar{\nabla}_i W_j + \bar{\nabla}_j W_i) \\ &\quad + \frac{\partial}{\partial s} \Phi_{t+s}^*(\bar{g}(x,t)) \Big|_{s=0} \\ &= -2\,\mathrm{Ric}(g)(x,t). \end{aligned}$$

This approach avoids appealing to the Nash–Moser inverse function theorem which is the original method adapted by Hamilton in his groundbreaking paper [15].

Tedious, but straightforward calculations show that the curvature tensor of $g(x,t)$ satisfies

$$\begin{aligned} \frac{\partial}{\partial t} R_{ijkl} &= \Delta R_{ijkl} + g^{pq} g^{st} R_{ijpt} R_{klqs} + 2 \left(B_{ikjl} - B_{iljk} \right) \\ &\quad - g^{pq} \left(R_{pjkl} r_{qi} + R_{ipkl} r_{qj} + R_{ijpl} r_{qk} + R_{ijkp} r_{ql} \right). \end{aligned}$$

Here $B_{ijkl} = g^{ps} g^{qt} R_{piqj} R_{sktl}$, r_{ij} being the Ricci curvature.

To make the computation easier, Hamilton in [16] introduced the gauge fixing trick (due to Karen Uhlebeck) to get rid of the last four terms. Let E denote a vector bundle which is isomorphic to TM. Then consider the map $u : E \to TM$ satisfying $\frac{\partial u}{\partial t} = \mathrm{Ric}\, u$. Here, by abusing notation, $\mathrm{Ric}_i^j = r_i^j = r_{ik} g^{jk}$ is viewed as a symmetric transformation of TM.

If we pull back the changing metric on TM by u and call it h, it is easy to see that

$$
\begin{aligned}
\frac{\partial}{\partial t} h(X,Y) &= \frac{\partial}{\partial t} g(u(X),u(Y)) \\
&= -2\,\mathrm{Ric}(u(X),u(Y)) + g(\mathrm{Ric}\,u(X),u(Y)) \\
&\quad + g(u(X),\mathrm{Ric}\,u(Y)) \\
&= 0.
\end{aligned}
$$

As long as the flow exists, u is an isometry between the fixed metric h on E and the changing metric $g(t)$ on TM. Again by possibly abusing notation, we pull back the curvature tensor R at time t, and denote by $\widetilde{R}$,

$$
\widetilde{R}(e_a,e_b,e_c,e_d) = R(u(e_a),u(e_b),u(e_c),u(e_d)).
$$

Using the previous convention we simply abbreviate it as $\widetilde{R}(a,b,c,d)$ or $\widetilde{R}_{abcd}$.

The connection (which shall be denoted by D) can also be pulled back through $u(D_i a) = \nabla_i u(a)$. Hence there exists a time-dependent metric connection D on the vector bundle E. It is easy to see that u is invariant, namely $Du = 0$.

Direct calculation shows that

$$
D_i \widetilde{R}(a,b,c,d) = (\nabla_i R)(u(a),u(b),u(c),u(d)).
$$

On the other hand,

$$
\begin{aligned}
\frac{\partial}{\partial t}\widetilde{R}_{abcd} &= \frac{\partial}{\partial t} R_{u(a)u(b)u(c)u(d)} + R(\mathrm{Ric}\,u(a),u(b),u(c),u(d)) \\
&\quad + R(u(a),\mathrm{Ric}\,u(b),c,d) \\
&\quad + R(u(a),u(b),\mathrm{Ric}\,u(c)),u(d)) \\
&\quad + R(u(a),u(b),u(c),\mathrm{Ric}\,u(d)).
\end{aligned}
$$

Hence

$$
\frac{\partial}{\partial t}\widetilde{R}_{abcd} = \Delta \widetilde{R}_{abcd} + 2\widetilde{\mathrm{Rm}}^2_{abcd} + 2\widetilde{\mathrm{Rm}}^\#_{abcd}. \tag{3}
$$

Here $\widetilde{\mathrm{Rm}}^2$ and $\widetilde{\mathrm{Rm}}^\#$ are the corresponding quadratic operations on $\widetilde{R}$ with

$$
\begin{aligned}
\widetilde{\mathrm{Rm}}^2_{ijkl} &= g^{pq} g^{st} \widetilde{R}_{ijpt} \widetilde{R}_{klqs} \\
\widetilde{\mathrm{Rm}}^\#_{ijkl} &= 2\left(B_{ikjl} - B_{iljk}\right).
\end{aligned}
$$

In [16], Hamilton also observed that there is a Lie algebraic interpretation on the second reaction term in the diffusion reaction equation (3) satisfied by the curvature tensor. First, there exists a natural identification between $\wedge^2\mathbb{R}^n$ and

$\mathfrak{so}(n)$, the Lie algebra of $\mathsf{SO}(n)$. The identification can be done by first defining $X \otimes Y(Z) = \langle Y, Z\rangle X$. Then $e_i \wedge e_j$ can be identified with $E_{ij} - E_{ji}$, where E_{ij} is the matrix with 0 components, except 1 at the (i, j)-th position. The product on $\mathfrak{so}(n)$ is taken to be $\langle v, w\rangle = \frac{1}{2}\operatorname{trace}(v^{\mathrm{tr}} w)$, so that the identification is an isometry.

