Solutions for Math 20E Midterm 1, Fall 98, Lindblad.

1. (a)  \( \mathbf{A} = \overrightarrow{PQ} = 2i + 2k \) and \( \mathbf{B} = \overrightarrow{PR} = i + j + 2k \) are parallel to the plane so \( \mathbf{N} = \mathbf{A} \times \mathbf{B} = ... = -2i - 2j + 2k \) is normal to the plane. The equation for the plane is \(-2(x - 1) - 2(y - 2) + 2(z - 0) = 0\).

(b) The area of the triangle is \( |\mathbf{A} \times \mathbf{B}|/2 = \sqrt{2^2 + 2^2 + 2^2}/2 = \sqrt{3}. \)

2. We will need \( \mathbf{R}'(t) = \frac{3}{2}(1 + t)^{1/2}i - \frac{3}{2}(1 - t)^{1/2}j + k. \)

(a) \[ \int_C ds = \int_0^1 |\mathbf{R}'(t)| dt = \int_0^1 \sqrt{\frac{9}{4}(1 + t) + \frac{9}{4}(1 - t) + 1} dt = \sqrt{\frac{11}{2}}. \]

(b) \[ \int_C \mathbf{F} \cdot d\mathbf{R} = \int_C y \, dx + x \, dy + x \, dz = \int_0^1 \left( y \frac{dx}{dt} + x \frac{dy}{dt} + x \frac{dz}{dt} \right) dt = ... = \int_0^1 \left( (1 - t)^{3/2} \frac{3}{2}(1 + t)^{1/2} - (1 + t)^{3/2} \frac{3}{2}(1 - t)^{1/2} + (1 + t)^{3/2} \right) dt \]
\[ = \int_0^1 \left( -3t(1-t^2)^{1/2} + (1+t)^{3/2} \right) dt = (1-t^2)^{3/2} + \frac{2}{5}(1+t)^{5/2}\bigg|_0^1 = -1 + \frac{2}{5}(2^{5/2} - 1) \]

3. (a) A flow line satisfy \( \mathbf{R}'(t) = \beta(t)\mathbf{F}(\mathbf{R}(t)) \) for some scalar function \( \beta \). i.e.
\[ \frac{dx}{dt} = \beta F_1, \quad \frac{dy}{dt} = \beta F_2, \quad \frac{dz}{dt} = \beta F_3 \quad \implies \quad \frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \]

In our case
\[ \frac{dx}{y} = \frac{dy}{x}, \quad dz = 0 \quad \implies \quad x \, dx = y \, dy, \quad dz = 0 \quad \implies \quad y^2 = x^2 + C_1, \quad z = C_2 \]

Substituting in \( (x, y, z) = (1, 2, 2) \) gives \( C_1 = 3, \quad C_2 = 2 \) so \( y = \sqrt{x^2 + 3} \) and \( z = 2 \).

(b) \( \nabla \times \mathbf{F} = ... = 0 \) and \( \mathbf{F} \) is continuous everywhere so there is a potential.

(c) The potential satisfy \( \left\{ \begin{array}{c}
\phi_x = y \\
\phi_y = x \\
\phi_z = 0
\end{array} \right. \implies \left\{ \begin{array}{c}
\phi = xy + f(y, z) \\
\phi = xy + g(y, z) \\
\phi = h(x, y)
\end{array} \right. \)

for any functions \( f, g \) and \( h \). This has the solution \( f = g = 0 \) and \( h = xy \), in which case \( \phi(x, y, z) = xy \). It follows that \( \int_C \mathbf{F} \cdot d\mathbf{R} = \phi(1, 2, 2) - \phi(0, 0, 0) = 2 \).

4 (a) \( \nabla \times \mathbf{F} = ... = 1/\sqrt{x^2 + y^2} \). (b) \( \nabla \cdot \mathbf{F} = ... = 0 \). (c) No since, \( \nabla \times \mathbf{F} \neq 0 \).