2. If a meromorphic function is defined by a discrete set $E \subset \Omega$ and $f \in \mathcal{H}(\Omega - E)$, then any pole of this function is contained in E.

Definition 5. Let Ω be an open set in $\mathbb{C}$ and f a meromorphic function on Ω. Let $a \in \Omega$ be a pole of f. We define the *order of the pole of f at a* as follows:

Let $f(z) = \sum\limits_{n=-N}^{\infty} c_n(z-a)^n$ be the Laurent expansion of f at a (it has the form given because of Lemma 3). We may assume that $c_{-N} \neq 0$. Then $-N < 0$ (since otherwise f would be holomorphic at a). The order of the pole of f at a is then N by definition.

Note. This is equivalent to requiring that $g(z) = (z-a)^N f(z)$ is holomorphic at a and that $g(a) \neq 0$.

Definition 6. Let Ω be a connected open set in $\mathbb{C}$ and let f be meromorphic on Ω, $f \not\equiv 0$.

Let $a \in \Omega$, and let $f(z) = \sum\limits_{-\infty}^{\infty} c_n(z-a)^n$ be the Laurent expansion of f at a.

We define the *order* $\operatorname{ord}_a(f)$ of f at a by

$$\operatorname{ord}_a(f) = \inf\{n \mid c_n \neq 0\}$$

(This is meaningful because of Lemma 3). If $f \equiv 0$, we set $\operatorname{ord}_a(0) = \infty$.

The next lemma contains results which are easy consequences of the definitions, so that the proofs will be omitted.

Lemma 4. *Let Ω be a connected open set in $\mathbb{C}$ and let f be a meromorphic function on Ω, $f \not\equiv 0$. Let $a \in \Omega$.*

1°. *a is a pole of f if and only if $\operatorname{ord}_a(f) < 0$. In this case, f has a pole of order $-\operatorname{ord}_a(f)$ at a.*

2°. *f is holomorphic and has a zero at a if and only if $\operatorname{ord}_a(f) > 0$. In this case $\operatorname{ord}_a(f)$ is called the order of the zero of f at a.*

3°. *f is holomorphic at $a \in \Omega$ if and only if $\operatorname{ord}_a(f) \geq 0$.*

4°. *f is holomorphic at a and $f(a) \neq 0$ if and only if $\operatorname{ord}_a(f) = 0$.*

5°. *If f and g are meromorphic functions on Ω, we have*

$$\operatorname{ord}_a(f \cdot g) = \operatorname{ord}_a(f) + \operatorname{ord}_a(g).$$

6°. *If f and g are meromorphic functions on Ω and $\lambda \in \mathbb{C}$, $\lambda \neq 0$, we have*

$$\operatorname{ord}_a(\lambda f) = \operatorname{ord}_a(f),$$
$$\operatorname{ord}_a(f+g) \geq \min\bigl(\operatorname{ord}_a(f), \operatorname{ord}_a(g)\bigr).$$

Further, if $\operatorname{ord}_a(f) \neq \operatorname{ord}_a(g)$, *we have*

$$\operatorname{ord}_a(f+g) = \min\bigl(\operatorname{ord}_a(f), \operatorname{ord}_a(g)\bigr).$$

Definition 7. Let f be meromorphic on the open set $\Omega \subset \mathbb{C}$, and let $a \in \Omega$.

We say that f has a *simple pole* at a if $\operatorname{ord}_a(f) = -1$.

We say that f has a *simple zero* at a if $\operatorname{ord}_a(f) = +1$.

If $\operatorname{ord}_a(f) = k > 0$, we call k the *order of the zero* of f at a. Thus a simple zero is a zero of order 1.

Definition 8. Let $a \in \mathbb{C}$, $r > 0$ and let f be holomorphic on $\{z \in \mathbb{C} \mid 0 < |z-a| < r\}$. We say that a is an *essential singularity of f* (or that f has an *essential singularity at a*) if, in the Laurent expansion $\sum\limits_{n=-\infty}^{\infty} c_n(z-a)^n$ of f at a, there are infinitely many $n < 0$ with $c_n \neq 0$.

