Problem 1 (WR Ch 7 #18). Let \( \{f_n\} \) be a uniformly bounded sequence of functions which are Riemann-integrable on \([a, b]\), and put
\[
F_n(x) = \int_a^x f_n(t) \, dt \quad (a \leq x \leq b).
\]
Prove that there exists a subsequence \( \{F_{n_k}\} \) which converges uniformly on \([a, b]\).

Solution. By Theorem 7.25, all we need to show is that \( \{F_n\} \) is pointwise bounded and equicontinuous. Since \( \{f_n\} \) is uniformly bounded, there exists some \( M > 0 \) such that \( |f_n(t)| < M \) for all \( t \in [a, b] \) and all \( n \). Therefore,
\[
|F_n(x)| = \left| \int_a^x f_n(t) \, dt \right| \leq \int_a^x |f_n(t)| \, dt \leq \int_a^x M \, dt = M(x - a) \quad \text{for all} \ n,
\]
which proves pointwise boundedness of \( \{F_n\} \). To prove equicontinuity, given some \( \epsilon > 0 \), we choose \( \delta = \epsilon/M \) so that for any \( x, y \in [a, b] \) such that \( |x - y| < \delta \), we have
\[
|F_n(x) - F_n(y)| = \left| \int_a^x f_n(t) \, dt - \int_a^y f_n(t) \, dt \right| = \left| \int_y^x f_n(t) \, dt \right| \leq \int_y^x |f_n(t)| \, dt < M\delta = \epsilon.
\]

Problem 2 (WR Ch 7 #19). Let \( K \) be a compact metric space, let \( S \) be a subset of \( C(K) \). Prove that \( S \) is compact (with respect to the metric defined in Section 7.14) if and only if \( S \) is uniformly closed, pointwise bounded, and equicontinuous. (If \( S \) is not equicontinuous, then \( S \) contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on \( K \).)

Solution.
\[ \implies \] Assume \( S \) is compact in \( C(K) \). By Theorem 2.34, \( S \) is uniformly closed. Let \( x \in K \). Define the sets
\[
U_n = \{ f \in C(K) : \|f\| < n \},
\]
(which are the open balls of radius \( n \) around the function \( g \equiv 0 \) in the uniform metric). Notice that \( \bigcup U_n \supset C(K) \supset S \). By compactness (and the fact that \( U_n \subset U_{n+1} \)), there is some \( N \) such that \( S \subset U_N \), which means \( \|f\| \leq N \) for all \( f \in S \). This means that \( S \) is uniformly bounded, and thus pointwise bounded.
Lastly, we need to check equicontinuity. Set $\epsilon > 0$. First we want to prove that there are some finite number of functions $f_1, \ldots, f_n \in S$ such that for each $f \in S$, $\|f - f_i\| < \epsilon/3$ for some $i$. Assume otherwise. Then the balls $B(f; \epsilon/3)$ form an open cover of $S$ which does not have a finite subcover, contradicting compactness.

Next, since each $f_i$ for $1 \leq i \leq n$ is continuous on a compact set $K$, each one is uniformly continuous, so that there exists some $\delta_i > 0$ such that $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$ whenever $d(x, y) < \delta_i$.

Now, given any $f \in S$, we choose $i$ such that $\|f - f_i\| < \frac{\epsilon}{3}$, and then for $\delta = \max\{\delta_1, \ldots, \delta_n\}$ we have

$|x - y| < \delta \implies |f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$

$\Rightarrow$ Assume $S$ is uniformly closed, pointwise bounded, and equicontinuous. By Theorem 7.25, any infinite subset of $S$ has a limit point, and since $S$ is uniformly closed, this limit point is in $S$. Therefore, by Exercise 2.26 (which we proved last quarter), $S$ is compact.

**Problem 3** (WR Ch 7 #20). If $f$ is continuous on $[0,1]$ and if

$$\int_0^1 f(x) x^n \, dx = 0 \quad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0,1]$.

**Solution.** Since $f$ is continuous on a compact set, it is bounded. So there exists some $M$ such that $|f(x)| \leq M$ on $[0,1]$. By the Weierstrass Approximation Theorem, there exists some sequence of polynomials $\{p_n\}$ such that $p_n \Rightarrow f$. Thus or any $\epsilon > 0$ we can choose some $N \in \mathbb{N}$ s.t. $|f(x) - p_n(x)| < \frac{\epsilon}{M}$ for $n \geq N$. But this means that

$$|f(x)p_n(x) - f^2(x)| = |f(x)||f(x) - p_n(x)| \leq M|f(x) - p_n(x)| < \epsilon \quad \text{for } n \geq N,$$

so $f_n \Rightarrow f^2$. At this point we should mention that, by the linearity of the integral, $\int_0^1 f(x)p(x) \, dx = 0$ for any polynomial. Therefore by Theorem 7.16 we have

$$\int_0^1 f^2(x) \, dx = \lim_{n \to \infty} \int_0^1 f(x)p_n(x) \, dx = \lim_{n \to \infty} 0 = 0,$$

and thus $f(x) = 0$ on $[0,1]$ by Exercise 6.2.
Problem 4 (WR Ch 7 #22). Assume \( f \in \mathcal{R}(\alpha) \) on \([a, b]\), and prove that there are polynomials \( P_n \) such that
\[
\lim_{n \to \infty} \int_a^b |f - P_n|^2 \, d\alpha = 0.
\]

Solution. We need to show that for every \( \epsilon > 0 \) there exists some polynomial \( P(x) \) such that \( \int_a^b |f - P|^2 \, d\alpha < \epsilon \). Since \( f \in \mathcal{R}(\alpha) \) on \([a, b]\), by Exercise 6.12 there exists a continuous function \( g \) such that
\[
\left\{ \int_a^b |f - g|^2 \, d\alpha \right\}^{1/2} < \sqrt{\frac{\epsilon}{2}}.
\]
By the Weierstrass Approximation Theorem there exists a polynomial \( P(X) \) such that
\[
\|g - P\| < \sqrt{\frac{\epsilon}{2 \int_a^b d\alpha}}.
\]
Putting this together, we have
\[
\int_a^b |f - P|^2 \, d\alpha \leq \int_a^b |f - g|^2 \, d\alpha + \int_a^b |g - P|^2 \, d\alpha \leq \frac{\epsilon}{2} + \frac{\epsilon}{2 \int_a^b d\alpha} \int_a^b d\alpha = \epsilon.
\]