Lecture 3: 1.4: Cylindrical and Spherical Coordinates. Recall that in the plane it is sometimes useful to introduce polar coordinates. There are two possible natural and useful generalizations of this to space:

**Cylindrical coordinates** \((r, \theta, z)\) of a point \(P(x, y, z)\) are obtained by using polar coordinates \((r, \theta)\) of the projection in the \(x\)-\(y\) plane and leaving \(z\) unchanged:

\[
x = r \cos \theta, \quad y = r \sin \theta, \quad z = z
\]

To convert from rectangular to cylindrical coordinate we use:

\[
r^2 = x^2 + y^2, \quad \tan \theta = \frac{y}{x}, \quad z = z.
\]

Note that the surfaces \(r = c\) are cylinders of all points at distance \(c\) from the \(z\)-axis. The surfaces \(\theta = c\) are half planes from the \(z\)-axis at an angle \(c\) with the \(x\)-\(z\) plane. The surface \(z = c\) are the planes parallel to and at a distance \(c\) from the \(x\)-\(y\) plane.

**Spherical coordinates** \((\rho, \theta, \phi)\) of a point \(P(x, y, z)\) are obtained by first using polar coordinates of the projection in the \(x\)-\(y\) plane. We use the angle \(\theta\) in the \(x\)-\(y\) plane but instead of the distance \(r\) to the \(z\) axis use the distance to the origin \(\rho\):

\[
\rho^2 = x^2 + y^2 + z^2
\]

Then \(\rho^2 = r^2 + z^2\) and we introduce another angle \(\phi\) to describe the height above the \(x\)-\(y\) coordinate plane: \(z = \rho \cos \phi\) and \(r = \rho \sin \phi\). We obtain the coordinates:

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \cos \theta, \quad z = \rho \cos \phi
\]

where we must have

\[
\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \pi
\]

Note that the surfaces \(\rho = c\) are spheres of radius \(c\).

The surfaces \(\theta = c\) are half planes from the \(z\)-axis at an angle \(c\) with the \(x\)-\(z\) plane. The surfaces \(\phi = c\) are half cones at an angle \(c\) with the positive \(z\)-axis.

**Ex.** Write down the equations in both cylindrical and spherical coordinates.

(a) \(z^2 = x^2 + y^2\); \(z \geq 0\), (b) \(z^2 = x^2 + y^2 + 1\), (c) \(x^2 + y^2 + z^2 = 4\).

**Sol.** (a) \(z^2 = r^2\) in cylindrical coordinates and \(\phi = \pi/4\) in spherical coordinates.

(b) \(z^2 = r^2 + 1\) in cylindrical coordinates and \(\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi + 1\) in spherical.

(c) \(z^2 + r^2 = 4\) in cylindrical coordinates and \(\rho^2 = 4\) in spherical coordinates.

**Ex.** Write down in in cylindrical and spherical coordinates the inequalities for the solid region \(\{(x, y, z); x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}\).

**Sol.** In cylindrical coordinates: \(\{(r, \theta, z); 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq z \leq \sqrt{1 - r^2}\}\) and in spherical coordinates it is \(\{(\rho, \theta, \phi); 0 \leq \rho \leq 1, 0 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/2\}\).
2.1 Functions and graphs.

A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) sends each point \( x \in \mathbb{R}^n \) to a specific point \( f(x) \in \mathbb{R}^m \).

If \( m = 1 \) it is called a scalar valued function and if \( m > 1 \) a vector valued function.

**Ex** The temperature at each point in space is a scalar function (or field) \( f : \mathbb{R}^3 \to \mathbb{R} \).

**Ex** The velocity of a fluid is a vector that depends on the point in space as well as the time so it is a vector function (or field) \( f : \mathbb{R}^4 \to \mathbb{R} \).

The **graph** of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) is the set of all points \((x, y, z)\) where \( z = f(x, y) \).

An **level set** of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is a set of all points \((x_1, ..., x_n)\) such that \( f(x_1, ..., x_n) = c \). If \( n = 2 \) it is called a level curve \( f(x, y) = c \) and if \( n = 3 \) it is called a level surface \( f(x, y, z) = 0 \).

**Ex** Draw the level curve \( f(x, y) = x^2 + y^2 = 1 \).

**Sol** The level curve is a circle.

**Ex** Draw the level surface \( f(x, y, z) = x^2 + y^2 = 1 \).

**Sol** The level surface is a cylinder since \( z \) is arbitrary.

A **section** of a graph of a function \( f : \mathbb{R}^2 \to \mathbb{R} \) is obtained by intersecting the graph with vertical planes, e.g. setting \( x = 0 \) produces the section \( z = f(0, y) \) which is the graph of a function of one variable.

One can also think of a level set of a function \( f(x, y) = c \) raised to the graph \( f(x, y) = c \) and \( z = c \) as the intersection of the graph with the horizontal planes \( z = c \).

Level sets and sections are useful tools to sketch a graph of a function \( z = f(x, y) \).

**Ex** Sketch the elliptic paraboloid \( z = x^2 + y^2 \).

**Sol** (1) Draw the intersection with the \( y-z \) plane when \( x = 0 \), \( z = y^2 \).

(2) Draw the level curves \( z = 1, 2, 3, 4 \) and raise them to the graph.

(2b) Alternatively, draw the intersection with the \( x-z \) plane when \( y = 0 \), \( z = x^2 \).

**Ex** Sketch the hyperbolic paraboloid (or saddle) \( z = y^2 - x^2 \).

**Sol** One draws the intersection with the \( y-z \) plane, when \( x = 0 \) then \( z = y^2 \), and the intersection with the \( x-z \) plane, when \( y = 0 \) then \( z = -x^2 \).
2.2 Limits.

Ex Let \( f(x) = \begin{cases} 
  x, & x > 0, \\
  0, & x < 0
\end{cases} \)

What is the limit of \( f(x) \) as \( x \to 0 \)? \( \lim_{x \to 0} f(x) = 0! \)

Ex Let \( f(x) = \begin{cases} 
  1, & x > 0, \\
  0, & x < 0
\end{cases} \)

What is the limit of \( f(x) \) as \( x \to 0 \)? \( f(x) \) does not have a limit as \( x \to 0 \) since its different from the different sides.

**Intuitive definition of limit** Suppose that \( f : \mathbb{R}^n \to \mathbb{R}^m \). The equation

\[
\lim_{x \to a} f(x) = L
\]

means that we can make \( \|f(x) - L\| \) arbitrarily small (i.e. near zero) by keeping \( \|x - a\| \) sufficiently small (but not zero). The precise statement is: For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \|f(x) - L\| < \varepsilon \) whenever \( 0 < \|x - a\| < \delta \).

Ex. Does \( f(x, y) = \frac{x^2}{x^2 + y^2} \) have a limit when \( (x, y) \to (0, 0) \)?

Sol. Note that \( \lim_{x \to 0} f(x, 0) = 1 \) and \( \lim_{y \to 0} f(0, y) = 0 \). Hence the limit of \( f(x, y) \) is different in different directions. Therefore the limit does not exist.

**Def** A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is called **continuous** at \( a \) if

\[
\lim_{x \to a} f(x) = f(a).
\]

Ex. The function defined by \( f(x) = 1 \) for \( x \geq 0 \) and \( f(x) = 0 \) for \( x < 0 \) is not continuous at 0.