This is equivalent to saying that f is not meromorphic on $D(a, r)$.

Theorem 4 (The Casorati–Weierstrass Theorem). *Let $a \in \mathbb{C}$, $r > 0$, $D^* = \{z \in \mathbb{C} \mid 0 < |z-a| < r\}$. Let $f \in \mathcal{H}(D^*)$ and suppose that a is an essential singularity of f. Then $f(D^*)$ is dense in $\mathbb{C}$.*

Proof. Suppose that this is false. Then, there exist $c \in \mathbb{C}$ and $\delta > 0$ such that

$$f(D^*) \cap \{w \in \mathbb{C} \mid |w-c| < \delta\} = \emptyset.$$

Hence, if $g = (f-c)^{-1}$, $g \in \mathcal{H}(D^*)$ and $|g(z)| \leq \delta^{-1}$ for $z \in D^*$. By the Riemann extension theorem, there is $G \in \mathcal{H}(D(a, r))$ such that $G | D^* = g$. Clearly then $G \not\equiv 0$ ($G \cdot (f-c) = 1$ on D^*) and $f = c + G^{-1}$ on D^*, so that f would be meromorphic on $D(a, r)$, contrary to our assumption that f has an essential singularity at a.

Very much more than this is true. The so-called "Big Picard Theorem" asserts that there can exist at most one $c \in \mathbb{C}$ such that $c \notin f(D^*)$. We shall prove this theorem in Chapter 4.

6 The Looman–Menchoff theorem

In this section, we shall prove the Looman–Menchoff theorem stated in §1.1.

We shall need, just in this section, to allow rectangles to degenerate into segments (or even points).

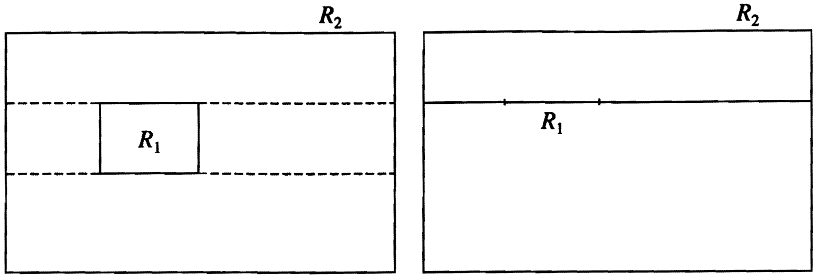

Figure 5.

Let $a \leq b$, $c \leq d$ be real numbers. The set

$$R = \{z \in \mathbb{C} | a \leq \mathfrak{Re}\, z \leq b, c \leq \mathfrak{Im}\, z \leq d\} = [a, b] \times [c, d]$$

will be called a *degenerate rectangle* if either $b = a$ or $d = c$ (or both). We define the boundary ∂R exactly as before (see §1.2, Example (c) and Definition 7). Note that if R is a degenerate rectangle and f is a continuous function defined on an open set containing $\mathrm{Im}(\partial R)$, then $\int_{\partial R} f\, dz = 0$.

Lemma 1. *Let R be a closed rectangle in $\mathbb{C}$ and let f be a continuous function on R which is holomorphic on the interior $\mathring{R}$ of R. Then $\int_{\partial R} f\, dz = 0$.*

Proof. Let $\varepsilon > 0$ and R_ε be a closed rectangle, $R_\varepsilon \subset \mathring{R}$, whose vertices converge to those of R as $\varepsilon \to 0$. Since f is uniformly continuous on R, we have

$$\int_{\partial R} f\, dz = \lim_{\varepsilon \to 0} \int_{\partial R_\varepsilon} f\, dz;$$

by §1.2, Theorem 1, $\int_{\partial R_\varepsilon} f\, dz = 0$. (The argument is the same as the one given in the proof of §1.2, Theorem 2.)

