Lecture 3: 1.4: Cylindrical and Spherical Coordinates. Recall that in the plane it is sometimes useful to introduce polar coordinates. There are two possible natural and useful generalizations of this to space:

Cylindrical coordinates (r, θ, z) of a point $P(x, y, z)$ are obtained by using polar coordinates (r, θ) of the projection in the x-y plane and leaving z unchanged:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

To convert from rectangular to cylindrical coordinate we use:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z.$$

Note that the surfaces $r = c$ are cylinders of all points at distance c from the z-axis. The surfaces $\theta = c$ are half planes from the z-axis at an angle c with the x-z plane. The surface $z = c$ are the planes parallel to and at a distance c from the x-y plane.

Spherical coordinates (ρ, θ, ϕ) of a point $P(x, y, z)$ are obtained by first using polar coordinates of the projection in the x-y plane. We use the angle θ in the x-y plane but instead of the distance r to the z axis use the distance to the origin ρ:

$$\rho^2 = x^2 + y^2 + z^2$$

Then $\rho^2 = r^2 + z^2$ and we introduce another angle ϕ to describe the hight above the x-y coordinate plane: $z = \rho\cos\phi$ and $r = \rho\sin\phi$. We obtain the coordinates:

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\cos\theta, \qquad z = \rho\cos\phi$$

where we must have

$$\rho \geq 0, \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi\pi$$

Note that the surfaces $\rho = c$ are spheres of radius c.
The surfaces $\theta = c$ are half planes from the z-axis at an angle c with the x-z plane.
The surfaces $\phi = c$ are half cones at an angle c with the positive z-axis.

Ex. Write down the equations in both cylindrical and spherical coordinates.
(a) $z^2 = x^2 + y^2$, $z \geq 0$, (b) $z^2 = x^2 + y^2 + 1$, (c) $x^2 + y^2 + z^2 = 4$.

Sol. (a) $z^2 = r^2$ in cylindrical coordinates and $\phi = \pi/4$ in spherical coordinates.
(b) $z^2 = r^2 + 1$ in cylindrical coordinates and $\rho^2\cos^2\phi = \rho^2\sin^2\phi + 1$ in spherical.
(c) $z^2 + r^2 = 4$ in cylindrical coordinates and $\rho^2 = 4$ in spherical coordinates.

Ex. Write down in in cylindrical and spherical coordinates the inequalities for the solid region $\{(x, y, z); x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$.

Sol. In cylindrical coordinates: $\{(r, \theta, z);\ 0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/2,\ 0 \leq z \leq \sqrt{1 - r^2}\}$ and in spherical coordinates it is $\{(\rho, \theta, \phi);\ 0 \leq \rho \leq 1,\ 0 \leq \theta \leq \pi/2,\ 0 \leq \phi \leq \pi/2\}$.

2.1 Functions and graphs.

A function $f : \mathbf{R}^n \to \mathbf{R}^m$ send each point $\mathbf{x} \in \mathbf{R}^n$ to a specific point $f(\mathbf{x}) \in \mathbf{R}^m$. If $m = 1$ it is called a scalar valued function and if $m > 1$ a vector valued function.
Ex The temperature at each point in space is a scalar function (or field) $f:\mathbf{R}^3 \to \mathbf{R}$.
Ex The velocity of a fluid is a vector that depends on the point in space as well as the time so it is a vector function (or field) $f : \mathbf{R}^4 \to \mathbf{R}$.

The **graph** of a function $f:\mathbf{R}^2 \to \mathbf{R}$ is the set of all points (x, y, z) where $z = f(x, y)$.

A **level set** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is a set of all points $(x_1, ..., x_n)$ such that $f(x_1, ..., x_n) = c$. If $n = 2$ it is called a level curve $f(x, y) = c$ and if $n = 3$ it is called a level surface $f(x, y, z) = 0$.

Ex Draw the level curve $f(x, y) = x^2 + y^2 = 1$.
Sol The level curve is a circle.

Ex Draw the level surface $f(x, y, z) = x^2 + y^2 = 1$.
Sol The level surface is a cylinder since z is arbitrary.

A **section** of a graph of a function $f:\mathbf{R}^2 \to \mathbf{R}$ is obtained by intersecting the graph with vertical planes, e.g. setting $x = 0$ produces the section $z = f(0, y)$ which is the graph of a function of one variable.

One can also think of a level set of a function $f(x, y) = c$ raised to the graph $f(x, y) = c$ and $z = c$ as the intersection of the graph with the horizontal planes $z = c$.

Level sets and sections are useful tools to sketch a graph of a function $z = f(x, y)$.

Ex Sketch the elliptic paraboloid $z = x^2 + y^2$.
Sol (1) Draw the intersection with the $y - z$ plane when $x = 0$, $z = y^2$.
(2) Draw the level curves $z = 1, 2, 3, 4$ and raise them to the graph.
(2b) Alternatively, draw the intersection with the $x - z$ plane when $y = 0$ $z = x^2$.

Ex Sketch the hyperbolic paraboloid (or saddle) $z = y^2 - x^2$.
Sol One draws the intersection with the y-z plane, when $x = 0$ then $z = y^2$, and the intersection with the x-z plane, when $y = 0$ then $z = -x^2$.

2.2 Limits.

Ex Let $f(x) = \begin{cases} x, & x > 0, \\ 0, & x < 0 \end{cases}$
What is the limit of $f(x)$ as $x \to 0$? $\lim_{x\to 0} f(x) = 0$!

Ex Let $f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0 \end{cases}$
What is the limit of $f(x)$ as $x \to 0$? $f(x)$ does not have a limit as $x \to 0$ since its different from the different sides.

Intuitive definition of limit Suppose that $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$. The equation

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{L}$$

means that we can make $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\|$ arbitrarily small (i.e. near zero) by keeping $\|\mathbf{x} - \mathbf{a}\|$ sufficiently small (but not zero). The precise statement is: For every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|\mathbf{f}(\mathbf{x}) - \mathbf{L}\| < \varepsilon$ whenever $0 < \|\mathbf{x} - \mathbf{a}\| < \delta$.

Ex. Does $f(x, y) = \dfrac{x^2}{x^2 + y^2}$ have a limit when $(x, y) \to (0, 0)$?
Sol. Note that $\lim\limits_{x\to 0} f(x, 0) = 1$ and $\lim\limits_{y\to 0} f(0, y) = 0$. Hence the limit of $f(x, y)$ is different in different directions. Therefore the limit does not exist.

Def A function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is called **continuous** at $\mathbf{a}$ if

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a}).$$

Ex The function defined by $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$ is not continuous at 0.