Monotonicity and holomorphic functions

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Abstract

This is a survey on some recent works, mainly by the author on the relation between holomorphic functions on Kähler manifolds, monotonicity and the geometry of complex manifolds. We also use this opportunity to give details of a sketched step in the proof of a previously established result.

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The study of the properties of holomorphic functions has been a venerable subject for many decades. The purpose of this article is two-folded. First we would like to survey recent progresses in understanding the existence of holomorphic functions and the dimension estimate of the space of holomorphic functions of polynomial growth on a complete noncompact Kähler manifolds. As the second purpose, we explain its connection with the monotonicity and the Kähler-Ricci flow. Since most results mentioned in this article are intimately related to problems proposed by Yau, it seems appropriate that we contribute this survey on the occasion of Professor Yau's 60th birthday.

There are at least two problems, Problem 48 and Problem 63 from [Y2], address the existence, the dimension estimate and the finite generation of holomorphic functions of polynomial growth on Kähler manifolds. They are all motivated by the uniformizations of the Kähler manifolds with positive curvature. (See for example, page 117 of [Y1].)

First we address the existence. The following result was proved in [N3].

Theorem 1 *Let (M^m, g_0) be a complete Kähler manifold with bounded nonnegative holomorphic bisectional curvature and maximum volume growth. Then the transcendence degree of the rational function field $\mathcal{M}(M)$ (the quotient field of the ring of the holomorphic functions of polynomial growth) is equal to m.*

Although the above result appears not to be related to either Problem 48 or 63 mentioned above, it is connected to another problem raised by Yau. In [Y3], the following question was asked, motivated by W.-X. Shi's work on the long time existence of Kähler-Ricci flow [Shi]:

Assume that (M, g) is a complete Kähler manifold of complex dimension m with bounded nonnegative bisectional curvature. Given that M is of maximum volume growth, namely

$$\frac{V_x(r)}{r^{2m}} \geq \delta > 0,$$

where $V_x(r)$ is the volume of the ball of radius r, does it imply that M has quadratic curvature decay in average sense, namely does it imply that the scalar curvature $S(y)$ satisfies

$$\frac{1}{V_x(r)}\int_{B_x(r)} S(y)\, d\mu(y) \le \frac{C}{(1+r)^2} \tag{1}$$

for some C, independent of x?

In [N3], Corollary 1, the author gave an affirmative answer to this question. Before this resolution, there are some previous related works. In a paper of 2004, Chen, Tang and Zhu, for the case of dimension 2, embedding in several steps leading to a generalization of a uniformization result of Mok on Kähler surfaces, obtained a weaker average scalar curvature decay result. Namely they showed that, for Kähler surfaces with bounded nonnegative bisectional curvature, the scalar curvature satisfies the estimate $\int_{B_x(r)} S(y)\frac{1}{r^2(y)}\, d\mu(y) \le C\log(2+r)$, under the assumption of maximum volume growth. In a paper by L.-F. Tam and the author [NT1] appeared in 2003, results on the average scalar curvature decay have already been proved under the assumption on the existence of nontrivial holomorphic functions of certain growth. The existence of the nonconstant holomorphic functions of polynomial growth is related to the solution of the above problem of Yau in the following way (reserving the earlier logic in [NT1]):