The curvature tensor can be viewed as a symmetric transformation between $\wedge^2\mathbb{R}^n$ via the equation

$$\langle \mathrm{Rm}(X \wedge Y), Z \wedge W\rangle = R(X, Y, Z, W).$$

We denote all such transformations by $S^2_B(\wedge^2(\mathbb{R}^n))$, where B stands for the first Bianchi identity. For any Rm_1 and $\mathrm{Rm}_2 \in S^2(\wedge^2(\mathbb{R}^n))$ we define $\langle \mathrm{Rm}_1, \mathrm{Rm}_2\rangle = \sum\langle \mathrm{Rm}_1(b^\alpha), \mathrm{Rm}_2(b^\alpha)\rangle$. Here $\{b^\alpha\}$, with $1 \le \alpha \le \frac{n(n-1)}{2}$, is an orthonormal basis of $\mathfrak{so}(n)$.

Lemma 2.1 (Hamilton). *With the above notation,* $\mathrm{Rm}^\#$ *is given via the following equation:*

$$\langle \mathrm{Rm}^\#(v), w\rangle = \frac{1}{2}\sum_{\alpha,\beta}\langle [\mathrm{Rm}(b^\alpha), \mathrm{Rm}(b^\beta)], v\rangle\langle [b^\alpha, b^\beta], w\rangle. \tag{4}$$

This, together with Hamilton's tensor maximum principle which, roughly put, asserts that the "nonnegativity" condition is preserved by the diffusion reaction equation as long as it is preserved by the ODE with the reaction term as the vector fields. This fact immediately implies that the Ricci flow preserves the nonnegativity of Rm, namely the nonnegativity of *the curvature operator*, since clearly $\mathrm{Rm}^2 \ge 0$, and if $\mathrm{Rm} \ge 0$, the above lemma asserts that $\mathrm{Rm}^\# \ge 0$. Thus the reaction term

$$\mathrm{Rm}^2 + \mathrm{Rm}^\# \ge 0$$

as long as $\mathrm{Rm} \ge 0$. This was first obtained in [16].

The second preserved positivity is on the *complex sectional curvature*. To define the terms we need to complexify the tangent bundle at any given point p and denote it as $T^{\mathbb{C}}_p M = T_p M \otimes \mathbb{C}$. Now extend linearly the curvature tensor to $\otimes^4 T^{\mathbb{C}}_p M$. Then we say that Rm has nonnegative complex sectional curvature if for any $X, Y \in T^{\mathbb{C}}_p M$,

$$\langle \mathrm{Rm}(X \wedge Y), \overline{X \wedge Y}\rangle = R(X, Y, \bar{X}, \bar{Y}) \ge 0. \tag{5}$$

It seems that the *nonpositivity of complex sectional curvature* was first introduced in [24] (1985) for Riemannian manifolds. For a Kähler manifold, given any nonzero $X \in T'_p M$ (holomorphic), $Y \in T''_p M$ (anti-holomorphic), then

$$\langle \mathrm{Rm}(X \wedge Y), \overline{X \wedge Y}\rangle < 0$$

is equivalent to *Siu's strong negativity* [25] (the condition introduced a few years earlier, under which Siu proved the holomorphicity of harmonic maps between Kähler manifolds). The following proof first appeared in [22].

Proposition 2.2. *Let (M, g_0) be a compact Riemannian manifold. Assume that $g(t)$ is a solution to (RF) on $M \times [0, T]$ with $g(0) = g_0$. Suppose that g_0 has nonnegative complex sectional curvature. Then $g(t), 0 \leq t \leq T$, has nonnegative complex sectional curvature.*

Proof. View $\langle \mathrm{Rm}(U \wedge V), \overline{U} \wedge \overline{V} \rangle$ as a linear functional $\ell_{U\wedge V}(\cdot)$ on $\mathrm{Rm} \in \mathbb{R}^N$ with N being the dimension of $S^2_B(\wedge^2\mathbb{R}^n)$. The cone $\mathcal{C}_{PCS}$ is defined as the set $\{\mathrm{Rm} \in \mathbb{R}^N \,|\, \ell_{U\wedge V}(\mathrm{Rm}) \geq 0, \text{ for all } U \wedge V\}$. By Hamilton's tensor maximum principle, it suffices to check that the ODE

$$\frac{d\,\mathrm{Rm}}{dt} = Q(\mathrm{Rm}) := \mathrm{Rm}^2 + \mathrm{Rm}^{\#} \tag{6}$$