Lemma 2. *Let R_2 be a closed rectangle in $\mathbb{C}$ and R_1 a closed rectangle or a degenerate rectangle such that $R_1 \subset R_2$. Suppose that f is continuous on $R_2 - \mathring{R}_1$ and holomorphic on $\mathring{R}_2 - R_1$. Then*

$$\int_{\partial R_1} f\, dz = \int_{\partial R_2} f\, dz.$$

In particular, if R_1 is a degenerate rectangle, $\int_{\partial R_2} f\, dz = 0$.

Proof. $\int_{\partial R_2} f\, dz - \int_{\partial R_1} f\, dz$ can be written as the sum of four integrals of the form $\int_{\partial R} f\, dz$, R being a closed rectangle contained in $R_2 - \mathring{R}_1$ and such that f is holo-

---

Lemma 3. *Let $S \subset \mathbb{R}$ and let S_0 be the set of isolated points of S. Then S_0 is either nite or countable.*

roof. Let $\{I_\nu\}$ be the set of open intervals with rational endpoints arranged in a quence, $\nu = 1, 2, \ldots$. For $s \in S_0$, let $\nu(s)$ be the smallest integer $\nu \geq 1$ such that $\cap S = \{s\}$. Then the map $s \mapsto \nu(s)$ of S_0 into $\mathbb{N}$ is injective; in fact, if $\nu(s) = \nu(s')$, e have $\{s\} = I_{\nu(s)} \cap S = I_{\nu(s')} \cap S = \{s'\}$.

mma 4. *Let $I = [a, b] \subset \mathbb{R}$ be a closed interval and let ϕ be a complex valued ction defined on I. Suppose that ϕ has a derivative ϕ' at every point of I. Let E be a nonempty closed subset of I and $M > 0$ a constant such that*

$$|\phi(x) - \phi(y)| \leq M|x - y| \quad \textit{whenever } x \in E, y \in I.$$

en

$$\left|\phi(b) - \phi(a) - \int_E \phi'(x)\, dx\right| \leq M \cdot m_1(I - E),$$

ere m_1 is Lebesgue measure on $\mathbb{R}$. (Note that $|\phi'| \leq M$ on E, so that the integral sts.)

of. Let $J = [\alpha, \beta] \subset I$. We define the function $\phi_J : \mathbb{R} \to \mathbb{C}$ to be the unique ction of the form $x \mapsto \lambda x + \mu$ ($\lambda, \mu \in \mathbb{C}$ being constants) for which $\phi_J(\alpha) =$), $\phi_J(\beta) = \phi(\beta)$. (Explicitly $\lambda = (\phi(\beta) - \phi(\alpha))/(\beta - \alpha)$, $\mu = (\beta\phi(\alpha) -$ $(\beta))/(\beta - \alpha)$.) We have

$$|\phi_J(x) - \phi_J(y)| \leq \frac{|\phi(\beta) - \phi(\alpha)|}{\beta - \alpha} |x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

$E_0 = E \cup \{a\} \cup \{b\}$, and define a function ψ on I as follows:

) $\psi | E_0 = \phi | E_0$;

) if J is a complementary interval of E_0 in I (i.e., J is the closure of a connected component of $I - E_0$),

$$\psi | J = \phi_J | J.$$

that both endpoints of such a J lie in E_0, and at least one of them lies in E (since rwise $J = [a, b]$ and $E = \emptyset$). We claim that

$$|\psi(x) - \psi(y)| \leq M|x - y| \quad \text{for all } x, y \in I.$$

may suppose that $x < y$. There are two possible cases.

Case 1. x, y lie in the same complementary interval $J = [\alpha, \beta]$ of $I - E_0$. In this case

$$|\psi(x) - \psi(y)| \leq \frac{|\phi(\beta) - \phi(\alpha)|}{\beta - \alpha} |x - y|,$$

one of α, β lies in E. By hypothesis, $|\phi(\beta) - \phi(\alpha)| \leq M(\beta - \alpha)$, so that

$$|\psi(x) - \psi(y)| \leq M|x - y|.$$

Case 2. x, y do not both lie in the same complementary interval of $I - E_0$.