The construction of holomorphic functions/sections of a line bundle is usually via the L^2-estimates of the $\bar\partial$-operator (see for example, [D]), which in turn relies on the existence of a (strictly) plurisubharmonic function of certain growth, serving as the weight/singular metric in the basic formulation of the L^2-estimate. By some standard elliptic estimates, the growth of the constructed holomorphic functions is determined by the growth of the plurisubharmonic function. In particular, the existence of holomorphic functions of polynomial growth demands a plurisubharmonic function of logarithmic growth as the weight. Since the usual geometric construction related to the Busemann functions only provide plurisubharmonic functions of linear growth (which however is sufficient to construct holomorphic functions of order not above 1), one has to appeal to other means, such as solving some partial differential equations, to construct such a plurisubharmonic function. In this case it is the so-called Poincaré-Lelong equation, a over-determined linear equation, looking for solution u such that $\sqrt{-1}\partial\bar\partial u$ equals to a given closed real $(1,1)$-form. In this case, the natural choice of the given closed form is the Ricci form, $\mathrm{Ric} = \sqrt{-1}R_{i\bar j}dz^i \wedge d\bar z^j$. The general ground work of solving the Poincaré-Lelong equation goes back, at least (possibly much earlier) to the work of [MSY]. Since the solution is not unique, the focus is on the minimal solution and the sharp estimates related. The result needed for our purpose is first proved in [NST], which concludes that if the trace of the Ricci form, the scalar curvature $S(y)$ satisfies (1), then the best solution to $\sqrt{-1}\partial\bar\partial u = \mathrm{Ric}$ is of logarithmic growth. (The best result concerning solving the Poincaré-Lelong equation and the sharp estimate on the optimal solution is the one obtained in [NT1].) Hence the affirmative answer to the third problem of Yau mentioned above supplies exactly the condition to ensure the existence of a holomorphic functions of polynomial growth.

After explaining the relevance of Theorem 1 with the third problem of Yau addressed in this article, we supply some details on the solution to this problem. This was mainly done in [N3], by generalizing a result of Perelman on Ricci flow with nonnegative curvature operator to Kähler-Ricci flow with non-negative bisectional curvature. This is Theorem 2 of [N3], which in particular, implies that *if (M, g) is a complete Kähler manifold with bounded nonnegative bisectional curvature and of maximum volume growth, then the Kähler-Ricci* *flow $g(y,t)$ with such metric as the initial data has a long time solution. Moreover, its scalar curvature $S(y,t)$ satisfies the estimate:*

$$S(y,t) \le \frac{C_1}{t+1} \tag{2}$$

for some $C_1 = C_1(M) > 0$. In [N3], the author proceeded by first recalling some estimates from Theorem 2.1 of [NT2] to conclude that

$$\int_0^r sk(x,s)\, ds \le C_2 \log(r+1) \tag{3}$$

for some $C_2 = C_2(M) > 0$. Here

$$k(x,s) := \frac{1}{V_x(s)} \int_{B_x(s)} S(y)\, d\mu(y).$$

For the practical purpose of constructing a plurisubharmonic function u with logarithmic growth, the estimate (3) in fact is enough to apply the existence theorem on solutions to the Poincaré-Lelong equation by Shi, Tam and the author. To solve Yau's third problem, namely going from (3) to (1), some technicalities need to be overcome. In [N3] a much involved argument was applied with many details grossed over. Here we take this opportunity to present two direct/detailed proofs on obtaining (1) from (3). Both proofs rely on certain monotonicity formulae. Let's first recall a monotonicity result from [N5] and give a heuristic argument to illustrate the idea. In [N5], the author proved the following (weak form) monotonicity result on the relative volume of analytic subvarieties in a compete Kähler manifold with nonnegative bisectional curvature.

Proposition 1 *Let $\mathcal{V}$ be a subvariety of M of complex dimension s. Let $\mathcal{A}_{\mathcal{V},x_0}(\rho)$ be $2s$-dimensional Hausdorff measure of set $\mathcal{V} \cap B_{x_0}(\rho)$. Let $\delta(s) = \frac{1}{\sqrt{2+4s}}$. There exists $C = C(m,s)$ such that for any $\rho' \in (0, \delta(s)\rho)$*

$$\frac{\mathcal{A}_{\mathcal{V},x_0}(\rho')(\rho')^{2(m-s)}}{V_{x_0}(\rho')} \le C(m,s) \frac{\mathcal{A}_{\mathcal{V},x_0}(\rho)\rho^{2(m-s)}}{V_{x_0}(\rho)}. \tag{4}$$

