Math 220, Fall **Midterm Exam** **Nov, 2010**

Instructor: Professor Ni

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Session #:	____________________

INSTRUCTIONS

1. **NO CALCULATOR.**
2. **CLOSE BOOK, CLOSE NOTES.**
3. **ID WILL BE CHECKED. GET IT READY!**

Problem	Points
Page 2 (40 points)	
Page 3 (12 points)	
Page 4 (12 points)	
Page 5 (12 points)	
Page 6 (24 points)	
Total (100 points)	

General guidelines: You may cite without proof any theorem given in the text (unless explicitly stated otherwise). You need to reprove any result given in an exercise. If in doubt, please ask!

1. (40 points.) True or False. For each of the following statement, determine if it is true or false. No proof or counter-example is needed.

(a) For any n positive integer, let $z = e^{\frac{2\pi}{n}\sqrt{-1}}$. Then $1 + z + \cdots + z^{n-1} = 0$.

False. It is not true for $n = 1$. *However it is true for* $n \geq 2$ *since* $z^n - 1 = 0$ *and* $z^n - 1 = (z-1)(1 + z + \cdots + z^{n-1}) = 0$, *but* $z \neq 1$.

(b) If $z \in \mathbf{C}$ and $\text{Re}(z^n) \geq 0$ for every positive integer n, then $z \geq 0$.

True. Assume that $z = r \cdot e^{\sqrt{-1}\theta}$ *with* $-\pi \leq \theta \leq \pi$. *Since* $\text{Re}(z) \geq 0$, $\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$. *Since* $\text{Re}(z^2) \geq 0$ *we in fact has* $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. *If* $\theta \neq 0$, *either* $\frac{-\pi}{2} < \theta < 0$ *or* $0 < \theta < \frac{\pi}{2}$. *For each of them we can show a contradiction by choosing suitable* n *so that* $n\theta \geq \frac{\pi}{2}$ *for the first time or* $n\theta \leq \frac{-\pi}{2}$ *for the first time.*

(c) A subset $G \subset \mathbf{C}$ is connected if and only if for any a, b in G, there is a polygon from a to b lying entirely inside G.

False. There exist a closed connected set of $\mathbf{R}^2$ *which is not path-connected.*

(d) Let (X, d) be a metric space. Then X is compact if and only if X is bounded and closed.

False. A counter example is the unit ball in a infinite dimensional normed space such as $L^2(a, b)$.

(e) If $A \subset X$ is connected, then A^- is connected.

True. We showed this in the class. See Proposition 2.8 of Ch2

2. (12 points.) Show that the closure of a totally bounded set is totally bounded.

Proof. *Let A be the concerned set. For any $\epsilon > 0$, since A is totally bounded, there exist finite many points $\{x_1\} \subset A$ for $1 \leq i \leq K$ such that $\cup_{i=1}^{K} B(x_i, \epsilon/2)$ covers A.*

For any $y \in A^-$, $B(y, \epsilon/2)$ contains some $x \in A$. But $x \in B(x_j, \epsilon/2)$. This implies that $y \in B(x_j, \epsilon)$. Hence $\cup_{i=1}^{K} B(x_i, \epsilon)$ covers A^-.

3. (12 points.) Find a conformal map which maps the unit disc onto $\{z \,|\, 0 < |z| < 1\}$.

First we have $f_1(z) = \frac{z+1}{z-1}$ *which maps* $0 \to -1$, $-1 \to 0$ *and* $1 \to \infty$. *Hence maps the unit disk to the left half plane* $\{w = u + \sqrt{-1}v | \, u \leq 0\}$.

Now let $f_2(w) = e^w$. *The composition* $e^{\frac{z+1}{z-1}}$ *does the job.*

4. (12 points.) Let A be a subset of a metric space (X, d). Prove that the set consisting of limit points of A is always closed.

Proof. *Let* L_A *be the set consisting of the limit points. Recall that* $x \in L_A$ *if and only if that for any neighborhood* $\mathcal{N}$ *of* x*,* $\mathcal{N} \cap \mathcal{A}$ *is a set of infinite many elements, which can be seen easily from the definition.*

Assume that $x_n \in L_A$ *and* $x_n \to x$*. We want to prove that* $x \in L_A$*. For any neighborhood* $\mathcal{N}$ *of* x*, we know that there exists* N *such that* $x_N \in \mathcal{N}$*. Hence there exists* $B(x_N, \delta) \subset \mathcal{N}$ *for some small* $\delta > 0$*. This then implies that* $B(x_N, \delta) \cap A$ *is a set of infinite elements since* $x_N \in L_A$*, which implies that* $\mathcal{N} \cap \mathcal{A}$ *is a set of infinite many elements.*

5. (24 points.) For C^1-function $f : \Omega \to \mathbf{C}$ with $f(x,y) = u(x,y) + \sqrt{-1}v(x,y)$, define $f_z = \frac{1}{2}(f_x - \sqrt{-1}f_y)$ and $f_{\bar{z}} = \frac{1}{2}(f_x + \sqrt{-1}f_y)$.

a) Check that the Cauchy-Riemann equation amounts to $f_{\bar{z}} = 0$.

b) Let $z(t) = x(t) + \sqrt{-1}y(t)$ be a curve, and $w(t) = f(z(t))$ be its image. check that

$$\dot{w}(t) = f_z(z(t))\dot{z}(t) + f_{\bar{z}}(z(t))\dot{\bar{z}}(t)$$

c) Using the above to show that if f has the angle preserving property, then f is holomorphic.

a) and b) are direct verifications.

c) Preserving the angle means that for two curves $z_1(t)$ and $z_2(t)$ passing z_0 at $t = 0$, angle between $w_2'(0)$ and $w_1'(0)$ is the same as $z_2'(0)$ and $z_1'(0)$. Here $w_i(t) = f(z_i(t)$ are the images.

Precisely $arg(w_2'(0)) - arg(w_1'(0)) = arg(z_2'(0)) - arg(z_1'(0))$ for any two curves passing z_0, which is equivalent to $arg(w'(0)) - arg(z'(0))$ is independent of $z'(0)$. This implies that

$$arg\left(\frac{w'(0)}{z'(0)}\right) = arg\left(f_z(z_0) + f_{\bar{z}}(z_0) \cdot \frac{\bar{z}'(0)}{z'(0)}\right)$$

is independent of $z'(0)$.

Observing that $a + b\frac{\bar{\eta}}{\eta}$ for any fixed complex numbers a and b, with η being all possible complex numbers traces out a circle centered at a with radius of $|b|$. The only possibility that all the points on this circle have a same argument is that $|b| = 0$, which amounts to $f_{\bar{z}}(z_0) = 0$. Since the argument holds for any z_0, it proves that f is holomorphic.

END OF EXAM