In this case, there is a point $\xi \in E$ with $x < \xi < y$, since otherwise, the open interval (x, y) would be disjoint from $E \cup \{a\} \cup \{b\}$, so that the closed interval $[x, y]$ would be contained in a complementary interval of E_0.

If $x \in E_0$, we have

$$|\psi(x) - \psi(\xi)| = |\phi(x) - \phi(\xi)| \leq M(\xi - x) \quad \text{since } \xi \in E.$$

If $x \notin E_0$, let J be the complementary interval of E_0 containing x, and let x' be the right-hand endpoint of J. Then

$$|\psi(x) - \psi(\xi)| \leq |\psi(x) - \psi(x')| + |\psi(x') - \psi(\xi)|;$$

now $|\psi(x) - \psi(x')| \leq M(x' - x)$ by Case 1, and $|\psi(x') - \psi(\xi)| \leq M(\xi - x')$ since $\xi \in E$, $x' \in E_0$. Hence

$$|\psi(x) - \psi(\xi)| \leq M(\xi - x).$$

Similarly, $|\psi(\xi) - \psi(y)| \leq M(y - \xi)$; adding these two inequalities gives $|\psi(x) - \psi(y)| \leq M(y - x)$. This proves that

$$|\psi(x) - \psi(y)| \leq M|x - y| \quad \text{for all } x, y \in I.$$

In particular, ψ is absolutely continuous, so that, by Lebesgue's theorem

$$\psi(b) - \psi(a) = \int_E \psi'(x)\,dx + \int_{I-E} \psi'(x)\,dx.$$

Now, $\psi(a) = \phi(a)$, $\psi(b) = \phi(b)$. Further, since ϕ' exists at every point of E, we have $\phi'(x) = \psi'(x)$ if x is a nonisolated point of E at which $\psi'(x)$ exists. Hence, by Lemma 3 and Lebesgue's theorem, $\phi' = \psi'$ almost everywhere on E. Finally, $|\psi'| \leq M$ almost everywhere on I. This gives

$$|\phi(b) - \phi(a) - \int_E \phi'(x)\,dx| = \left|\int_{I-E} \psi'(x)\,dx\right| \leq M \cdot m_1(I - E).$$

Lemma 5. *Let Ω be an open set in $\mathbb{C}$ and let f be a continuous function on Ω. Let $R = [a, b] \times [c, d]$ be a closed rectangle, $R \subset \Omega$, and let $A > 0$ be such that $A^{-1} \leq (d - c)/(b - a) \leq A$.*

Suppose that there is a nonempty closed set $E \subset \Omega$ and a constant $M > 0$ such that

$$\left\{\begin{aligned} &|f(x', y) - f(x, y)| \leq M|x' - x| \text{ whenever } (x, y) \in E \text{ and } (x', y) \in \Omega \\ \text{and } &|f(x, y') - f(x, y)| \leq M|y' - y| \text{ whenever } (x, y) \in E \text{ and } (x, y') \in \Omega. \end{aligned}\right.$$

Suppose also that $\partial f/\partial x$, $\partial f/\partial y$ exist at every point $z = x + iy \in \Omega$, $x, y \in \mathbb{R}$.

Let $R_0 \subset R$ be the intersection of all closed rectangles containing $E \cap R$; if $\cap R \neq \emptyset$, then R_0 is either a closed rectangle or a degenerate rectangle.

Under these conditions, we have

$$\left|\int_{\partial R_0} f\,dz - 2i \iint_{E\cap R} \frac{\partial f}{\partial \bar{z}}\,dz\,dy\right| \leq 8 \cdot A \cdot M \cdot m_2(R - R \cap E),$$

here m_2 denotes Lebesgue measure in $\mathbb{R}^2 = \mathbb{C}$.

oof. We may suppose that $E \cap R \neq \emptyset$, since otherwise the statement is trivial.

