1. (20 points) The figure below shows the graphs of $y = x$ and $y = \sin(\pi x)$. Set $x_0 = 0.4$ and $x_{n+1} = \sin(\pi x_n)$ ($n = 0, 1, 2, \ldots$). Construct the cobweb to find $x_1, x_2, x_3, x_4$. These numbers should be marked on the $x$-axis.
2. (30 points) Consider the system \( \dot{x} = y - 2x, \dot{y} = \mu + x^2 - y. \)

(a) Sketch the nullclines and find the fixed points. (The nullclines and fixed points may depend on the parameter \( \mu \). You need to discuss different cases.)

(b) For each fixed point, discuss the linear stability. (You can use the stability diagram that is provided.)

(c) Find the critical value \( \mu_c \) at which a bifurcation occurs and classify the bifurcation.
3. (30 points) Consider the vector field given in the polar coordinates by \( \dot{r} = r - r^2, \dot{\theta} = 1 \).

(a) Let \( S \) be the positive \( x \)-axis and compute the Poincaré map from \( S \) to itself.

(b) Show that the system has a unique periodic orbit and classify its stability.
4. (20 points) The Lorenz equations are

\[
\begin{align*}
\dot{x} &= \sigma(y - x), \\
\dot{y} &= rx - y - xz, \\
\dot{z} &= xy - bz,
\end{align*}
\]

where \(\sigma, r, b > 0\). Prove that volume in the phase space contracts.
5. (30 points)
(a) The von Koch curve is defined through a sequence of simple curves. Draw the first three of them.
(b) Calculate the box dimension of the von Koch curve.
6. (20 points) Consider a smooth map \( x_{n+1} = f(x_n) \). Define \( f^1(x) = f(x) \), \( f^2(x) = f(f(x)) \), \( f^3(x) = f(f(f(x))) \), etc. Let \( k \geq 2 \) be any integer. Prove the following:

(a) If \( x^* \) is a fixed point of \( f \), then it is also a fixed point of \( f^k \);

(b) \( (d/dx)(f^k)(x_0) = \Pi_{i=0}^{k-1} f'(x_i) \), where \( x_{i+1} = f(x_i) \) (\( i = 0, 1, \ldots, k - 1 \)).
7. (20 points)
(a) State the definition of the Liapunov exponent for the orbit \( x_{n+1} = f(x_n) \) starting from a point \( x_0 \);
(b) Let \( r \) be a real number. Calculate the Liapunov exponent for the linear map \( x_{n+1} = rx_n \) at a given point \( x_0 \).
8. (30 points) Consider the map \( x_{n+1} = f(x_n) \) with \( f(x) = x^2 - 1 \).
(a) Find all its fixed points and classify their linear stabilities.
(b) Prove this map has a superstable 2-cycle.