preserves the cone. It is then sufficient to show the following. If $\mathrm{Rm}_0 \in \partial\mathcal{C}_{PCS}$, which amounts to $\ell_{U_0\wedge V_0}(\mathrm{Rm}_0) = 0$ for some $U_0 \wedge V_0$ and $\ell_{U\wedge V}(\mathrm{Rm}_0) \geq 0$ for all $U \wedge V$, then we need to check that $Q(\mathrm{Rm}_0) \in T_{\mathrm{Rm}_0}\mathcal{C}_{PCS}$. Let K be the collection of all $U_0 \wedge V_0$ satisfying $\ell_{U_0\wedge V_0}(\mathrm{Rm}_0) = 0$. Then at Rm_0, the tangent cone is given by the intersection of halfplanes $\ell_{U\wedge V}(\mathrm{Rm} - \mathrm{Rm}_0) \geq 0$ for all $U \wedge V \in K$. Hence in order to show that the ODE (6) preserves $\mathcal{C}_{PCS}$ it suffices to verify the *null vector condition*: If, for some $\mathrm{Rm} \in \mathcal{C}_{PCS}$, there exists $U \wedge V$ satisfying $\langle \mathrm{Rm}(U \wedge V), \overline{U} \wedge \overline{V} \rangle = 0$, then $\langle Q(\mathrm{Rm})(U \wedge V), \overline{U} \wedge \overline{V} \rangle \geq 0$. Since $\langle \mathrm{Rm}^2(U \wedge V), \overline{U} \wedge \overline{V} \rangle = \langle \mathrm{Rm}(U \wedge V), \overline{\mathrm{Rm}(U \wedge V)} \rangle \geq 0$ always, it suffices to show that $\langle \mathrm{Rm}^{\#}(U \wedge V), \overline{U} \wedge \overline{V} \rangle \geq 0$, which, via the definition, amounts to

$$R_{Up\overline{U}q}R_{Vp\overline{V}q} - R_{Up\overline{V}q}R_{Vp\overline{U}q} \geq 0, \tag{7}$$

where $\{e_p\}$ is a orthonormal basis of T_pM (which is identified to $\mathbb{R}^n$). Now for any U_1 and V_1, consider the function

$$I(z) := \langle \mathrm{Rm}((U + zU_1) \wedge (V + zV_1)), \overline{(U + zU_1) \wedge (V + zV_1)} \rangle$$

which satisfies that $I(z) \geq 0$ and $I(0) = 0$. Hence $\frac{\partial^2}{\partial z \partial \bar{z}} I(z)|_0 \geq 0$, which implies that

$$\begin{aligned} \langle \mathrm{Rm}(U \wedge V_1), \overline{U \wedge V_1} \rangle + 2\mathcal{R}e(\langle \mathrm{Rm}(U \wedge V_1), \overline{U_1 \wedge V} \rangle) \\ + \langle \mathrm{Rm}(U_1 \wedge V), \overline{U_1 \wedge V} \rangle \geq 0. \end{aligned} \tag{8}$$

Let $A_{ij} = R_{iVj\overline{V}} = R_{Vi\overline{V}j}$, $B_{ik} = R_{iV\overline{U}k}$, $C_{kl} = R_{Uk\overline{U}l}$ and $A = (A_{ij})$, $B = (B_{ik})$, $C = (C_{kl})$; then (7) asserts that

$$\mathcal{M}_1 := \begin{pmatrix} A & B \\ \overline{B}^{\mathrm{tr}} & C \end{pmatrix} \geq 0.$$

It is easy to check that (8) is equivalent to $\mathrm{trace}(A\overline{C} - B\overline{B}) \geq 0$, since $\mathcal{M}_1 \geq 0$ implies that

$$\mathcal{M}_2 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ \overline{B}^{\mathrm{tr}} & C \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} C & -\overline{B}^{\mathrm{tr}} \\ B & A \end{pmatrix} \geq 0.$$

The theorem follows from $2\,\mathrm{trace}(A\overline{C} - B\overline{B}) = \mathrm{trace}(\mathcal{M}_1\overline{\mathcal{M}_2}) \geq 0$, a simple fact from the linear algebra. □

This proof was discovered shortly after the proof on the invariance of nonnegativity of isotropic curvature in [6] and [19], which seemed a bit mysterious at the time. We were led to such a notion since at that time it was the only condition left in a table of [14, page 18], whose invariance was not yet clear at the point before the above proof (in 2007) and further development afterwards.

Recall that $X \wedge Y$ is called an isotropic plane if for any $W \in \sigma$ where σ is the plane $\mathrm{span}\{X, Y\}$, $\langle W, W\rangle = 0$. The curvature operator is said to have nonnegative isotropic curvature if (5) holds for any isotropic plane $X \wedge Y$. In [6], it has been observed that $M \times \mathbb{R}^2$ has nonnegative isotropic curvature and is also preserved under the Ricci flow. After we discovered the above presented proof, we suspected that (M, g) having nonnegative complex sectional curvature is equivalent to $M \times \mathbb{R}^2$ nonnegative isotropic curvature. Our speculation was also motivated by an observation of Brendle and Schoen at that time that (M, g) having nonnegative complex sectional curvature is equivalent to $M \times \mathbb{R}^4$ has nonnegative isotropic curvature. When I discussed our speculation with Nolan, I got the confirmed answer the same day! Interested readers are referred to [22] for Nolan's simple proof of this equivalence. In view of this equivalence, the first proof to the proposition was obtained in [6] via the more involved isotropic curvature invariance. The above proof provides a simple alternative.

