

Name:	_____
Student #:	_____
TA's Name:	_____
Session #:	_____

INSTRUCTIONS

- 1. NO CALCULATOR.**
- 2. CLOSE BOOK, CLOSE NOTES.**
- 3. ID WILL BE CHECKED. GET IT READY!**

Problem	Points
Page 2 (55 points)	
Page 3 (25 points)	
Page 4 (20 points)	
Total (100 points)	

General guidelines: You may cite without proof any theorem given in the text (unless explicitly stated otherwise). You need to reprove any result given in an exercise. If in doubt, please ask!

1. (55 points.) True or False. For each of the following statement, determine if it is true or false. No proof or counter-example is needed.

(a) A subset in a complete metric space is complete if and only if it is closed.

(b) For any complex number $|a| < 1$ and $|b| \leq 1$

$$\left| \frac{a-b}{1-\bar{a}b} \right| \leq 1.$$

(c) Let $f(z)$ be a real valued function (of complex variable). Assume that $f'(a)$ exists for some complex number a . Then $f'(a) = 0$.

(d) If A and B are closed subsets of a metric space X with $A \cap B = \emptyset$, then $d(A, B) > 0$, where $d(A, B)$ denotes the distance between A and B .

(e) The series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

is uniformly convergent for all $x \geq \frac{1}{2005}$.

(f) Let $P(z) = (z-1)(z-2)(z-3)\cdots(z-2004)(z-2005)$. Let $P^{(1000)}(z)$ be the 1000-th derivative of P (which is a polynomial of degree 1005). Then $P^{(1000)}(z)$ must have 1005 real roots.

(g) The radius of convergence of the power series $\sum n^p z^n$ (with $p > 0$) is p .

(h) Let $\{x_n\}$ be a sequence in a metric space. Then x is called a limit point of $\{x_n\}$ if any neighborhood of x contains infinity many elements from the set $X = \{x_n\}$.

(i) Let $E_1 \supset E_2 \supset E_3 \supset \cdots$ is a decreasing sequence of nonempty bounded closed sets in the complex plane, then the intersection $\bigcap_1^{\infty} E_i$ can not be empty.

(j) Let A be the set of points $z = (x, y)$ in the complex plane with $x = 0, |y| \leq 1$ and B be the set of points with $x > 0, y = \sin 1/x$. Then $A \cup B$ is not connected.

(k) The accumulation points of any set form a closed set.

2. (25 points.) Assume that $R(z)$ is a rational function so that $|R(z)| = 1$ for $|z| = 1$. Determine $R(z)$.

3. (20 points.) Do one of the following:

a) Prove the Abel's limit theorem: *If $\sum_0^\infty a_n$ converges, then $f(z) = \sum_0^\infty a_n z^n$ tends to $f(1)$ if $z \rightarrow 1$ in such a way that $|1 - z|/(1 - |z|)$ remains bounded.* (Hint: Obviously you can not just cite itself)

b) Let z and z' be two points on the complex plane. Denote $d(z, z')$ the (Euclidean) distance of their images under the stereographic projection. Prove that

$$d(z, z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}.$$

END OF EXAM