The heuristic argument goes as follows. If the Ricci form is dual to some analytic subvariety $\mathcal{V}$ of complex dimension $m-1$, then

$$\int_{B_x(\rho)} \mathrm{Ric} \wedge \omega^{m-1} = \mathcal{A}_{\mathcal{V},x}(\rho).$$

Here ω is the Kähler form of (M,g). Hence the monotonicity (4) gives that

$$(\rho')^2 k(x,\rho') \le C_3(m)\rho^2 k(x,\rho).$$

for all $\rho' \le \delta(m)\rho$, for some $C_3(m) > 0$. Now (1) follows from (3) by a contradiction argument. Namely if there exists $\rho' > 0$ such that

$$k(x,\rho') \ge \frac{2C_2C_3}{(\rho')^2}$$

then we have that

$$\rho^2 k(x,\rho) \geq 2C_2$$

for all $\rho \geq \frac{1}{\delta(m)}\rho'$, which implies that

$$\int_{\frac{1}{\delta}\rho'}^{\rho} \rho k(x,\rho)\, d\rho \geq 2C_2 \left(\log \rho - \log(\frac{1}{\delta}\rho')\right).$$

This is a contradiction to (3).

The heuristic argument can not work in general since we do not know if there exists such analytic subvariety $\mathcal{V}$ which is dual to Ric. However this can be circumvented with a monotonicity on a parabolic equation, Theorem 2.1 of [N2], which we state below.

Theorem 2 *Let (E, H) be a holomorphic vector bundle on M. Consider the Hermitian metric $H(x,t)$ deformed by the Hermitian-Einstein flow:*

$$\frac{\partial H}{\partial t} H^{-1} = -\Lambda F_H + \lambda I.$$

Here Λ means the contraction by the Kähler form ω, λ is a constant, which is a holomorphic invariant in the case M is compact, and F_H is the curvature of the metric H, which locally can be written as $F^j_{i\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta e^*_i \otimes e_j$ with $\{e_i\}$ a local frame for E. The transition rule for H under the frame change is $H^U_{i\bar{j}} = f^k_i \overline{f^k_j} H^V_{k\bar{l}}$ with transition functions f^j_i satisfying $e^U_i = f^j_i e^V_j$. Let

$$\rho = \frac{\sqrt{-1}}{2\pi} \Omega_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta = \frac{\sqrt{-1}}{2\pi} \sum_i F^i_{i\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Assume that $\Omega_{\alpha\bar{\beta}}$ is smooth on $M \times (0,T]$. Then $\Omega_{\alpha\bar{\beta}}(x,t)$ satisfies the heat equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right) \Omega_{\alpha\bar{\beta}} = R_{\alpha\bar{\beta}\delta\bar{\gamma}} \Omega_{\bar{\delta}\gamma} - \frac{1}{2}\left(R_{\alpha\bar{\gamma}} \Omega_{\gamma\bar{\beta}} + \Omega_{\alpha\bar{\gamma}} R_{\gamma\bar{\beta}}\right).$$

Therefore, if $\Omega_{\alpha\bar{\beta}}(x,t) \geq 0$, then $Z_\Omega(x,t) \geq 0$, provided that $\Omega_{\alpha\bar{\beta}}(x,t)$ satisfies the growth assumption

$$\int_\epsilon^T \int_M e^{-ar^2(y)} \|\Omega_{\alpha\bar{\beta}}\|^2(y)\, d\mu(y)\, dt < \infty,$$

for any $\epsilon > 0$ and some $a > 0$, when the manifold is noncompact. In particular, if $\Omega(x,t) = g^{\alpha\bar{\beta}}(x)\Omega_{\alpha\bar{\beta}}(x,t) > 0$, one has that

$$\Omega_t - \frac{|\nabla \Omega|^2}{\Omega} + \frac{\Omega}{t} \geq 0.$$

If the equality ever holds somewhere for positive t and $\Omega_{\alpha\bar{\beta}} > 0$, then M is flat.

