

COMPLEX INTEGRATION

UNDAMENTAL THEOREMS

Many important properties of analytic functions are very difficult to prove without use of complex integration. For instance, it is only recently that it became possible to prove, without resorting to complex integrals or equivalent tools, that the derivative of an analytic function is continuous, or that the higher derivatives exist. At present the integration-free proofs are, to say the least, much more difficult than the classical proofs.†

As in the real case we distinguish between *definite* and *indefinite integrals*. An indefinite integral is a function whose derivative equals a given analytic function in a region; in many elementary cases indefinite integrals can be found by inversion of known derivation formulas. The definite integrals are taken over differentiable or piecewise differentiable arcs and are not limited to analytic functions. They can be defined by a limit process which mimics the definition of a real definite integral. Actually, we shall prefer to define complex definite integrals in terms of real integrals. This will save us from repeating existence proofs which are essentially the same as in the real case. Naturally, the reader must be thoroughly familiar with the theory of definite integrals of real continuous functions.

1.1. Line Integrals. The most immediate generalization of a real integral is to the definite integral of a complex function over a real interval. If $f(t) = u(t) + iv(t)$ is a continuous function,

† Without use of integration R. L. Plunkett proved the continuity of the derivative (*Bull. Am. Math. Soc.* 65, 1959). E. H. Connell and P. Porcelli proved the existence of all derivatives (*Bull. Am. Math. Soc.* 67, 1961). Both proofs lean on a topological theorem due to G. T. Whyburn.

defined in an interval (a, b) , we set by definition

$$(1) \quad \int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

This integral has most of the properties of the real integral. In particular, if $c = \alpha + i\beta$ is a complex constant we obtain

$$(2) \quad \int_a^b cf(t) dt = c \int_a^b f(t) dt,$$

for both members are equal to

$$\int_a^b (\alpha u - \beta v) dt + i \int_a^b (\alpha v + \beta u) dt.$$

When $a \leq b$, the fundamental inequality

$$(3) \quad \left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt$$

holds for arbitrary complex $f(t)$. To see this we choose $c = e^{-i\theta}$ with a real θ in (2) and find

$$\operatorname{Re} \left[e^{-i\theta} \int_a^b f(t) dt \right] = \int_a^b \operatorname{Re} [e^{-i\theta} f(t)] dt \leq \int_a^b |f(t)| dt.$$

For $\theta = \arg \int_a^b f(t) dt$ the expression on the left reduces to the absolute value of the integral, and (3) results.†

We consider now a piecewise differentiable arc γ with the equation $z = z(t)$, $a \leq t \leq b$. If the function $f(z)$ is defined and continuous on γ , then $f(z(t))$ is also continuous and we can set

$$(4) \quad \int_{\gamma} f(z) dz = \int_a^b f(z(t)) z'(t) dt.$$

This is our *definition* of the complex line integral of $f(z)$ extended over the arc γ . In the right-hand member of (4), if $z'(t)$ is not continuous throughout, the interval of integration has to be subdivided in the obvious manner. Whenever a line integral over an arc γ is considered, let it be tacitly understood that γ is piecewise differentiable.

The most important property of the integral (4) is its invariance under a change of parameter. A change of parameter is determined by an increasing function $t = t(\tau)$ which maps an interval $\alpha \leq \tau \leq \beta$ onto $a \leq t \leq b$; we assume that $t(\tau)$ is piecewise differentiable. By the rule

† θ is not defined if $\int_a^b f dt = 0$, but then there is nothing to prove.

for changing the variable of integration we have

$$\int_a^b f(z(t))z'(t) dt = \int_a^b f(z(t(\tau)))z'(t(\tau))t'(\tau) d\tau.$$

But $z'(t(\tau))t'(\tau)$ is the derivative of $z(t(\tau))$ with respect to τ , and hence the integral (4) has the same value whether γ be represented by the equation $z = z(t)$ or by the equation $z = z(t(\tau))$.