The *complex sectional curvature* not only has a long root in the study of geometry as pointed out above, but also motivated (according to [30]) the formulation of the following general invariant cone result of Wilking, which provides so far the most general result on invariant conditions, while with the simplest proof (at the same time illuminating the possible previous mystery related to the isotropic curvature).

First we set up some notation. The complexified Lie algebra $\mathfrak{so}(n, \mathbb{C})$ can be identified with $\wedge^2(\mathbb{C}^n)$. Its associated Lie group is $\mathsf{SO}(n, \mathbb{C})$, namely all complex matrices A satisfying $A \cdot A^{\mathrm{tr}} = A^{\mathrm{tr}} \cdot A = \mathrm{id}$. Recall that there exists the natural action of $\mathsf{SO}(n, \mathbb{C})$ on $\wedge^2(\mathbb{C}^n)$ by extending the adjoint action $g \in \mathsf{SO}(n)$ on $x \otimes y$ ($g(x \otimes y) = gx \otimes gy$). For any $a \in \mathbb{R}$, let $\Sigma_a \subset \wedge^2(\mathbb{C}^n)$ be a subset which is invariant under the adjoint action of $\mathsf{SO}(n, \mathbb{C})$. Let $\widetilde{\mathcal{C}}_{\Sigma_a}$ be the cone of curvature operators satisfying that $\langle \mathrm{Rm}(v), \bar{v}\rangle \geq a$ for any $v \in \Sigma_a$. Here we view the space of algebraic curvature operators as a subspace of $S^2(\wedge^2(\mathbb{R}^n))$ satisfying the first Bianchi identity. In [30], the following result is proved.

Theorem 2.3 (Wilking). *Assume that* $(M, g(t))$*, for* $0 \le t \le T$*, is a solution of Ricci flow on a compact manifold. Assume that* $\mathrm{Rm}(g(0)) \in \widetilde{\mathcal{C}}_{\Sigma_a}$*. Then* $\mathrm{Rm}(g(t)) \in \widetilde{\mathcal{C}}_{\Sigma_a}$ *for all* $t \in [0, T]$.

3 Manifolds with positive and nonnegative sectional curvature

Unfortunately, the Ricci flow does not preserve the nonnegativity of the sectional curvature when the dimension is greater than three. This fact may have been known for the ODE (6) long before the concrete geometric example illustrated in [20]. But nothing was written down explicitly before [20]. Moreover, the geometric example says more than that the ODE (6) does not preserve such a condition. Compact examples were constructed later in [7]. But before we present these examples we recall the examples of Wallach [27] and Aloff–Wallach [2] on manifolds with positive sectional curvature since this, together with the above mentioned examples (about Ricci flow non-invariance), shows the subtlety of the sectional curvature.

We say that (M, g) is δ-pinched if $K(\sigma) > 0$ for all σ such two planes and if

$$\frac{\inf_\sigma(K(\sigma))}{\sup_\sigma(K(\sigma))} = r > \delta.$$

By compactness, it is easy to see that if (M, g) has positive sectional curvature, there must be some $\delta > 0$ such that (M, g) is δ-pinched.

Until the work of Marcel Berger ([4], 1961) the only known simply connected manifolds that admitted a $\delta > 0$ pinched structure were the spheres and projective spaces over $\mathbb{C}$ and $\mathbb{H}$ (the quaternions) and the projective plane over the octonions $\mathbb{O}$. Berger proved that two new examples have this property. One is of dimension 7 and another of dimension 13.

In 1969, Wallach set out to classify the homogeneous, simply connected, examples of positive pinching. In 1970, in the Bulletin of AMS he announced a partial result, which, in particular, asserted that in even dimensions the spaces had to be diffeomorphic with spheres and projective spaces over $\mathbb{C}$, $\mathbb{H}$ and the projective plane over the octonions or the full flag variety in $\mathbb{C}^3$ or $\mathbb{H}^3$. A breakthrough came when Wallach realized that he had overlooked one possible example: $F_4/\mathrm{Spin}(8)$, the manifold of flags in the 2-dimensional octonion projective plane.

Theorem 3.1 (Wallach). *The flag varieties in the* 2*-dimensional projective plane over* $\mathbb{C}, \mathbb{H}$ *and the octonions (dimensions* 6*,* 12 *and* 24*), namely* $\mathsf{SU}(3)/T^2, \mathsf{Sp}(3)/(\mathsf{Sp}(1) \times \mathsf{Sp}(1) \times \mathsf{Sp}(1)), \mathsf{F}_4/\mathsf{Spin}(8)$ *admit a homogeneous positive pinching metric.*

He also considered $\mathsf{SU}(3)/T$ with T a circle group embedded in $\mathsf{SU}(3)$. Up to conjugacy these are of the form

$$T_{k,l} = \left\{ \sigma_{k,l}(z) = \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & z^{-(k+l)} \end{pmatrix}, |z| = 1 \right\}$$

where $k, l \in \mathbb{Z}$, gives rise to the spaces $W^7_{k,l} = \mathsf{SU}(3)/T_{k,l}$. The following was the main result of [2].