In the theorem Z_Ω is a quadratic form involving Ω along with its first and second covariant derivatives. We refer the interested readers to [N2] for details. Here we only need to use a special case of the above theorem that $E = K_M^{-1}$. Let us first recall the discussion from page 922 of [N2]. Deforming the Hermitian metric on K_M^{-1} by the Hermitian-Einstein flow reduces to solving the heat equation

$$\left(\frac{\partial}{\partial t} - \Delta\right) v(y,t) = S(y)$$

with $v(x,0) = 0$. Here $e^{-v(y,t)}$ is the quotient of the deformed metric over the initial metric on K_M^{-1} induced from (M, g). Let $w(y,t) = \frac{\partial}{\partial t} v(y,t)$. Let us emphasize here that Δ is the Laplacian operator with respect to the *fixed* metric g (no Kähler-Ricci flow is involved). Then a special case of Theorem 2 concludes the following monotonicity.

Proposition 2 *With the above notations*

$$\frac{\partial}{\partial t}\left(t w(x,t)\right) \geq 0. \tag{5}$$

It is easy to see, as pointed out on page 922 of [N2], that $w(x,t)$ is a solution to the heat equation with $w(y,0) = S(y)$.

The desired estimate (1) will follow from this monotonicity and the following technical lemma.

Lemma 1 *There exists a constant $C_4(M)$, depending on C_2 and m, such that for $t >> 1$,*

$$v(x,t) \leq C_4 \log t.$$

Assuming the above lemma the estimate (1) can be derived as follows: First by the monotonicity (5) it is easy to show that $w(x,t) \leq \frac{C_4}{t}$ for all t, arguing by contradiction as in the heuristic argument above. Then (1) follows from the so-called 'moment' estimate Theorem 3.1 of [N1], noting that $w(y,t)$ solves the heat equation with initial data $S(y)$.

The first proof of the lemma is computational and is based on the heat kernel estimate of Li-Yau:

$$\frac{C(m)^{-1}}{t^m}\exp\left(-\frac{r^2(x,y)}{3t}\right) \leq H(x,y,t) \leq \frac{C(m)}{t^m}\exp\left(-\frac{r^2(x,y)}{5t}\right). \tag{6}$$

Since M is of the maximum volume growth, we also have that $V_x(r) \geq \delta(M) r^{2m}$, $A_x(r) \geq \delta(M) r^{2m-1}$ for some $\delta(M) > 0$. Here $A_x(r)$ is the surface area of $\partial B_x(r)$. So there is no harm to replace $V_x(\sqrt{t})$ (or $A_x(r)$) by t^m (respectively r^{2m-1}) and vice versa.

To estimate the $v(x,t)$ from above we use the representation formula:

$$v(x,t) = \int_0^t \int_M H(x,y,s) S(y)\, d\mu(y)\, ds.$$

Estimate

$$\begin{aligned} \int_M H(x,y,s)S(y)\,d\mu(y) &\leq \frac{C(m)}{V_x(\sqrt{s})}\int_0^{\sqrt{s}} + \int_{\sqrt{s}}^{\infty}\left(\int_{\partial B_x(r)} S(y) e^{-\frac{r^2}{5s}}\,dA(y)\right) dr \\ &\leq C(m)k(x,\sqrt{s}) + \frac{C(m)}{V_x(\sqrt{s})}\int_{\sqrt{s}}^{\infty}\int_{\partial B_x(r)} S(y)e^{-\frac{r^2}{5s}}\,dA(y)\,dr. \end{aligned}$$

Hence

$$v(x,t) \le C(m)\int_0^t k(x,\sqrt{s})\,ds + \int_0^t \frac{C(m)}{V_x(\sqrt{s})}\int_{\sqrt{s}}^\infty \int_{\partial B_x(r)} S(y)e^{-\frac{r^2}{5s}}\,dA(y)\,dr\,ds.$$

