1. [20 points] Show that the system

\[ \begin{align*}
\dot{x} &= x + y - x^3 \\
\dot{y} &= -x
\end{align*} \]

has a closed orbit. Hint: Show that \(|x| \leq 2\) and \(|y - x| \leq 4\) defines a trapping region.
2. Consider the system
\[ \dot{r} = r((r - 1)^2 - \mu r) \]
\[ \dot{\theta} = 1. \]

a. [5 points] For which values of \( \mu \) does this system undergo a bifurcation?

b. [7 points] Describe how the solutions change as \( \mu \) crosses each of its critical values, i.e., list any fixed points and cycles and determine their types. Draw a bifurcation diagram illustrating your results.
3. Consider the system
\[
\begin{align*}
\dot{x} &= -\mu x + yz \\
\dot{y} &= -\mu y + (z - a)x \\
\dot{z} &= 1 - xy,
\end{align*}
\]
where $a, \mu > 0$.

a. [5 points] Show that this system is dissipative.

b. [5 points] Find the fixed points of this system.

c. [14 points] Classify the fixed points. Hint: Do the algebra carefully and show that two of the eigenvalues of the Jacobian are $\pm i\omega$, $\omega \in \mathbb{R}$. 
4. Let $x_{n+1} = f(x_n) = 2x_n - \lfloor 2x_n \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less that or equal to $x$.
   
a. [2 points] Draw the graph of $f$.
   
b. [2 points] Find the fixed points of $f$.
   
c. [2 points] If $0.b_1b_2\ldots$ is the binary expansion of $x_n$, what is $x_{n+1}$?
   
d. [3 points] Draw the graph of $f \circ f$.
   
e. [3 points] Find the period 2 cycles of $f$. 

4. (continued)
   f. [4 points] Show that $f$ has periodic cycles of all periods $p \geq 1$ and that all of them are unstable.
   g. [4 points] Show that $f$ has infinitely many aperiodic trajectories.
   h. [4 points] Show that $f$ has countably many periodic orbits and uncountably many aperiodic trajectories.
5. Let $f(x) = 2x - \lfloor 2x \rfloor$ and let $C$ be the “remove middle thirds” Cantor set.
   a. [10 points] What is $f(C)$?
   b. [10 points] What is the box dimension of $f(C)$? Justify your answer.
      [Extra Credit, 15 points] Show that the box dimension of any set is invariant under invertible linear maps.