In Chap. 3, Sec. 2.1, we defined the opposite arc $-\gamma$ by the equation $z = z(-t)$, $-b \leq t \leq -a$. We have thus

$$\int_{-\gamma} f(z) dz = \int_{-b}^{-a} f(z(-t))(-z'(-t)) dt,$$

and by a change of variable the last integral can be brought to the form

$$\int_b^a f(z(t))z'(t) dt.$$

We conclude that

$$(5) \quad \int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz.$$

The integral (4) has also a very obvious additive property. It is quite clear what is meant by subdividing an arc γ into a finite number of subarcs. A subdivision can be indicated by a symbolic equation

$$\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n,$$

and the corresponding integrals satisfy the relation

$$(6) \quad \int_{\gamma_1 + \gamma_2 + \cdots + \gamma_n} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz + \cdots + \int_{\gamma_n} f dz.$$

Finally, the integral over a closed curve is also invariant under a shift of parameter. The old and the new initial point determine two subarcs γ_1, γ_2 , and the invariance follows from the fact that the integral over $\gamma_1 + \gamma_2$ is equal to the integral over $\gamma_2 + \gamma_1$.

In addition to integrals of the form (4) we can also consider line integrals with respect to \bar{z} . The most convenient definition is by double conjugation

$$\int_{\gamma} f \bar{dz} = \overline{\int_{\gamma} \bar{f} dz}.$$

Using this notation, line integrals with respect to x or y can be introduced by

$$\begin{aligned} \int_{\gamma} f dx &= \frac{1}{2} \left(\int_{\gamma} f dz + \int_{\gamma} f \bar{dz} \right) \\ \int_{\gamma} f dy &= \frac{1}{2i} \left(\int_{\gamma} f dz - \int_{\gamma} f \bar{dz} \right). \end{aligned}$$

With $f = u + iv$ we find that the integral (4) can be written in the form

$$(7) \quad \int_{\gamma} (u \, dx - v \, dy) + i \int_{\gamma} (u \, dy + v \, dx)$$

which separates the real and imaginary part.

Of course we could just as well have started by defining integrals of the form

$$\int_{\gamma} p \, dx + q \, dy,$$

in which case formula (7) would serve as definition of the integral (4). It is a matter of taste which one prefers.

An essentially different line integral is obtained by integration with respect to *arc length*. Two notations are in common use, and the definition is

$$(8) \quad \int_{\gamma} f \, ds = \int_{\gamma} f |dz| = \int_{\gamma} f(z(t)) |z'(t)| \, dt.$$

This integral is again independent of the choice of parameter. In contrast to (5) we have now

$$\int_{-\gamma} f |dz| = \int_{\gamma} f |dz|$$

while (6) remains valid in the same form. The inequality

$$(9) \quad \left| \int_{\gamma} f \, dz \right| \leq \int_{\gamma} |f| \cdot |dz|$$

is a consequence of (3).

For $f = 1$ the integral (8) reduces to $\int_{\gamma} |dz|$ which is by definition the *length* of γ . As an example we compute the length of a circle. From the parametric equation $z = z(t) = a + \rho e^{it}$, $0 \leq t \leq 2\pi$, of a full circle we obtain $z'(t) = i\rho e^{it}$ and hence

$$\int_0^{2\pi} |z'(t)| \, dt = \int_0^{2\pi} \rho \, dt = 2\pi\rho$$

as expected.

1.2. Rectifiable Arcs. The length of an arc can also be defined as the least upper bound of all sums

$$(10) \quad |z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \cdots + |z(t_n) - z(t_{n-1})|$$

where $a = t_0 < t_1 < \cdots < t_n = b$. If this least upper bound is finite we say that the arc is *rectifiable*. It is quite easy to show that piecewise differentiable arcs are rectifiable, and that the two definitions of length coincide.

Because $|x(t_k) - x(t_{k-1})| \leq |z(t_k) - z(t_{k-1})|$, $|y(t_k) - y(t_{k-1})| \leq |z(t_k) - z(t_{k-1})|$ and $|z(t_k) - z(t_{k-1})| \leq |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})|$ it is clear that the sums (10) and the corresponding sums

$$\begin{aligned} &|x(t_1) - x(t_0)| + \cdots + |x(t_n) - x(t_{n-1})| \\ &|y(t_1) - y(t_0)| + \cdots + |y(t_n) - y(t_{n-1})| \end{aligned}$$

are bounded at the same time. When the latter sums are bounded, one says that the functions $x(t)$ and $y(t)$ are of *bounded variation*. An arc $z = z(t)$ is *rectifiable* if and only if the real and imaginary parts of $z(t)$ are of *bounded variation*.

If γ is rectifiable and $f(z)$ continuous on γ it is possible to define integrals of type (8) as a limit

$$\int_{\gamma} f ds = \lim \sum_{k=1}^n f(z(t_k)) |z(t_k) - z(t_{k-1})|.$$

Here the limit is of the same kind as that encountered in the definition of a definite integral.