Having (3), the first term is in line with the upper bound claimed in the lemma. Now we focus on estimating the second term above, which we shall denote by I. Changing the order of the integrations we have that

$$I = \int_0^{\sqrt{t}}\int_0^{r^2} \frac{e^{-\frac{r^2}{5s}}}{V_x(\sqrt{s})}\,ds \int_{\partial B_x(r)} S(y)\,dA(y)\,dr + \int_{\sqrt{t}}^\infty \int_0^t \frac{e^{-\frac{r^2}{5s}}}{V_x(\sqrt{s})}\,ds\int_{\partial B_x(r)} S(y)\,dA(y)\,dr.$$

Straight forward calculation shows that there exists $C_5(m) > 0$ such that

$$\int_0^{r^2} \frac{e^{-\frac{r^2}{5s}}}{V_x(\sqrt{s})}\,ds + \sup_{t\le r^2}\int_0^t \frac{e^{-\frac{r^2}{10s}}}{V_x(\sqrt{s})}\,ds \le \frac{C_5(m)}{r^{2m-2}}. \tag{7}$$

On the other hand, it is easy to derive from (3) that for $r >> 1$,

$$k(x,r) \le C_2'(m)\frac{\log(r+2)}{(r+1)^2}. \tag{8}$$

Using these estimates,

$$\begin{aligned} II &:= \int_0^{\sqrt{t}}\int_0^{r^2} \frac{e^{-\frac{r^2}{5s}}}{V_x(\sqrt{s})}\,ds\int_{\partial B_x(r)} S(y)\,dA(y)\,dr \\ &\le \int_0^{\sqrt{t}} \frac{C_6(m)}{r^{2m-2}}\int_{\partial B_x(r)} S(y)\,dA(y)\,dr \\ &\le C_6(m)tk(x,\sqrt{t}) + C_6(m)\int_0^{\sqrt{t}} rk(x,r)\,dr \\ &\le C_6(m)\log(t+1). \end{aligned}$$

From the second line to the third line we have done the integration by parts. For the last line we have used (3) and (7). On the other hand,

$$\begin{aligned} III &:= \int_{\sqrt{t}}^\infty \int_0^t \frac{e^{-\frac{r^2}{5s}}}{V_x(\sqrt{s})}\,ds\int_{\partial B_x(r)} S(y)\,dA(y)\,dr \\ &\le \int_{\sqrt{t}}^\infty e^{-\frac{r^2}{10t}}\int_0^t \frac{e^{-\frac{r^2}{10s}}}{V_x(\sqrt{s})}\,ds\int_{\partial B_x(r)} S(y)\,dA(y)\,dr \\ &\le C_5(m)\int_{\sqrt{t}}^\infty \frac{e^{-\frac{r^2}{10t}}}{r^{2m-2}}\int_{\partial B_x(r)} S(y)\,dA(y)\,dr \\ &\le C_5(m)\int_{\sqrt{t}}^\infty \frac{e^{-\frac{r^2}{10t}}}{r^{2m-2}}\left(\frac{r}{5t}+\frac{2m-2}{r}\right)\int_{B_x(r)} S(y)d\mu(y)\,dr \\ &\le C_7(m)\int_{\sqrt{t}}^\infty e^{-\frac{r^2}{10t}}\left(\frac{r}{5t}r^2 + (2m-2)r\right)k(x,r)\,dr. \end{aligned}$$

Using (8) elementary computation gives that $III \le C_8(M) \log t$ for $t >> 1$. This finishes the proof of the lemma.

There exists also a less computational proof of Lemma 1. This makes use of (2) and a computation of Shi [Shi]. First we have to define

$$F(x,t) := \log \frac{\det(g_{i\bar{j}}(x,t))}{\det(g_{i\bar{j}}(x,0))}.$$

It is easy to check that $-\frac{\partial}{\partial t} F(x,t) = S(x,t)$. Hence (2) implies that

$$-F(x,t) \le 2C_1(M) \log t \tag{9}$$

for $t >> 1$.

On the other hand, it was shown by W.-X. Shi (see also Lemma 2.3 of [NT2]) that

$$\left(\frac{\partial}{\partial t} - \Delta\right)(-F(y,t)) \ge S(y)$$

and $F(y,0) = 0$. By the comparison principle we have that $v(x,t) \le -F(x,t)$, hence Lemma 1 follows from (9).

