Math 220A Complex Analysis
Solutions to Homework #4
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Conway, Page 33, Problem 7.

Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1$, -1,and i.

Proof. The sequence

$$b_n = \left(\frac{1}{n}\right)^{\frac{1}{n(n+1)}}$$

is the subsequence of $|a_n|^{1/n}$ which consists of exactly the nonzero terms. It is easy to see that $\lim_{n\to\infty} b_n = 1$. (For example, take logarithms and use results from calculus.) Since the lim sup of a sequence is its largest subsequential limit, we have $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ and hence $R = 1^{-1} = 1$.

At $z = 1$ and $z = -1$, the series reduces to

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \, ,$$

which converges by the alternating series test. Of course the convergence here is conditional since $\sum 1/n$ diverges. At $z = i$, we obtain the sum

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots \, .$$

We consider two groupings of the above sum, both of which converge by the alternating series test:

$$1 - \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5}\right) - \left(\frac{1}{6} + \frac{1}{7}\right) + \dots \, ,$$

$$\left(1 - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) - \dots$$

It follows that the series converges at $z = i$. Indeed, we have shown that the partial sums S_{2n} and S_{2n+1} both converge; further it is clear that

$$|S_{2n+1} - S_{2n}| = 1/(2n+1) \to 0$$

and hence they must converge to the same limit. This convergence is conditional for the same reason as before. □

Conway, Page 44, Problem 13. Let $G = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\}$ and $n \in \mathbb{Z}^+$. Find all analytic functions $f : G \to \mathbb{C}$ such that $z = f(z)^n$ for all $z \in G$.

Proof. Let $\mathrm{Log}(z)$ denote the principal branch of the logarithm in G. Consider the function

$$f(z) := \exp\left(\frac{1}{n}\,\mathrm{Log}(z)\right).$$

(This is the "principal branch" of $z^{1/n}$ in G.) Observe that

$$\begin{aligned} f(z)^n &= \prod_{k=1}^{n} \exp\left(\frac{1}{n}\,\mathrm{Log}(z)\right) \\ &= \exp\left(n \cdot \frac{1}{n}\,\mathrm{Log}(z)\right) \\ &= \exp(\mathrm{Log}(z)) \\ &= z\,. \end{aligned}$$

Moreover, it is clear that $f(z) \neq 0$ for all $z \in G$ since $\exp(z)$ is never zero. Let $g(z)$ be any function which satisfies $g(z)^n = z$ for all $z \in G$. Then we have

$$\left(\frac{g(z)}{f(z)}\right)^n = 1$$

for all $z \in G$. Hence $g(z) = \zeta(z)f(z)$ where ζ_z is an n-th root of unity which a priori depends upon z. But since the function $g(z)/f(z)$ is continuous on G and the n-th roots of unity form a discrete subset of $\mathbb{C}$, we must have $\zeta_z = \zeta$ for all z for some n-th root of unity ζ. Thus we have shown that $g(z) = \zeta f(z)$ for some n-th root of unity ζ.

If we set $\zeta := \exp(2\pi i/n)$, we know that the n-th roots of unity are exactly $1, \zeta, \zeta^2, \ldots, \zeta^{n-1}$. Thus there are exactly n functions satisfying our criterion and they are given by

$$f_k(z) := \zeta^k f(z), \quad k = 0, 1, \ldots, n-1\,.$$

This completes the proof. ■

Conway, Page 44, Problem 21. Prove there is no branch of the logarithm defined on $G = \mathbb{C} \setminus \{0\}$.

Proof. Let $G' = \mathbb{C} \setminus \{z \in \mathbb{R} \mid z \leq 0\} \subseteq G$. Write Log to denote the principal branch of the log on G'. That is,

$$\mathrm{Log}(z) = \log|z| + i \arg(z)$$

with $\arg(z) \in (-\pi, \pi)$. By way of contradiction, suppose $f(z)$ is a branch of the logarithm defined on G. Clearly $f|_{G'}$ is a branch of the log on G'. Hence it differs from the principal branch by $2\pi i k$ for some $k \in \mathbb{Z}$. This gives

$$f(z) = \log|z| + i \arg(z) + 2\pi i k$$

for all $z \in G'$. By assumption f is analytic in G; in particular, f is continuous at -1. Therefore we must have

$$\lim_{\substack{\Im(z)>0 \\ z \to -1}} f(z) = \lim_{\substack{\Im(z)<0 \\ z \to -1}} f(z)\,.$$

However, we compute

$$\begin{aligned} \lim_{\substack{\Im(z)>0 \\ z \to -1}} f(z) &= i\pi + 2\pi i k \\ \lim_{\substack{\Im(z)<0 \\ z \to -1}} f(z) &= -i\pi + 2\pi i k \end{aligned}$$

and hence $i\pi = -i\pi$, a contradiction. $\blacksquare$