On the Denogardus Great Number and Hooke's Law

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Recently, an international mathematical congress took place in a small German town called Sturmgard. The main topic discussed at the congress happened to be a somewhat charred manuscript entitled *Principia* (First Principles) written by the medieval scholastic, alchemist, and astrologer Denogardus. It had been discovered quite by chance when some old documents were being sorted out in the Sturmgard town hall. It turned out that the manuscript contained a few extraordinary facts that are of great importance to contemporary mathematics.

Persecuted by the Inquisition, Denogardus used anagrams and drawings to disguise his ideas. The Inquisition failed to expose the scientist during his lifetime. However, after his death the authorities, acting on information from Denogardus's nephew, confiscated the manuscript and threw it onto the fire. But the manuscript was not completely destroyed due to its thickness.

It is common knowledge that the astrologer devoted considerable effort to computing the "Great Number," which he assumed to be responsible for the movement of all heavenly bodies (planets, comets, and so on) belonging to the solar system. In fact, this number is changing all the time. Observing its pattern of change, Denogardus could forecast various calamities (depending on the pattern of change), such as floods, fires, wars, and epidemics. Although his forecasts were often confirmed, the ideas behind them were neglected. For example, on page 666 of his *Principia* a great astronomical catastrophe was foretold. This catastrophe was identified by the participants at the congress as the Tungus meteorite. Denogardus was off by only nine years. The manuscript also contained a few rigorous mathematical theorems. They were called *scholia* by Denogardus. We shall reproduce them in contemporary form.

**Scholium 1.** *The motion of each planet is planar, its orbit being in a plane that contains the Sun.*

In the manuscript two other scholia were given as drawings (see Figures 1 and 2). Their meaning is expressed by the following.

**Scholium 2.** *Between two intersections with an arbitrary ray OA a comet must intersect every additional ray OB exactly once.*

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1The Russian original is published in *Kvant* 1989, no. 8, pp. 8–16.

2An anagram is a word formed by changing the order of the letters in another word. (*Translation editor’s note: For example, “cocaine” and “oceanic” are anagrams of each other.*)

2This word is of Greek origin (*scholia* = “lecture”) and is related to the word “school.”
Scholium 3. If a comet is repulsed from the Sun, then its trajectory cannot intersect more than once any straight line passing through the Sun.

But the participants at the congress were especially excited over the following theorem.

Denogardus’s Great Theorem. The Great Number can be found by observing a comet.

The Sturmgard congress addressed an inquiry regarding Denogardus to the European association of history and archeology “Archeologe” but obtained very scanty information. They could determine not a single reliable fact about his life. In one of the chronicles there was a story about a citizen who was bitten by a rat while observing a solar eclipse from the tower of the town hall. However, there was no confirmation of the strong suspicion that it was Denogardus who was making the astronomical observation. There is more information on Denogardus’s views. The distinguished astrologer (quite in the spirit of his time) believed that the planets move about the Sun, attached to it by invisible strings. Aside from direct statements to this effect that were found in the manuscript, there is an accompanying drawing confirming Denogardus’s views on the subject.

The participants at the congress were confronted with a complicated problem of rendering Denogardus’s statements into the language of contemporary mathematics.
and establishing the mathematical significance of his scholia. This task was carried out through the joint efforts of mathematicians, physicists, and historians.

According to Denogardus, each celestial body is joined to the Sun by a spring, so that the force of attraction is proportional to the distance between the body and the Sun. (We know this fact concerning forces in springs as Hooke's law. However, Denogardus seemed to have known this long before Hooke.) The force of attraction may be either positive (the spring tends to contract) or negative (the spring is compressed and tends to expand). Moreover, according to Denogardus, the stiffness of the invisible springs that are joined to the planets varies with time. Most researchers now identify this stiffness (the spring constant) with the Great Number governing the motion of celestial bodies.

Denogardus did not use mathematical symbols (such notation for describing physical phenomena was still in the distant future). However, the participants at the Sturmard congress managed to formulate Denogardus's results in conventional mathematical terms.

In the “Denogardus model” (such was the name given at the congress to the model described in the manuscript), the force acting on a planet obeys Hooke's law:

\[ F = k(t) m \vec{r}. \] (1)

In this formula \( m \) is the mass of the planet, \( \vec{r} \) is the radius vector with initial point at the Sun’s center and terminal point at the center of the given planet, and the time-dependent coefficient \( k(t) \) is the stiffness (the spring constant) of the invisible spring, or Denogardus’s Great Number.