In Problem 48 of [Y2], Yau asked if the Euclidean space of the same dimension has the most polynomial growth harmonic functions. We devote the rest to describe a result addressing the same question for the holomorphic functions of polynomial growth. Before we describe the result let us first introduce some notations. Let $\mathcal{O}_d(M)$ be the space of holomorphic functions of polynomial growth with degree not greater than d. Precisely, fixing $x \in M$,

$$\mathcal{O}_d(M) := \{f \,|\, f \text{ is holomorphic and } |f|(y) \le C_f(r_x(y)+1)^d\}$$

where $r_x(y) = r(x,y)$ the distance between x and y. We also denote

$$\mathcal{O}_P(M) := \cup_{d \ge 1} \mathcal{O}_d(M).$$

Theorem 3 *Let M^m be a complete Kähler manifold with nonnegative holomorphic bisectional curvature. Then*

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \le \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^m)). \tag{10}$$

Here $[d]$ is the greatest integer less than or equal to d. In the case that equality holds in (0.1), M is biholomorphic-isometric to $\mathbb{C}^m$.

This result was first proved in [N2] under the assumption that M is of maximum volume growth. This assumption was removed by Chen, Fu, Yin and Zhu [CFYZ]. Below we shall give a short account on the strategy of [N2] and the steps of the proof. Along the way we also describe the part of contribution from [CFYZ]. In [N2], the proof was divided into three lemmas. The first lemma is the key and again a special case of Theorem 2.

Lemma 2 *Let $f \in \mathcal{O}(M)$ be a nonconstant holomorphic function of order less than one, in the sense of Hadamard. Denote $u(x) = \log(|f|(x))$. Then there exists a solution $v(x,t)$ to the heat equation $\left(\frac{\partial}{\partial t} - \Delta\right) v(x,t) = 0$ such that $v(x,0) = u(x)$, where $v(x,t)$ is plurisubharmonic. Moreover, the function $w(x,t) := \Delta v(x,t) > 0$, for $t > 0$, and*

$$\frac{\partial}{\partial t}(t\, w(x,t)) \ge 0. \tag{11}$$

This monotonicity provides the comparison between the value of $tw(x,t)$ at $t=0$ and the limiting value as $t\to\infty$. The other two lemmas compute the asymptotic values. The asymptotic at $t=0$ relies on the short time behavior of the heat kernel, which is true for general Kähler manifolds.

Lemma 3 *Let u, v, w be as in Lemma 2. Then*

$$\lim_{t\to 0} tw(x,t) = \frac{1}{2} ord_x(f). \tag{12}$$

If the equality ever holds somewhere and $v_{\alpha\bar\beta}(x,t)>0$, M must be flat.

The asymptotic value of $tw(x,t)$ as $t\to\infty$ is related to the growth order of f.