Let $R_0 = [a_0, b_0] \times [c_0, d_0] = I \times J$. For $x \in I$, let $E_x = \{y \in J | (x, y) \in E\}$. hypothesis we have

$$|f(x, y) - f(x, y')| \leq M|y' - y| \quad \text{if } y \in E_x,\ y' \in J.$$

nce, if $E_x \neq \emptyset$, Lemma 4 gives

$$\left|f(x, d_0) - f(x, c_0) - \int_{E_x} \frac{\partial f}{\partial y}\,dy\right| \leq M \cdot m_1(J - E_x) \leq 4AMm_1(J - E_x).$$

however, $E_x = \emptyset$, we can find $\xi, \xi' \in I$ such that $(\xi, c_0) \in E \cap R$, $(\xi', d_0) \in E \cap R$ definition of R_0 as the smallest, possibly degenerate, closed rectangle containing R). We then have

$$\begin{aligned} |f(x, d_0) - f(x, c_0)| &\leq |f(x, d_0) - f(\xi', d_0)| + |f(\xi', d_0) - f(\xi', c_0)| \\ &\quad + |f(\xi', c_0) - f(\xi, c_0)| + |f(\xi, c_0) - f(x, c_0)| \\ &\leq M\{|x - \xi'| + |d_0 - c_0| + |\xi' - \xi| + |\xi - x|\}. \end{aligned}$$

, $|d_0 - c_0| \leq (d - c)$, while $|x - \xi'| + |\xi' - \xi| + |\xi - x| \leq 3(b_0 - a_0) \leq 3(b - a) \leq$ $d - c)$.

Thus, if $E_x = \emptyset$, we have $|f(x, d_0) - f(x, c_0)| \leq (3A+1)M(d-c) \leq 4AM(d-c)$.

s, in either case,

$$\left|f(x, d_0) - f(x, c_0) - \int_{E_x} \frac{\partial f}{\partial y}\,dy\right| \leq 4A \cdot M \cdot \big(d - c - m_1(E_x)\big).$$

grating this inequality with respect to x over I, we obtain

$$\begin{aligned} &\left|\int_{a_0}^{b_0} \big(f(x, d_0) - f(x, c_0)\big)\,dx - \iint_{E\cap R} \frac{\partial f}{\partial y}\,dx\,dy\right| \\ &\leq 4A \cdot M \cdot \{(b_0 - a_0)(d - c) - m_2(E \cap R_0)\} \\ &\leq 4A \cdot M \cdot m_2(R - E \cap R), \quad \text{since } E \cap R_0 = E \cap R. \end{aligned} \tag{6.1}$$

he same way

$$\left|\int_{c_0}^{d_0} \big(f(b_0, y) - f(a_0, y)\big)\,dy - \iint_{E\cap R} \frac{\partial f}{\partial x}\,dx\,dy\right| \leq 4AMm_2(R - E \cap R). \tag{6.2}$$

Now

$$\int_{\partial R_0} f\,dz = i\int_{c_0}^{d_0}\big(f(b_0,y)-f(a_0,y)\big)\,dy + \int_{a_0}^{b_0}\big(f(x,c_0)-f(x,d_0)\big)\,dx$$

and $2i\,\partial f/\partial\overline{z} = i\,\partial f/\partial x - \partial f/\partial y$. Hence, if we multiply (6.2) by i and add the result to (6.1), we obtain

$$\left|\int_{\partial R_0} f\,dz - 2i\iint_{E\cap R}\frac{\partial f}{\partial\overline{z}}\,dx\,dy\right| \le 8AMm_2(R - R\cap E).$$

We pass now to the Looman–Menchoff theorem.

Theorem 1 (The Looman–Menchoff Theorem). *Let Ω be an open set in $\mathbb{C}$ and let f be a continuous function on Ω. Suppose that $\partial f/\partial x$, $\partial f/\partial y$ exist at every point of Ω and satisfy*

$$\frac{\partial f}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \equiv 0 \quad on\ \Omega.$$

Then f is holomorphic on Ω.

Proof. Let $\Omega' \subset \Omega$ be the set of $a \in \Omega$ which have neighborhoods U such that $f|U \in \mathcal{H}(U)$, and let $E = \Omega - \Omega'$; E is the smallest closed subset of Ω such that $f|\Omega - E$ is holomorphic.