The obvious meaning of relation (1) makes possible an immediate proof of Denogardus’s first scholium. Fix a moment in time and construct a plane \( l \) passing through the Sun \( O \), the planet \( A \), and the vector of the planet’s velocity \( \vec{v} \). According to (1), the force vector acting on the planet is proportional to the vector \( OA \) and lies in the same plane \( l \). Therefore, there is no force component perpendicular to the plane that could move the planet \( A \) away from the plane \( l \). Hence further motion of the planet can occur only in the plane under consideration.

The proof given here is the result of a painstaking reconstruction of the arguments that are furnished on the pages of *Principia*. The author himself widely used the “principle of adequate justification” and the “postulate of purposefulness.” From the modern viewpoint, these formulations are somewhat obscure, and their scientific significance is doubtful.3

To verify Denogardus’s second and third scholia, we shall again turn to equation (1). By Newton’s second law (which was probably also known to Denogardus), the acceleration of a planet is proportional to the force acting on it: \( \vec{a} = \vec{F}/m \). It follows that the acceleration vector is given by

\[ \vec{a} = k(t) \vec{r}. \] (2)

By the first scholium, both vectors \( \vec{a} \) and \( \vec{r} \) lie in some plane. Introducing a coordinate system in this plane, we can write \( \vec{a} = (a_1, a_2) \) and \( \vec{r} = (x_1, x_2) \). Equation (2) then transforms into two similar equations,

\[ a_i(t) = k(t) x_i(t), \quad i = 1, 2, \]

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3In another place Denogardus writes, “The amazing harmony of the world composed only of the two elements, celestial bodies and springs, is the best confirmation of the fact that the universe was created by God in the same way as a skillful watchmaker makes a watch.”
where \( a \) and \( x \) are no longer vectors but ordinary (scalar) functions of time \( t \). Recall that the acceleration \( a(t) \) is the second derivative of the coordinate \( x(t) \) with respect to time. Therefore, the Hooke–Denogardus law may be written as the following differential equation in each planetary coordinate:

\[
x''(t) = k(t)x(t).
\]

While thinking that he had stated the laws governing the motion of celestial bodies, Denogardus had in fact constructed a theory of the differential equation (3). This is the mathematical significance of his work.

The solutions of equation (3) may be represented as functions of time. They can be multiplied by numbers and added—the results are again solutions. The totality of solutions can be described by the assertion that follows.

**Assertion.** The solutions of equation (3) may be represented as vectors in the plane: Each solution can be obtained from any two specific solutions by multiplying each of them by a number and adding.

This is a complicated mathematical theorem, since it is not at all clear why even one solution should exist. However, Denogardus understood the theorem well enough and almost proved it by the following elegant argument. Since each planet is in motion, it follows that its coordinates represent two solutions of equation (3) (!). But if we assume that there exists a third solution that is independent of these two, then we would immediately obtain a contradiction. Indeed, a point in space all three of whose coordinates are solutions of equation (3) must necessarily move in one and only one plane (!!).

In order to become better acquainted with equation (3), it is advisable to treat the simplest particular case in which the coefficient \( k(t) \) does not depend on time.

**Exercise 1.** Solve the equation \( x'' = -x \). **Hint:** consider a point in the plane that is moving uniformly in a circle.

Denogardus's second and third scholia acquire significance as remarkable theorems on the properties of the solutions of the differential equation (3).

**Theorem.** In the interval \([t_1, t_2]\) at whose endpoints one solution of the equation \( x'' = k(t)x \) is zero, any other solution must necessarily be zero at some point in the interval.

In other words, the zeros of two solutions \( x_1(t) \) and \( x_2(t) \) must alternate.

In order to deduce this theorem from the second scholium, we shall consider the plane motion of a point whose coordinates change in time in accordance with the functions \( x_1(t) \) and \( x_2(t) \). In this case the motion of the point will satisfy equation (3), and consequently equation (2); i.e., it will obey Denogardus's second scholium. Choose the ray \( OA \) as the positive \( x \)-axis and the ray \( OB \) as the negative \( y \)-axis. When the point intersects one axis, its projection on the other axis (i.e., the second solution) becomes zero. Since the instances of intersection of the axes must alternate, it follows that the zeros of the solutions \( x_1(t) \) and \( x_2(t) \) must alternate, too.

