Math 220, Fall **Final Exam** **Dec, 2005**

Instructor: Professor Ni

Name: ____________________

Student #: ____________________

Problem	Points
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Total (100 points)	
Extra Credit (10 points)	

General guidelines: Unless stated otherwise, you may cite without proof any theorem given in the text (unless explicitly stated otherwise). You need to reprove any result given in an exercise. If in doubt, please ask!

1. (30 points.) Simple proofs. For the following give a complete proof on the following statements as instructed.

   (a) (8 points) Let $f(t)$ be a complex valued continuous function defined on $[a, b]$. Prove that

$$|\int_a^b f(t)\,dt| \leq \int_a^b |f(t)|\,dt$$

(**Short proof-continued**)

(b) (8 points) Let γ be a smooth closed curve. Let a be a point of complex plane not contained in γ. Prove that $\int_\gamma \frac{dz}{z-a}$ is a multiple of $2\pi i$.

(**Short proof-continued**)

(c) (6 points) Using the fact that *any linear (fractional) transformation maps the real line onto either a circle or a straight line* to prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on either a circle or a straight line.

(**Short proof-continued**)

(d) (8 points) Prove the Liouville's theorem: *A function which is analytic and bounded in the whole plane must reduce to a constant.* (Hint: Clearly you can not just cited itself. However you may use Cauchy's integral formula.)

2. (15 points.) Suppose that a linear transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.

3. (15 points.) Find the conformal map, which maps the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ onto the disk $|w| < 1$.

4. (15 points) If $f(z)$ is analytic for $|z| < 1$ and satisfies $|f(z)| \leq 1$. Prove that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

5. (10 points) For any rectifiable curve $\gamma : [a, b] \to C$ (the complex plane) the following theorem was proved in the class.

Theorem. *Suppose that $f : [a, b] \to C$ is continuous. Then there exists a complex number I such that for every $\epsilon > 0$ there is a $\delta > 0$ such that when $P = \{t_0 < t_1 < \cdots < t_m\}$ is a partition of $[a, b]$ with $\|P\| \le \delta$ then*

$$|I - \sum_{k=1}^{m} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1}))| \le \epsilon$$

for whatever choice of point $\tau_k, t_{k-1} \le \tau_k \le t_k$.

The complex number I in the theorem is designated by $I = \int_a^b f d\gamma$.

Prove that if γ is piecewise smooth and f is continuous then

$$\int_a^b f\, d\gamma = \int_a^b f(t)\gamma'(t)\, dt.$$

6. (15 points) Let f and g be analytic functions on a region Ω (in complex plane) such that $f(z) \neq 0$ and $g(z) \neq 0$. Assume that there exists a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = a$ and $a_n \neq a$ for all n. If for all n

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)}$$

then there exists a constant c such that $f = cg$.

7. (Extra credit 10 points) Assume that $R(z)$ is a rational function so that $|R(z)| = 1$ for $|z| = 1$. Determine $R(z)$.

   Hint: 1) You have to prove your claim. Guessing the form does not earn you any credits.

END OF EXAM