In the elementary theory of analytic functions it is seldom necessary to consider arcs which are rectifiable, but not piecewise differentiable. However, the notion of rectifiable arc is one that every mathematician should know.

1.3. Line Integrals as Functions of Arcs. General line integrals of the form $\int_{\gamma} p dx + q dy$ are often studied as functions (or *functionals*) of the arc γ . It is then assumed that p and q are defined and continuous in a region Ω and that γ is free to vary in Ω . An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. In other words, if γ_1 and γ_2 have the same initial point and the same end point, we require that $\int_{\gamma_1} p dx + q dy = \int_{\gamma_2} p dx + q dy$. To say that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if γ is a closed curve, then γ and $-\gamma$ have the same end points, and if the integral depends only on the end points, we obtain

$$\int_{\gamma} = \int_{-\gamma} = - \int_{\gamma}$$

and consequently $\int_{\gamma} = 0$. Conversely, if γ_1 and γ_2 have the same end points, then $\gamma_1 - \gamma_2$ is a closed curve, and if the integral over any closed curve vanishes, it follows that $\int_{\gamma_1} = \int_{\gamma_2}$.

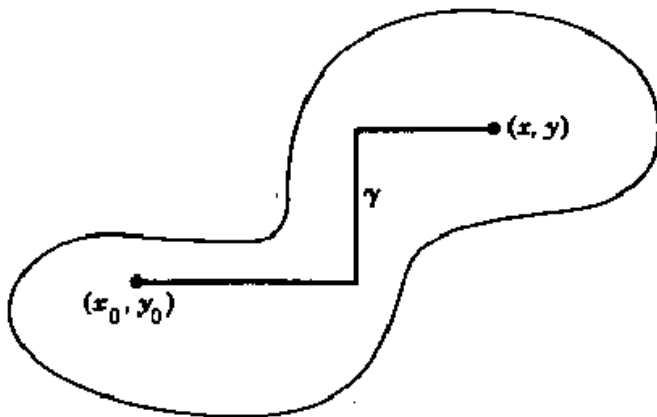


FIG. 4-1

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem 1. *The line integral $\int_{\gamma} p dx + q dy$, defined in Ω , depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with the partial derivatives $\partial U/\partial x = p$, $\partial U/\partial y = q$.*

The sufficiency follows at once, for if the condition is fulfilled we can write, with the usual notations,

$$\begin{aligned} \int_{\gamma} p dx + q dy &= \int_a^b \left(\frac{\partial U}{\partial x} x'(t) + \frac{\partial U}{\partial y} y'(t) \right) dt = \int_a^b \frac{d}{dt} U(x(t), y(t)) dt \\ &= U(x(b), y(b)) - U(x(a), y(a)), \end{aligned}$$

and the value of this difference depends only on the end points. To prove the necessity we choose a fixed point $(x_0, y_0) \in \Omega$, join it to (x, y) by a polygon γ , contained in Ω , whose sides are parallel to the coordinate axes (Fig. 4-1) and define a function by

$$U(x, y) = \int_{\gamma} p dx + q dy.$$

Since the integral depends only on the end points, the function is well defined. Moreover, if we choose the last segment of γ horizontal, we can keep y constant and let x vary without changing the other segments. On the last segment we can choose x for parameter and obtain

$$U(x, y) = \int^x p(x, y) dx + \text{const.},$$

the lower limit of the integral being irrelevant. From this expression it

follows at once that $\partial U/\partial x = p$. In the same way, by choosing the last segment vertical, we can show that $\partial U/\partial y = q$.

It is customary to write $dU = (\partial U/\partial x) dx + (\partial U/\partial y) dy$ and to say that an expression $p dx + q dy$ which can be written in this form is an *exact differential*. Thus an integral depends only on the end points if and only if the integrand is an exact differential. Observe that p, q and U can be either real or complex. The function U , if it exists, is uniquely determined up to an additive constant, for if two functions have the same partial derivatives their difference must be constant.