Theorem 3.2 (Aloff–Wallach). *For each k, l such that $k, l, k+l$ are not 0, there exists a one parameter family of positively pinched metrics $\langle\cdot,\cdot\rangle$ with $0 < t < 1$, yielding $W^7_{k,l,t}$. Moreover*

$$H_4(W^7_{k,l}, \mathbb{Z}) = \mathbb{Z}/(k^2 + l^2 + kl)\mathbb{Z}.$$

This result asserting the infinite topological type of 7-dimensional manifolds with positive sectional curvature shows that the subject is quite intricate since Gromov [13] showed that there exists $C(n)$ such that the Betti numbers of any compact Riemannian manifold with *nonnegative* sectional curvature is bounded by $C(n)$.

Now we explain the examples on Ricci flow invariance. After we told Nolan about our noncompact example [20] and pointed out the question on possible compact examples, he immediately suggested that we study some of the metrics on $\mathsf{SU}(3)/T^2$, which admit nonnegative sectional curvature and share a very similar Lie algebraic structure as the compact examples in [7], which we state below.

Theorem 3.3 (Böhm–Wilking). *On the* 12*-dimensional flag manifold*

$$M = \mathsf{Sp}(3)/(\mathsf{Sp}(1) \times \mathsf{Sp}(1) \times \mathsf{Sp}(1))$$

there exists an $\mathsf{Sp}(3)$*-adjoint homogenous metric g with which, as the initial data shows, the Ricci flow cannot preserve the positivity of the sectional curvature.*

The metrics in [20], where the Ricci flow does not preserve the nonnegativity of the sectional curvature, reside on noncompact manifolds. Precisely, they are complete metrics on the total space of the tangent bundle over spheres. The fact that the Ricci flow solution with bounded curvature does not preserve the nonnegativity follows from the following structure result proved in [20].

Theorem 3.4. *Let $(M, g_{ij}(x,t))$ be a solution to the Ricci flow with nonnegative sectional curvature. Assume also that M is simply-connected. Then M splits isometrically as $M = N \times M_1$, where N is a compact manifold with nonnegative sectional curvature. M_1 is diffeomorphic to $\mathbb{R}^k$ and for the restriction of metric $g_{ij}(x,t)$ on M_1 with $t > 0$, there is a strictly convex exhaustion function on M_1. Moreover, the soul of M_1 is a point and the soul of M is $N \times \{o\}$ if o is the soul of M_1.*

It was remarked in [7] that on the 6-dimensional manifold $\mathsf{SU}(3)/T^2$, there exists a metric of positive sectional curvature, which is not preserved by the Ricci flow. It would be interesting to find out whether or not such a four-dimensional compact example exists. Due to a general convergence theorem in the next section, one cannot expect that the ODE (6) preserves the nonnegativity of the sectional curvature. The intricacy of the problem in dimension 4 is of course also related to the celebrated Hopf conjecture on the existence of a positively curved metric on $\mathbb{S}^2 \times \mathbb{S}^2$. It is also interesting to find out if such a metric exists on the seven-dimensional examples of [2].

It has been computed that the pinching constant δ on the nonsymmetric examples with positive sectional curvature is rather small (considerably smaller than $1/4$ for example).

Recently, Cheung and Wallach [10] gave a detailed study on how the sectional curvature evolves under the Ricci flow of homogenous metrics on flag varieties.

4 Flag curvature pinching

First, we start with a general Ricci flow convergence theorem, which first appeared in [29], since this result and the above examples of Berger, Wallach and Aloff–Wallach also illuminate the reason why the Ricci flow does not preserve the sectional curvature.

Theorem 4.1 (Böhm–Wilking). *Let $\mathcal{C}$ be an $\mathsf{O}(n)$-invariant convex cone of full dimension in the vector space of algebraic curvature operators $S^2_B(\mathfrak{so}(n))$ with the following properties:*

(i) *$\mathcal{C}$ is invariant under the ODE* $\frac{d\,\mathrm{Rm}}{dt} = \mathrm{Rm}^2 + \mathrm{Rm}^\#$.
(ii) *$\mathcal{C}$ contains the cone of nonnegative curvature operators, or slightly weaker all nonnegative curvature operators of rank* 1.
(iii) *$\mathcal{C}$ is contained in the cone of curvature operators with nonnegative sectional curvature.*

Then for any compact manifold (M, g) whose curvature operator is contained in the interior of $\mathcal{C}$ at every point $p \in M$, the normalized Ricci flow evolves g to a limit metric of constant sectional curvature.

Assume that the nonnegativity of the sectional curvature is preserved (in the sense of ODE); then the above result would conclude that any manifold with positive sectional curvature is a space form.

We should remark that the above result was first proved in [8] for $\mathcal{C}$ being the cone of nonnegative curvature operators. Then, it was observed in [6] that the proof of [8] for the case of $\mathcal{C}$ being the nonnegative curvature operator cone can be transplanted, verbatim, to cover the case where $\mathcal{C}$ is the cone of nonnegative complex sectional curvatures. It appeared first in [29] with the above generality. In [5], a slightly different argument was adapted to prove the above result for the case of $\mathcal{C}$ being the cone of the nonnegative complex sectional curvatures.