We now turn to the proof of the second scholium. Its simple geometric proof is readily seen in Denogardus's treatise (see Figure 1). If a comet cuts the ray \( OA \) twice without intersecting the ray \( OB \), then at some intermediate moment in time its trajectory will touch a ray \( OC \). At this moment both the velocity of the comet
and the force acting on it are directed along the line $OC$. Thus there is no force that could move the comet away from the line, and so the comet would remain on that line, which contradicts the assumption. This elegant argument shows the skill with which Denogardus treated problems in geometry.

The third scholium can be proved in a similar way (see Figure 2). If the force acting on the comet is directed away from the Sun (Denogardus writes about a compressed spring about to straighten out), then the comet will stay on one side of any tangent to its trajectory and will not be able to cross the ray $OA$ again. Choosing the ray $OA$ as the positive $y$-axis, we can obtain the theorem that follows.

**Theorem.** If the coefficient $k(t)$ is greater than 0 for all $t$, then there is no solution of the equation $x'' = k(t)x$ that can attain the value zero more than once.

By doing the exercise that follows you can verify the theorem in a particular case.

**Exercise 2.** Solve the equation $x'' = x$. *Hint:* consider the functions $x = e^t$ and $x = e^{-t}$.

Having at his disposal only primitive astronomical instruments, Denogardus could measure angles between celestial bodies but not their distances to the Sun. Nevertheless, the astrologer could find the Great Number $k(t)$ using only the direction of the ray emanating from the comet to the Sun. The contemporary version of the proof of Denogardus's Great Theorem demonstrates the way this could be done. This proof is based on a simple observation.

**Exercise 3.** Prove that for any two solutions of the equation $x''(t) = k(t)x(t)$ the value of $W(x_1, x_2) = x_1 x_2' - x_2 x_1'$ does not depend on time $t$. *Hint:* find the derivative of $W$ with respect to time and use equation (3).

The result that $W(x_1, x_2)$ is a constant is not just a pure coincidence but has physical significance. Consider the plane motion of a body with acceleration $a(t) = k(t)\vec{r}(t)$. Its angular momentum does not vary with time, since the force acting on the body is always directed towards the center. The value of the angular momentum is equal to the area of the parallelogram in which adjacent sides represent the vectors $\vec{r}$ and $m\vec{v}$. Show that this area is equal to $m(x_1 v_2 - x_2 v_1)$. Since $x_2' = v_2$ and $x_1' = v_1$, the angular momentum being a constant implies that the value of $W(x_1, x_2)$ is also a constant.

Thus the Great Theorem states that the coefficient $k(t)$ can be evaluated by the direction of the radius vector $\vec{r}(t)$. We shall specify the direction of $\vec{r}(t)$ by the value of $\tan \varphi(t)$, where $\varphi(t)$ is the angle that the vector $\vec{r}(t)$ makes with the $x$-axis. We shall denote by $f(t)$ the function $\tan \varphi(t)$.

**Exercise 4.** Prove that

$$k(t) = -\frac{1}{2} \left[ \frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right]$$

(Denogardus’s “Great Formula”).

*Hint:* by the definition of the tangent we have $x_2(t) = f(t)x_1(t)$. Substitute the value of $x_2(t)$ in the formula for $W(x_1, x_2)$, which on simplification yields $W = x_2^2 f'$. It follows that the body’s first coordinate $x_1(t) = \sqrt{W/f'}$ is determined, up to a constant factor $W$, by $f$. The equation $x_1'' = k(t)x_1$ implies that $k(t) =$
\( x''_1 / x_1 \). The rest is a useful exercise in taking a derivative, using the chain rule for differentiation (see Kvant 1988, no. 4, p. 36).

There still remains the mystery of how Denogardus, with no knowledge of the differential calculus whatsoever, could define his Great Number. Perhaps it was the scientist's superior intuition and experience. Perhaps the prominent astrologer knew more than he dared to put in writing. The question remains open.

In conclusion, we pose another problem related to the Denogardus model.

**EXERCISE 5.** Prove that if the angular velocity of a celestial body relative to the Sun is constant, then the coefficient \( k(t) \) is independent of time.

Denogardus's *Principia* created quite a stir among mathematicians. Most participants at the Sturmgard congress were unanimous in the opinion that Denogardus should be considered the inventor of the differential calculus and the founder of the theory of differential equations.