When is $f(z) dz = f(z) dx + if(z) dy$ an exact differential? According to the definition there must exist a function $F(z)$ in Ω with the partial derivatives

$$\begin{aligned} \frac{\partial F(z)}{\partial x} &= f(z) \\ \frac{\partial F(z)}{\partial y} &= if(z). \end{aligned}$$

If this is so, $F(z)$ fulfills the Cauchy-Riemann equation

$$\frac{\partial F}{\partial x} = -i \frac{\partial F}{\partial y};$$

since $f(z)$ is by assumption continuous (otherwise $\int_{\gamma} f dz$ would not be defined) $F(z)$ is analytic with the derivative $f(z)$ (Chap. 2, Sec. 1.2).

The integral $\int_{\gamma} f dz$, with continuous f , depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω .

Under these circumstances we shall prove later that $f(z)$ is itself analytic.

As an immediate application of the above result we find that

$$(11) \quad \int_{\gamma} (z - a)^n dz = 0$$

for all closed curves γ , provided that the integer n is ≥ 0 . In fact, $(z - a)^n$ is the derivative of $(z - a)^{n+1}/(n + 1)$, a function which is analytic in the whole plane. If n is negative, but $\neq -1$, the same result holds for all closed curves which do not pass through a , for in the complementary region of the point a the indefinite integral is still analytic and single-valued. For $n = -1$, (11) does not always hold. Consider a circle C with the center a , represented by the equation $z = a + \rho e^{it}$, $0 \leq t \leq 2\pi$. We obtain

$$\int_C \frac{dz}{z - a} = \int_0^{2\pi} i dt = 2\pi i.$$

This result shows that it is impossible to define a single-valued branch of $\log(z - a)$ in an annulus $\rho_1 < |z - a| < \rho_2$. On the other hand, if the closed curve γ is contained in a half plane which does not contain a , the integral vanishes, for in such a half plane a single-valued and analytic branch of $\log(z - a)$ can be defined.

EXERCISES

1. Compute

$$\int_{\gamma} x \, dz$$

where γ is the directed line segment from 0 to $1 + i$.

2. Compute

$$\int_{|z|=r} x \, dz,$$

for the positive sense of the circle, in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.

3. Compute

$$\int_{|z|=2} \frac{dz}{z^2 - 1}$$

for the positive sense of the circle.

4. Compute

$$\int_{|z|=1} |z - 1| \cdot |dz|.$$

5. Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in a region that contains γ). Show that

$$\int_{\gamma} \overline{f(z)} f'(z) \, dz$$

is purely imaginary. (The continuity of $f'(z)$ is taken for granted.)

6. Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω . Show that

$$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$

for every closed curve in Ω . (The continuity of $f'(z)$ is taken for granted.)

7. If $P(z)$ is a polynomial and C denotes the circle $|z - a| = R$, what is the value of $\int_C P(z) \, dz$? *Answer:* $-2\pi i R^2 P'(a)$.

6. Describe a set of circumstances under which the formula

$$\int_{\gamma} \log z \, dz = 0$$

is meaningful and true.

1.4. Cauchy's Theorem for a Rectangle. There are several forms of Cauchy's theorem, but they differ in their topological rather than in their analytical content. It is natural to begin with a case in which the topological considerations are trivial.

We consider, specifically, a rectangle R defined by inequalities $a \leq x \leq b, c \leq y \leq d$. Its perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that R lies to the left of the directed segments. The order of the vertices is thus $(a,c), (b,c), (b,d), (a,d)$. We refer to this closed curve as the *boundary curve* or *contour* of R , and we denote it by ∂R . †

We emphasize that R is chosen as a closed point set and, hence, is not a region. In the theorem that follows we consider a function which is analytic on the rectangle R . We recall to the reader that such a function is by definition defined and analytic in an open set which contains R .

The following is a preliminary version of *Cauchy's theorem*:

Theorem 2. *If the function $f(z)$ is analytic on R , then*

$$(12) \quad \int_{\partial R} f(z) \, dz = 0.$$

The proof is based on the method of bisection. Let us introduce the notation

$$\eta(R) = \int_{\partial R} f(z) \, dz$$

which we will also use for any rectangle contained in the given one. If R is divided into four congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$, we find that

$$(13) \quad \eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}),$$

for the integrals over the common sides cancel each other. It is important to note that this fact can be verified explicitly and does not make illicit use of geometric intuition. Nevertheless, a reference to Fig. 4-2 is helpful.

† This is standard notation, and we shall use it repeatedly. Note that by earlier convention ∂R is also the boundary of R as a point set (Chap. 3, Sec. 1.2).