Flag curvature pinching was first introduced by Andrews–Nguyen [3], who proved a $1/4$-flag pinching condition is invariant under the Ricci flow in dimension four and obtained a classification result for such manifolds in dimension four. First we introduce the definition.

Assume that (M, g) has nonnegative sectional curvature. Fixing a point $x \in M$, for any nonzero vector $e \in T_x M$, we define the flag curvature in the direction e by the symmetric bilinear form $R_e(X, X) = R(e, X, e, X)$. Restricting $R_e(\cdot, \cdot)$ to the subspace orthogonal to e, it is semi-positive definite. We say that (M, g) *has λ-pinched flag curvature ($1 > \lambda \geq 0$) if the eigenvalues of the symmetric bilinear form $R_e(\cdot, \cdot)$, restricted to the subspace orthogonal to e, are λ-pinched for all nonzero vectors e*, namely

$$R_e(X, X) \geq \lambda(x) R_e(Y, Y) \tag{9}$$

for any X, Y in the subspace orthogonal to e, with $|X| = |Y|$.

The λ-pinched flag curvature condition is equivalent to saying that $K(\sigma_1) \geq \lambda K(\sigma_2)$ for a pair of planes σ_1 and σ_2 such that $\sigma_1 \cap \sigma_2 \neq \{0\}$.

It is easy to see that if an algebraic curvature operator has λ-pinched flag curvature, then its sectional curvature is λ^2-pinched. This estimate is indeed sharp. Precisely, in [21] there exists an example of an algebraic curvature operator, such that its $1/4$-flag pinched and its sectional curvature are no better than $1/16$-pinched.

The first result of [21] is a classification result.

Theorem 4.2. *Let (M^n, g) be a compact nonnegatively curved Riemannian manifold with $1/4$-pinched flag curvature and the scalar curvature $\mathrm{Scal}(x) > 0$ for some $x \in M$. Then (M, g) is diffeomorphic to a spherical space form or isometric to a finite quotient of a rank-one symmetric space.*

In view of the convergence result Theorem 4.1, the key towards the above result is the following.

Theorem 4.3. *Let (M^n, g) be a nonnegatively curved Riemannian manifold. If (M, g) has a quarter pinched flag curvature, then (M, g) has nonnegative complex sectional curvature.*

If we assume the stronger assumption that the sectional curvature is $1/4$-pinched, the nonnegativity of the complex sectional curvature was essentially proved earlier by Hernández [17] and Yau–Zheng [31] in the 1990s. (What was proved there is that if a curvature operator has negative sectional curvature and $1/4$-pinched sectional curvature, then it must have nonpositive complex sectional curvature. By flipping the sign, the argument can be transplanted to the case of nonnegative sectional curvature.) An immediate consequence of this fact, together with Proposition 2.2, Theorem 4.1, is Brendle–Schoen's sectional curvature $1/4$-pinching sphere theorem [6].

For the proof of Theorem 4.3, first observe the following lemma.

Lemma 4.4. *Given any complex plane $\sigma \subset \mathbb{C}^n = \mathbb{R}^n \otimes \mathbb{C}$, where $\mathbb{R}^n$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ which is extended bilinearly to $\mathbb{C}^n$, there must exist unit vectors $U, V \in \sigma$ such that*

$$\langle U, U\rangle, \langle V, V\rangle \in \mathbb{R} \text{ with } 1 \geq \langle U, U\rangle \geq \langle V, V\rangle \geq 0,\ \langle U, V\rangle = \langle U, \bar{V}\rangle = 0.$$

Particularly, if $U = X + \sqrt{-1}Y$, $V = Z + \sqrt{-1}W$, *it implies that*

$$|X| \geq |Y|, |Z| \geq |W| \text{ and } \{X, Y, Z, W\}$$

are mutually orthogonal.

Proof. Let $f(\widetilde{U}) \doteqdot Re\left(\langle \widetilde{U}, \widetilde{U}\rangle\right)$ be the functional defined on the unit sphere (with respect to the norm $|\widetilde{U}| = \sqrt{\langle \widetilde{U}, \bar{\widetilde{U}}\rangle}$) inside σ. Let U be the maximizing vector, at which f attains the maximum $\bar{\lambda}$, with $\bar{\lambda} \in [0, 1]$. Clearly for such U, $f(U) = |\langle U, U\rangle|$. Let V be a unit vector such that it is perpendicular to U (namely $\langle U, \bar{V}\rangle = 0$). By the maximizing property of U, from the first variation, it is easy to see that $\langle U, V\rangle = 0$ for any $V \in \sigma$ with $\langle U, \bar{V}\rangle = 0$ and $|V| = 1$. To see this let $h(\theta) = f(\cos\theta U + \sin\theta V)$. Since $h(0) = \bar{\lambda} \geq h(\theta)$, we have $h'(0) = 0$, which, together with the same conclusion with V replaced by $Ve^{\sqrt{-1}\pi/2}$, implies the claim. Among all possible choices of such V, which can be parametrized by S^1, there clearly exists one with $\langle V, V\rangle \geq 0$. □