**Commentary.**

The reader has probably already suspected that the above historical narration was nothing but balderdash—a hoax. The great alchemist and astrologer Denogardus is a fictitious character. The real founder of the theory of differential equations was Isaac Newton,\(^4\) who, by the way, also disguised his discovery in the form of a famous anagram, which can be loosely translated from Latin into English as "Solving differential equations is useful."\(^5\) However, Newton's reasons for secrecy were different from those of Denogardus. While Denogardus was afraid of the Inquisition, Newton was afraid of competition.

![Figure 3](image)

**FIGURE 3**

As for spring-loaded planets that are attracted to the Sun in accordance with Hooke's law, Denogardus here contradicts Newton. According to the law of universal gravitation, the force of attraction between a planet and the Sun is inversely proportional to the square of the distance between them. For fairness' sake, we note

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\(^4\)The name "Newton" means "new town," and "Denogardus" means essentially the same: "de noe gardus" can be taken as some sort of Germanic equivalent to "of the new town." Such patois, which was widespread in the former Roman empire of the sixth century, was the result of contact between German invaders and the local population. Some other variants of the same name are "Neustadt" (in German) and "Novgorodtsev" (in Russian). By the way, the main work of Newton is also called *Principia*.

\(^5\)See the relevant material (including the original version of the anagram and the message deciphered) in the article by V.I. Arnol'd in Kvant 1986, no. 2, p. 19 [see pp. 73–85 of this volume].
that the motion of each planet is governed by Hooke’s law \( \vec{F} = mk(t) \vec{r}'(t) \), the coefficient \( k(t) \) being different for each planet. Generally speaking, Newton’s law and Hooke’s law are, in a way, similar to each other. For example, the trajectories of bodies governed by Hooke’s law, with the coefficient \( k \) constant, are second-degree (or conic) curves, such as ellipses, hyperbolas, and parabolas, just as the trajectories of ordinary celestial bodies are governed by Newton’s law. Moreover, if a falling body continued to move inside the Earth due only to Earth’s gravitational pull, then its motion would obey Hooke’s law.

The relations between Newton and Hooke were as complicated as those between the laws bearing their names. Hooke was the first to surmise the law of “inverse squares,” and he informed Newton about his conjecture. Using this conjecture, Newton managed to interpret Kepler’s observations. For some time there was a priority conflict between Hooke and Newton. You can read about this as well as other interesting things in the book by V.I. Arnol’d, Huygens and Barrow, Newton and Hooke. Pioneers in mathematical analysis and catastrophe theory, from evolvents to quasi-crystals, “Nauka”, Moscow, 1989; English transl., Birkhäuser, Basel, 1990.

The equation \( x''(t) = k(t)x(t) \) is usually called the Sturm–Liouville equation, and the second and the third of Denogardus’s scholia are known as Sturm’s theorems. These theorems were proved 150 years ago. They gave rise to a new branch of mathematics now considered classical, namely, Sturm theory. The number \( W(x_1, x_2) = x_1x_2' - x_2x_1' \) (angular momentum) is called Wronski’s determinant, or simply the Wronskian.

If the coefficient \( k \) is time-independent and \( k < 0 \), then the equation \( x''(t) = k(t)x(t) \) transforms into the equation of small oscillations, which is treated in school textbooks (see also Exercise 1). This equation describes small-amplitude oscillations of any kind, where \( x \) is a small deviation from equilibrium.

**Exercise 6.** Evaluate the coefficient \( k \) in the cases illustrated in Figure 3 if:

(a) The spring constant (the coefficient of stiffness) is \( \kappa \), and the mass of the weight is \( m \).

(b) The length of the string, which is weightless and inextensible, is \( l \).

(c) The radius of the hole is \( R \), and the diameter of the ball is \( d \).

Last (but not least), Denogardus’s Great Number, or rather Denogardus’s Great Formula (see Exercise 4), is known as the Schwarz derivative. As a matter of fact, this formula was communicated to H.A. Schwarz in a letter written by the great German mathematician, C.F. Gauss. The Schwarz derivative is a popular character in contemporary mathematics. It is remarkable in that it turns up in a number of completely different theories.

To sum up, despite the fact that everything said about Denogardus was rubbish, the mathematical significance of his work is great.

Translated by N.K. KULMAN