**Math 240, Winter** **Midterm-Solution Exam** **February, 2015**
**Instructor: Professor Ni**

Name: ____________________

Student #: ____________________

TA's Name: ____________________

Session #: ____________________

**INSTRUCTIONS**

1. **NO CALCULATOR.**
2. **CLOSE BOOK, CLOSE NOTES.**
3. **ID WILL BE CHECKED. GET IT READY!**

| Problem | Points |
|---|---|
| Page 2 (16 points) | |
| Page 3 (16 points) | |
| Page 4 (16 points) | |
| Page 5 (16 points) | |
| Page 6 (16 points) | |
| Page 7 (20 points) | |
| Total (100 points) | |

Be sure to carefully motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text (as well as ones covered in the lecture). If you use a theorem from the text (or lecture), refer to it either by name (if it has one) or explain what it says. Also verify explicitly all hypotheses in the theorem. You need to reprove any result given as an exercise (unless it has been singled out and lectured upon).

1. State theorems/definitions, supply an example or supply a short proof.

   a) State the Tychonoff's theorem (regarding the compactness).

   b) Let $X$ be a topological space. Let $B(X, R)$ be the space of the bounded real-valued functions equipped with the uniform topology. Prove that $B(X, R)$ is complete.

   *Solution: I did this in the class. See the lecture notes*

c) State and prove the Urysohn's lemma for a metric space $(X, d)$.

*Solution: Given any two closed subset $A$ and $B$ so that $A \cap B = \emptyset$. Then there exists a continuous function such that $f|_A = 1$ and $f|_B = 0$.*

*Example:* $f(x) = \frac{d(x,B)}{d(x,A)+d(x,B)}$

d) Give an example of topological space $X$ and a subset $E$ along with a point $p \in X$ such that $p \in \overline{E}$, but there exists no sequence $\{x_n\} \subset E$ such that it converges to $p$ as $n \to \infty$.

*Solution: I did this in the class. Let $X = C^R$, namely all the complex valued function of real variable. Let $E = C(R)$, namely the continuous functions. Then if $p \in X$ is a non-measurable function. Then one can not have $\{f_n\} \subset C(R)$ with $f_n \to p$. But $p \in \overline{E}$ can be checked by definition.*

d) Define a subnet $\langle y_\beta \rangle_{\beta \in B}$ of a net $\langle x_\alpha \rangle_{\alpha \in A}$ (associated with a directed set $A$).

e) State the open mapping theorem.

2. Let $(X, \rho)$ be a metric space. A function $f \in C(X)$ is called Hölder continuous (of exponent $1 > \alpha > 0$) if

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)^\alpha}$$

if finite. If $X$ is compact, $\{f \in C(X) : \|f\|_u \leq 1; N_\alpha(f) \leq 1\}$ is compact in $C(X)$.

*Solution: Homework problem. See solutions of the Homework 2*

3. Let $X$ and $Y$ be Banach spaces. If $T : X \to Y$ is a linear map such that for any $f \in Y^*$, $x \to f(T(x))$ is an element of $X^*$, then $T$ is bounded.

*Solution: I did it in the lecture. The goal is to show that for $x$ with $\|x\| \le 1$, $\sup \|T(x)\| < \infty$.*

*We use the uniform boundedness principle to prove this. For $f \in Y^*$, $\widehat{T(x)}(f) = f(T(x))$. Here $\widehat{(\cdot)}$ is the isometric embedding of $Y$ into $(Y^*)^*$.*

Hence one have that

$$\sup_{\|x\|\le 1} |f(T(x))| \le \|f \cdot T\|\|x\| \le \|f \cdot T\|.$$

This shows that

$$\sup_{\|x\|\le 1} |\widehat{T(x)}(f)| < \infty$$

for any $f \in Y^*$.

Hence we have that

$$\sup_{\|x\|\le 1} \|T(x)\| = \sup_{\|x\|\le 1} \|\widehat{T(x)}\| < \infty$$

which is what we wanted.

*Another solution is via the closed graph theorem. Try to show that $G(T)$ is closed. Namely if $(x_n, T(x_n))$ converges to $(x, y)$ we show that $y = T(x)$. Prove by the contradiction. Assume not, by the consequences of the Hahn-Banach theorem there exists a linear functional $f \in Y^*$ such that $\|f\| = 1$, $f(y) \ne f(T(x))$.*

*Since $f(T(\cdot))$ is a bounded linear functional of $X$, $\lim_{n\to\infty} f(T(x_n)) = f(T(x))$. On the other hand if $y_n = T(x_n)$, $\lim_{n\to\infty} f(y_n) = f(y)$ since $y_n \to y$. Hence we have that $f(T(x)) = f(y)$, a contradiction!*

4. Let $X$ be a normed vector space.

   a) Assume that $E$ is a closed subspace such that $E \neq X$. Then for any $\epsilon > 0$, one can find $x \in X$ with $\|x\| = 1$ and $\|x + E\| > 1 - \epsilon$.

   b) Show that if $X$ is infinite dimensional, then the closed unit ball is not compact.

   *Solution: Both parts were covered in the lecture. For the first part it is also in the Homework 3.*

**END OF EXAM**