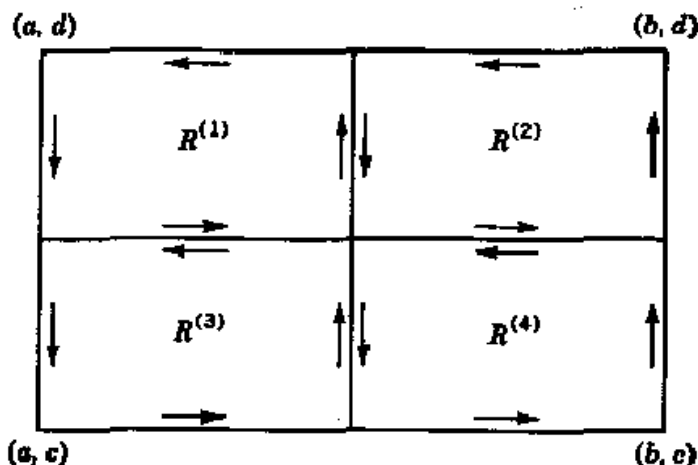


FIG. 4-2. Bisection of rectangle.

It follows from (13) that at least one of the rectangles $R^{(k)}$, $k = 1, 2, 3, 4$, must satisfy the condition

$$|\eta(R^{(k)})| \geq \frac{1}{4}|\eta(R)|.$$

We denote this rectangle by R_1 ; if several $R^{(k)}$ have this property, the choice shall be made according to some definite rule.

This process can be repeated indefinitely, and we obtain a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \dots \supset R_n \supset \dots$ with the property

$$|\eta(R_n)| \geq \frac{1}{4^n}|\eta(R_{n-1})|$$

and hence

$$(14) \quad |\eta(R_n)| \geq 4^{-n}|\eta(R)|.$$

The rectangles R_n converge to a point $z^* \in R$ in the sense that R_n will be contained in a prescribed neighborhood $|z - z^*| < \delta$ as soon as n is sufficiently large. First of all, we choose δ so small that $f(z)$ is defined and analytic in $|z - z^*| < \delta$. Secondly, if $\epsilon > 0$ is given, we can choose δ so that

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

or

$$(15) \quad |f(z) - f(z^*) - (z - z^*)f'(z^*)| < \epsilon|z - z^*|$$

for $|z - z^*| < \delta$. We assume that δ satisfies both conditions and that R_n is contained in $|z - z^*| < \delta$.

We make now the observation that

$$\int_{\partial R_n} dz = 0$$

$$\int_{\partial R_n} z dz = 0.$$

These trivial special cases of our theorem have already been proved in Sec. 1.1. We recall that the proof depended on the fact that 1 and z are the derivatives of z and $z^2/2$, respectively.

By virtue of these equations we are able to write

$$\eta(R_n) = \int_{\partial R_n} [f(z) - f(z^*) - (z - z^*)f'(z^*)] dz,$$

and it follows by (15) that

$$(16) \quad |\eta(R_n)| \leq \varepsilon \int_{\partial R_n} |z - z^*| \cdot |dz|.$$

In the last integral $|z - z^*|$ is at most equal to the length d_n of the diagonal of R_n . If L_n denotes the length of the perimeter of R_n , the integral is hence $\leq d_n L_n$. But if d and L are the corresponding quantities for the original rectangle R , it is clear that $d_n = 2^{-n}d$ and $L_n = 2^{-n}L$. By (16) we have hence

$$|\eta(R_n)| \leq 4^{-n} dL \varepsilon,$$

and comparison with (14) yields

$$|\eta(R)| \leq dL \varepsilon.$$

Since ε is arbitrary, we can only have $\eta(R) = 0$, and the theorem is proved.

This beautiful proof, which could hardly be simpler, is due to É. Goursat who discovered that the classical hypothesis of a continuous $f'(z)$ is redundant. At the same time the proof is simpler than the earlier proofs inasmuch as it leans neither on double integration nor on differentiation under the integral sign.

The hypothesis in Theorem 2 can be weakened considerably. We shall prove at once the following stronger theorem which will find very important use.

Theorem 3. *Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of interior points ζ_j . If it is true that*

$$\lim_{z \rightarrow \zeta_j} (z - \zeta_j)f(z) = 0$$

for all j , then

$$\int_{\partial R} f(z) dz = 0.$$

It is sufficient to consider the case of a single exceptional point ζ , for evidently R can be divided into smaller rectangles which contain at most one ζ_j .

We divide R into nine rectangles, as shown in Fig. 4-3, and apply