Lemma 4 *Let M be as in Theorem 3 and let u, v, w be as in Lemma 2. Then*

$$\limsup_{t\to\infty} \frac{v(x,t)}{\log t} \le \frac{1}{2} d. \tag{13}$$

which, in particular, implies that

$$\lim_{t\to\infty} tw(x,t) \le \frac{1}{2} d. \tag{14}$$

The most involved step is to prove Lemma 2, which in turn is derived from a matrix Li-Yau-Hamilton type estimate for the Hermitian-Einstein flow, Theorem 2. The novel part of [N2] is to prove this result and discover the method of the comparison via the above three lemmas. By now, there are two matrix Li-Yau-Hamilton type inequalities can be used for Lemma 2. One was proved in [N2]. The other was later shown in [N5]. In [N2], only in Lemma 4 the assumption that M has maximum volume growth is needed. After the appearance of [N2] on the arXiv, it was observed in [CFYZ] afterwards, that one also have Lemma 4, even without assuming the maximum volume growth. A shorter proof (less than half a page) of Lemma 4 for the general case can also be found on page 938 of [N2]. The proof of Lemma 2 and Lemma 3 from [N2] were essentially reproduced in [CFYZ]. The part that the equality in the dimension comparison implies that the manifold is Euclidean is proved by making use of Lemma 3, through constructing a v with $v_{\alpha\bar\beta}(x,t)>0$. The details can be found on pages 936–937 of [N2]. This was also proved in [CFYZ] via a more involved induction argument relying on a splitting theorem of L.-F. Tam and the author previously proved in [NT1]. The main ingredient of the argument (both that of [CFYZ] and [N2]) comes from [N4], which was made available on ArXiv since 2002, even though it was not in print until 2005. In [N4] the author gave a more direct proof to a previous result of Hamilton, as well as an Kähler analogue of H.-D. Cao, on the characterization of eternal/immortal solutions being gradient/expanding solitons. What was proved in [N4] in fact is a rigidity result on Kähler-Ricci flow solutions satisfying the equality in a linear trace Li-Yau-Hamilton estimate. Since the Euclidean space can be viewed as an expanding/shrinking soliton, the rigidity result implied by Lemma 3 can be viewed a special case of the more general result of [N4] of 2002.

In Theorem 0.3 of [N2] the following result was proved on the dimension bounding, which sharpens Theorem 3.

Theorem 4 *Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. If $k(M) = deg_{tr}(\mathcal{M}(M)) \leq m - 1$, we have that*

$$\dim_{\mathbb{C}}(\mathcal{O}_d(M)) \leq \dim_{\mathbb{C}}(\mathcal{O}_{[d]}(\mathbb{C}^{k(M)})). \tag{15}$$

The case of equality implies the splitting $M = M_1 \times \mathbb{C}^{k(M)}$, with $\mathcal{O}_P(M_1) = \mathbb{C}$.

Relating to the finite generation of the ring $\mathcal{O}_P(M)$, in [N2], it was also proved that the quotient fields $\mathcal{M}(M)$ of the ring of polynomial growth holomorphic functions must have its transcendence degree bounded by the complex dimension of the underlying Kähler manifold. (cf. Corollary 3.1 of [N2]). The finite generation of $\mathcal{O}_P(M)$ still remains a very interesting question.

In [Y2], Problem 48, the similar question was also asked for the space of holomorphic sections of line bundles over Kähler manifolds. For this we refer the interested readers to Theorem 4.3 of [N2]. For the holomorphic sections of line bundles, besides the growth order, another factor plays role in the dimension bounding is the Lelong number of the line bundle.

Despite the recent advances due to Chau and Tam [CT] on the complex structure of the Kähler manifolds with bounded nonnegative bisectional curvature and uniformly quadratic average decay, we think that the following result of [N2], as an application of Theorem 3 and the L^2-estimates construction, is still interesting since it assumes neither the uniform boundedness of the curvature nor the uniform average decay. On the other hand, all the result via Kähler-Ricci flow assumes that the curvature is *uniformly bounded* and the scalar curvature satisfies some *uniform* average decay condition.

Corollary 1 *Let M^m be a complete Kähler manifold with nonnegative bisectional curvature. Assume that the Ricci curvature is positive somewhere and the scalar curvature $\mathcal{S}(x)$ satisfies that for some $o \in M$,*

$$\sup_{r\geq 0}\left(\exp(-ar^2)\fint_{B_o(r)} \mathcal{S}^2(y)\, d\mu(y)\right) < \infty$$

and

$$\fint_{B_o(r)} \mathcal{S}(y)\, d\mu(y) \leq \frac{C}{r^2}$$

for some positive constants a and C. Then the transcendence degree of the rational function field $\mathcal{M}(M)$ (the quotient field of the ring of the holomorphic functions of polynomial growth) is equal to m. Moreover, $\pi_1(M)$ is finite.

Finally we should mention that there are many related works on harmonic functions of polynomial growth. We refer the readers to [CM, LW] for further information.

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