It is clear from the proof that σ is isotropic if and only if $\bar{\lambda}(\sigma) = 0$. It also makes sense to define $\underline{\lambda}(\sigma)$ to be the minimum of $|\langle U, U\rangle|$ for any $U \in \sigma$ with unit length. Since the inner product induces one on the space of 2-planes $\sigma = U \wedge V$, similarly one may define the $\mu(\sigma)$ as $|\langle U \wedge V, U \wedge V\rangle|$ among all $\sigma = U \wedge V$ of unit length. We may call σ *weakly isotropic* if $\underline{\lambda}(\sigma) = 0$. Clearly both σ being isotropic and σ being weakly isotropic are invariant under the adjoint action of $\mathsf{SO}(n, \mathbb{C})$. Hence Theorem 2.3 implies the Ricci flow invariance on the complex sectional curvature nonnegativity for all such 2-planes.

We may define for any $a, b \in [0, 1]$, $\Sigma^{\bar{\lambda}}_a = \{\sigma\,|\bar{\lambda}(\sigma) \leq a\}$, $\Sigma^{\mu}_b = \{\sigma\,|\mu(\sigma) \leq b\}$ and $\Sigma_{a,b} = \{\sigma\,|\bar{\lambda}(\sigma) \leq a, \mu(\sigma) \leq b\}$. It is a natural question to ask if the nonnegativity on $\Sigma^{\bar{\lambda}}_a$, Σ^{μ}_b, or $\Sigma_{a,b}$ is invariant for any $(a, b) \in [0, 1] \times [0, 1]$ since Theorem 2.3 implies that it is the case when $(a, b) = (0, 0)$ and $(a, b) = (1, 1)$. Related to this, it is also interesting to ask whether or not the condition

$$\langle \mathrm{Rm}(U \wedge V), \overline{U \wedge V}\rangle + \underline{\lambda}(\sigma)\|U \wedge V\|^2 \geq 0 \tag{10}$$

is preserved under the Ricci flow. Here σ is the plane spanned by $\{U, V\}$.

The key to Theorem 4.3 is the following result generalizing a useful lemma of Berger.

Proposition 4.5. *Assume that* (M, g) *has* λ*-pinched flag curvature with dimension* $n \geq 4$. *Assume that the sectional curvature is nonnegative at* x *and* $X, Y, Z, W \in T_xM$ *are four vectors mutually orthogonal. Then*

$$6\frac{1+\lambda}{1-\lambda}|R(X, Y, Z, W)| \leq k(X, Z) + k(Y, Z) + k(X, W) + k(Y, W) + 2k(X, Y) + 2k(Z, W).$$

If equality holds and $\mathrm{Rm}(x) \neq 0$, *then vectors* X, Y, Z, W *have the same norm.*

In [21], results were obtained for manifolds with flag-pinching constant below $1/4$ (note that flag curvature pinching is always *pointwise*).

Theorem 4.6. *For any dimension $n \geq 4$ and $C > 0$, there is an $\epsilon > 0$ such that the following holds. Let (M^n, g) be a nonnegatively curved Riemannian orbifold of dimension n with $\frac{1-\epsilon}{4}$ pinched-flag curvature and scalar curvature satisfying $1 \leq \text{Scal} \leq C$. Then the following holds.*

(i) When $n = 2m + 1$, M admits a metric of constant curvature;
(ii) When $n = 2m$, either M is diffeomorphic to the quotient of rank one symmetric space by a finite isometric group action or it is diffeomorphic to the quotient of a weighted complex projective space by a finite group action.

If one replaces the flag pinching (pointwise) condition by a global sectional curvature pinching, a similar result was obtained by Petersen and Tao [23] earlier.

Since here ϵ is depending on n, we would like to point out a related result and some open problems. A theorem of Abresch and Meyer [1] asserts that *any simply connected odd-dimensional manifold with sectional curvature K satisfying $\frac{1}{4(1+10^{-6})^2} \leq K \leq 1$ is homeomorphic to a sphere.* Note that here a *global* instead of *pointwise* pinching is assumed. An obvious question arises whether or not one can weaken the assumption to a pointwise one and improve the conclusion from the homeomorphism to the diffeomorphism.

Since Micallef–Moore [18] proved (using harmonic spheres) that *any simply-connected manifold with positive isotropic curvature is a homotopy sphere (hence homeomorphic)*, it is natural to ask if this can be improved to diffeomorphic.

In [28] Wilking obtained homotopic classification result for manifolds with positive curvature and "large" enough symmetry. Can the method of using the isometry group and the method of the Ricci flow be combined to get a better result?

Grove–Shiohama [12] (see also [9, Theorem 6.13]) proved *a sphere theorem by assuming that the sectional curvature is bounded from below by one (namely $K \geq 1$) and* $\text{diam}(M) > \frac{\pi}{2}$. Can this be upgraded to a "diffeomorphism"?

