1. What is a gradient system? Prove that a gradient system does not have a closed orbit.

2. Consider \( \dot{x} = y^2 + y \cos x, \ \dot{y} = 2xy + \sin x \). Determine if this is a gradient system. If it is, then find the potential \( V = V(x, y) \) so that \( \dot{x} = -\partial_x V \) and \( \dot{y} = -\partial_y V \).

3. Show that \( V(x, y) = x^2 + y^2 \) is a Liapunov function for the system \( \dot{x} = y - x^3 \) and \( \dot{y} = -x - y^3 \) at the fixed point \((0, 0)\).

4. Show by Dulac’s Criterion that the system \( \dot{x} = x(2 - x - y), \ \dot{y} = y(4x - x^2 - 3) \) has no closed orbits in the first quadrant \( x > 0, y > 0 \).

5. Show that the system \( \dot{x} = x - y - x^3, \ \dot{y} = x + y - y^3 \) has a periodic solution.

6. Consider the system \( \dot{x} = x(1 - 4x^2 - y^2) - 0.5y(1 + x), \ \dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x) \).
   (a) Show that the origin is an unstable fixed point.
   (b) Consider \( \dot{V} \) with \( V(x, y) = (1 - 4x^2 - y^2)^2 \) to show that all trajectories approach the ellipse \( 4x^2 + y^2 = 1 \) as \( t \to \infty \).

7. Consider the system \( \dot{x} = x - y - x^2 + 2y^2, \ \dot{y} = x + y - y^2 + 2y^2 \).
   (a) Write the system in the polar coordinates. (You can use \( rr' = x \dot{x} + y \dot{y} \) and \( \theta' = (y \dot{x} - x \dot{y})/r^2 \).
   (b) Use the trapping region method to show that this system has a closed orbit in the region defined by \( r_1 < r < r_2 \) with some positive numbers \( r_1 \) and \( r_2 \) with \( 0 < r_1 < r_2 \).

8. Consider the two-timing expansion \( x(t, \varepsilon) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + O(\varepsilon^2) \), where \( \tau = t \) and \( T = \varepsilon t \). Plug this expansion into the equation \( \ddot{x} + x + \varepsilon h(x, \dot{x}) = 0 \) to get the equations for \( x_0 \) and \( x_1 \).

9. Consider \( \dot{x} = y - 2x, \ \dot{y} = \mu + x^2 - y \).
   (a) Sketch the nullclines.
   (b) Find and classify the bifurcations that occur as \( \mu \) varies.
   (c) Sketch the phase portrait as a function of \( \mu \).

10. Consider the logistic equation \( \dot{N} = rN(1 - N/K) \), where \( K = K(t) \) is smooth, positive, and \( T \) periodic. Use the Poincaré map argument to show that this system has at least one limit cycle of period \( T \), contained in the strip \( K_{\min} \leq N \leq K_{\max} \).

11. Consider the system in polar coordinates \( \dot{r} = \mu r - r^3 \) and \( \dot{\theta} = 1 \), where \( \mu \) is a parameter. In the cartesian coordinates, this system is \( \dot{x} = (\mu - x^2 - y^2)x - y \) and \( \dot{y} = (\mu - x^2 - y^2)y + x \).
   (a) Suppose \( \mu < 0 \). Show that the origin is a stable spiral and explain why all trajectories approach the origin.
   (b) Suppose \( \mu > 0 \). Show that the origin is an unstable spiral and explain why the circle \( r = \sqrt{\mu} \) is a stable limit cycle.
   (This system has a supercritical Hopf bifurcation at \( \mu_c = 0 \).)

12. Consider the vector field given in the polar coordinates by \( \dot{r} = r - r^2, \ \dot{\theta} = 1 \).
   (a) Let \( S \) be the positive \( x \)-axis and compute the Poincaré map from \( S \) to itself.
   (b) Show that the system has a unique periodic orbit and classify its stability.
13. Prove for the Lorenz system that volume in the phase space contracts.

14. Prove that the $z$-axis is an invariant line for the Lorenz system, i.e., a trajectory starts on the $z$-axis stays on it forever.