The theorem asserts that, with the hypotheses made in Theorem 1, we have $E = \emptyset$. Suppose that the contrary holds: $E \ne \emptyset$.

We claim that there is an open set $W \subset \Omega$, and a constant $M > 0$ such that: $E \cap W \ne \emptyset$ and $|f(x',y) - f(x,y)| \le M|x'-x|$, $|f(x,y') - f(x,y)| \le M|y'-y|$ whenever $(x,y) \in E\cap W$ and (x',y) and $(x,y') \in W$.

We shall prove this claim using Baire's theorem (see the appendix to the book for this theorem). Let k be an integer ≥ 1, and set

$$\Omega_k^{(1)} = \left\{(x,y)\in\Omega \,\middle|\, |f(x',y)-f(x,y)| \le k|x'-x| \text{ for } |x'-x| \le \frac{1}{k}\right\}$$

$$\Omega_k^{(2)} = \left\{(x,y)\in\Omega \,\middle|\, |f(x,y')-f(x,y)| \le k|y'-y| \text{ for } |y'-y| \le \frac{1}{k}\right\}$$

and let $\Omega_k = \Omega_k^{(1)} \cap \Omega_k^{(2)}$. Then, since f is continuous, Ω_k is closed in Ω. Further, since $\partial f/\partial x$, $\partial f/\partial y$ exist everywhere, $(f(x',y) - f(x,y))/(x'-x)$ and $(f(x,y') - f(x,y))/(y'-y)$ converge as $x' \to x$ and $y' \to y$, for any $(x,y)\in\Omega$. Consequently $\bigcup_{k\ge1}\Omega_k = \Omega$; in particular, $\bigcup_{k\ge1}(\Omega_k\cap E) = E$. By Baire's theorem, at least one of the sets $\Omega_k \cap E$ must have nonempty interior in E, say $\Omega_{k_0}\cap E$. This means that there is an open set $W \subset \Omega$ such that

$$\emptyset \ne W\cap E \subset \Omega_{k_0}\cap E.$$

We may assume that W is relatively compact in Ω. There is then $c > 0$ such that $< c/2$ on W. Then, if $(x,y) \in E\cap W \subset \Omega_{k_0}\cap E$, and (x',y), $(x,y') \in W$, we

e

$$|f(x',y) - f(x,y)| \le \begin{cases} k_0|x'-x| & \text{if } |x'-x| \le \frac{1}{k_0} \\ ck_0|x'-x| & \text{if } |x'-x| > \frac{1}{k_0}, \end{cases}$$

similar inequalities for $|f(x,y') - f(x,y)|$. This proves our claim with $M =$ (k_0, ck_0).

To complete the proof of the theorem, it is sufficient to prove that $f|W$ is holomor- Because of Morera's theorem, we have only to prove that $\int_{\partial R} f\,dz = 0$ for any d rectangle $R = [a,b]\times[c,d] \subset W$ $(a < b, c < d)$.

hoose $A > 0$ so that $A^{-1} \le (d-c)/(b-a) \le A$. Let $\varepsilon > 0$ and let U be an set such that $E \subset U$ and $m_2(U - E) < \varepsilon$. (Such a U exists because any closed measurable, so that its outer measure equals its measure.)

et $N \ge 1$ be an integer. We divide R into 4^N congruent rectangles R_ν, $\nu =$ $\ldots, 4^N$ by iterating N times the operation of dividing a rectangle into four rectan- by joining the midpoints of opposite sides as in the proof of the Cauchy–Goursat em (§1.2, Theorem 1). If $R_\nu = [\alpha,\beta]\times[\gamma,\delta]$, we have $(\delta-\gamma)/(\beta-\alpha) =$ $c)/(b-a)$, so that $A^{-1} \le (\delta-\gamma)/(\beta-\alpha) \le A$.