References

1. Abresch, U.; Meyer, W.- T., A sphere theorem with a pinching constant below $\frac{1}{4}$. (English summary) *J. Differential Geom.* **44** (1996), no. 2, 214–261.
2. Aloff, S.; Wallach, N. R., An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures. *Bull. Amer. Math. Soc.* **81** (1975), 93–97.
3. Andrews, B.; Nguyen, H., Four-manifolds with 1/4-pinched flag curvatures. (English summary) *Asian J. Math.* **13** (2009), no. 2, 251–270.
4. Berger, M., Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. (French) *Ann. Scuola Norm. Sup. Pisa (3)* **15**(1961), 179–246.
5. Brendle, S., Ricci flow and the sphere theorem. *GSM* **111**, AMS 2010.
6. Brendle, S.; Schoen, R., Manifolds with $1/4$-pinched curvature are space forms. *J. Amer. Math. Soc.* **22** (2009), no. 1, 287–307.

7. Böhm, C.; Wilking, B., Nonnegatively curved manifolds with finite fundamental groups admit metrics with positive Ricci curvature. *Geom. Funct. Anal.* **17** (2007), no. 3, 665–681.
8. Böhm, C.; Wilking, B., Manifolds with positive curvature operators are space forms. *Ann. of Math.* (2) 167 (2008), no. 3, 1079–1097.
9. Cheeger, J.; Ebin, D., *Comparison theorems in Riemannian geometry.* North-Holland Mathematical Library, Vol. 9. North-Holland Publishing Co., New York, 1975.
10. Cheung, M.; Wallach, N., *Ricci flow and curvature on the variety of flags on the two dimensional projective space over the complexes, quaternions and the octonions.* Preprint, 2012. To appear in proceedings of AMS
11. Do Carmo, M., *Differential Curves and Surfaces.* Prentice Hall, 1976.
12. Grove, K.; Shiohama, K., A generalized sphere theorem. *Ann. of Math.* **106**(1977), 201–211.
13. Gromov, M. Curvature, diameter and Betti numbers. *Comment. Math. Helv.* **56** (1981), no. 2, 179–195.
14. Gromov, M., Positive curvature, macroscopic dimension, spectral gaps and higher signatures. (English summary) *Functional analysis on the eve of the 21st century*, Vol. *II* (New Brunswick, NJ, 1993), 1–213, Progr. Math., Vol. 132, Birkhäuser Boston, 1996.
15. Hamilton, R. S., Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), no. 2, 255–306.
16. Hamilton, R. S., Four-manifolds with positive curvature operator. *J. Differential Geom.* **24** (1986), no. 2, 153–179.
17. Hernández, L., Kähler manifolds and 1/4-pinching. *Duke Math. J.* **62** (1991), no. 3, 601–611.
18. Micallef, M. J.; Moore, J. D., Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. *Ann. of Math.* (2) **127** (1988), no. 1, 199–227.
19. Nguyen, H. T., Isotropic curvature and the Ricci flow. *Int. Math. Res. Not.*, 2010, no. 3, 536–558.
20. Ni, L., Ricci flow and nonnegativity of sectional curvature. (English summary) *Math. Res. Lett.* **11** (2004), no. 5–6, 883–904.
21. Ni, L.; Wilking, B., Manifolds with 1/4-pinched flag curvature. (English summary) *Geom. Funct. Anal.* **20** (2010), no. 2, 571–591.
22. Ni, L.; Wolfson, J., Positive complex sectional curvature, Ricci flow and the differential sphere theorem. ArXiv:0706.0332, 2007.
23. Petersen, P.; Tao, T., Classification of almost quarter-pinched manifolds. (English summary) *Proc. Amer. Math. Soc.* **137** (2009), no. 7, 2437–2440.
24. Sampson, J. H., Harmonic maps in Kähler geometry. Harmonic mappings and minimal immersions (Montecatini, 1984), 193–205, *Lecture Notes in Math.*, Vol. 1161, Springer, Berlin, 1985.
25. Siu, Y. T., The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. of Math.* (2) 112 (1980), no. 1, 73–111.
26. Spivak, M., *Differential Geometry.* 2nd Vol. Publish or Perish, 1979.
27. Wallach, N. R., Compact homogeneous Riemannian manifolds with strictly positive curvature. *Ann. of Math.* (2) 96 (1972), 277–295.
28. Wilking, B., Positively curved manifolds with symmetry. *Ann. of Math.* (2) **163** (2006), no. 2, 607–668.
29. Wilking, B., Nonnegatively and positively curved manifolds. *Surveys in differential geometry.* Vol. XI, 25–62, Surv. Differ. Geom., 11, Int. Press, Somerville, MA, 2007.
30. Wilking, B., A Lie algebraic approach to Ricci flow invariant curvature condition and Harnack inequalities. *J. Reine Angew. Math.* **679**(2013), 223–247.
31. Yau, S.-T.; Zheng, F., Negatively $\frac{1}{4}$-pinched Riemannian metric on a compact Kähler manifold. *Invent. Math.* **103** (1991), no. 3, 527–535.
32. Ziller, W., Examples of Riemannian manifolds with non-negative sectional curvature. *Surveys in differential geometry.* Vol. XI, 63–102, Int. Press, Somerville, MA, 2007.