N is sufficiently large, we have the following: if $R_\nu\cap E \ne \emptyset$, then $R_\nu \subset U$ $1,\ldots,4^N$). We have

$$\int_{\partial R} f\,dz = \sum_\nu \int_{\partial R_\nu} f\,dz = \sum_{R_\nu\cap E\ne\emptyset}\int_{\partial R_\nu} f\,dz$$

$\int_{\partial R_\nu} f\,dz = 0$ if $R_\nu \subset W - E$ (holomorphic functions being $\mathbb{C}$-differentiable, n apply the Cauchy–Goursat theorem).

et $R_\nu^{(0)}$ be the intersection of all closed rectangles containing $E\cap R_\nu$. $R_\nu^{(0)}$ is a bly degenerate closed rectangle and we have $\int_{\partial R_\nu} f\,dz = \int_{\partial R_\nu^{(0)}} f\,dz$ by Lemma plying Lemma 5 to a value of ν with $R_\nu\cap E\ne\emptyset$, we obtain

$$\begin{aligned}\left|\int_{\partial R_\nu} f\,dz\right| &= \left|\int_{\partial R_\nu^{(0)}} f\,dz\right| \\ &= \left|\int_{\partial R_\nu^{(0)}} f\,dz - 2i\iint_{E\cap R_\nu}\frac{\partial f}{\partial\overline{z}}\,dx\,dy\right| \le 8AMm_2(R_\nu - R_\nu\cap E).\end{aligned}$$

e $\partial f/\partial\overline{z} \equiv 0$). Hence

$$\left|\int_{\partial R} f\,dz\right| \le \sum_{R_\nu\cap E\ne\emptyset}\left|\int_{\partial R_\nu} f\,dz\right| \le 8AM\sum_{R_\nu\cap E\ne\emptyset} m_2(R_\nu - R_\nu\cap E).$$

$R_\nu\cap E\ne\emptyset$ implies that $R_\nu \subset U$. Further, the intersection of two distinct sets $R_{\nu'}$, $\nu\ne\nu'$ has two-dimensional Lebesgue measure 0. Hence

$$\sum_{R_\nu\cap E\ne\emptyset} m_2(R_\nu - R_\nu\cap E) \le m_2(U - U\cap E) < \varepsilon.$$

Thus

$$\left|\int_{\partial R} f\,dz\right| < 8AM\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_{\partial R} f\,dz = 0$. This proves that $f\,|\,W$ is holomorphic, contradicting our assumption that $W \cap E \neq \emptyset$. This contradiction proves the theorem.

References

[1] Ahlfors, L. V.: *Complex Analysis*, 3rd ed. New York, McGraw-Hill, 1979.

[2] Cartan, H.: *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Paris, 1961. (English translation: *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Addison-Wesley, 1963.)

[3] Connell, E. H. and P. Porcelli: A proof of the power series expansion without Cauchy's formula, *Bull. Amer. Math. Soc.* **67** (1961), 177–81.

[4] Conway, J. B.: *Functions of One Complex Variable*, Springer, 1973.

Hurwitz, A. and R. Courant: *Funktionentheorie*, 4th ed., with an appendix by H. Röhrl. Springer, 1964.

Looman, H.: Über die Cauchy–Riemannschen Differentialgleichungen, *Göttinger Nach.* (1923): 97–108.

Menchoff, D.: *Les conditions de monogénéité*, Paris, 1936.

Montel, P.: Sur les différentielles totales et les fonctions monogènes, *C. R. Acad. Sci. Paris* **156** (1913), 1820–1822.

Plunkett, R. L.: A topological proof of the continuity of the derivative of a function of a complex variable, *Bull. Amer. Math. Soc.* **65** (1959), 1–4.

Rudin, W.: *Real and Complex Analysis*, New York, McGraw-Hill, 1966.

Saks, S.: *Theory of the Integral*, 2nd ed. Warsaw, 1937, Dover reprint: 1954.

Saks, S. and A. Zygmund: *Analytic Functions*, Warsaw, 1952.

Whyburn, G. T.: *Topological analysis*, 2nd ed., Princeton, 1964.