Math 240, Winter Theorems on Strong Convergence in $L^p(X)$ by L. Ni, 2009

Theorem 0.1 *Let $1 < p < \infty$. Assume that $f_n \in L^p(X)$, $f \in L^p(X)$, and $f_n \to f$ almost everywhere, $\|f_n\|_p \to \|f\|_p$. Then $f_n \to f$ in L^p-norm.*

Namely, convergence almost everywhere and convergence in norm implies the convergence in L^p. This result is not hard to prove. In fact the result also hold for $p = 1$.

Proof. We evoke the *Generalized Dominate Convergence Theorem* which asserts that if F_n, g_n, F, g in L^1 satisfies that $F_n \to F$ and $g_n \to g$, almost everywhere, $|F_n| \le g_n$ and $\int g_n \to \int g$, then $\int F_n \to \int F$.

Consider $F_n = |f_n - f|^p$. Note that $F_n \le 2^p(|f_n|^p + |f|^p)$, which is denoted by g_n. Clearly $F_n \to 0$ and $g_n \to 2^{p+1}|f|^p$. Now the claimed result follows easily. □

There are two results which are related to the above. However, the proofs to them are both harder. The first is about the weak convergence.

Theorem 0.2 *Let $1 < p < \infty$. Assume that $\sup_n \|f_n\|_p < \infty$ and $f_n \to f$ a.e. Then $f_n \to f$ weakly.*

This is Exercise 6.20 of Folland's book. The other is relating the weak convergence to the strong convergence, on which we shall discuss about its proof.

Theorem 0.3 *Assume that $f_n \to f$ weakly in L^p. Assume further that $\|f_n\| \to \|f\|$. Then $f_n \to f$ strongly in L^p.*

We divide the proof into two cases: $p \ge 2$ and $1 < p < 2$. First note that we may assume that f_n, f are real-valued.

Proof. (For the case $p \ge 2$.) We first claim that there exists a positive number C_1 such that

$$|1+t|^p \ge 1 + pt + C_1|t|^p.$$

Let $h(t) = |1+t|^p - 1 - pt$. It is easy to see that $h(0) = 0$, $h'(0) = 0$, $h''(t) \ge 0$. In fact $h'(t) = p\,\mathrm{sgn}(1+t)|1+t|^{p-1} - p$, $h''(t) = p(p-1)|1+t|^{p-2} > 0$, unless $t = -1$. This implies that $h(t) > 0$ for all $t \neq 0$. Let

$$\eta(t) = \frac{h(t)}{|t|^p}$$

which is positive and well-defined for all $t \neq 0$. Since $\eta(t) \to 1$ as $|t| \to \infty$ and

$$\lim_{t\to 0} \eta(t) = \lim_{t \to 0} \frac{|t+1|^{p-2}}{|t|^{p-2}}$$

which is either 1 or ∞, we can be sure that there exists an $C_1 > 0$ such that $\eta(t) \ge C_1$.

Now we have that

$$\left|1 + \frac{f_n - f}{f}\right|^p \ge 1 + p\frac{f_n - f}{f} + C_1\left|\frac{f_n - f}{f}\right|^p$$

which is equivalent to

$$\begin{aligned}|f_n|^p &\geq |f|^p + p(f_n - f)|f|^{p-1}\mathrm{sgn}(f) + C_1\,|f_n - f|^p \\ &= |f|^p + p(f_n - f)|f|^{p-2}f + C_1\,|f_n - f|^p\,.\end{aligned}$$

Now noting that $|f|^{p-1}\mathrm{sgn}(f) \in L^q(X)$, we have the claim after integrating the above estimate and letting $n \to \infty$. $\square$

Proof. (For the case $1 < p < 2$.) Now we define $\eta(t)$ to be $\frac{h(t)}{|t|^p}$ for $|t| \geq 1$, and to be $\frac{h(t)}{t^2}$ for $|t| \leq 1$. Again we have that $\eta(t) \geq C_1$ for some $C_1 > 0$. As above we have that for $|f_n - f| \geq |f|$,

$$|f_n|^p \geq |f|^p + p(f_n - f)|f|^{p-2}f + C_1\,|f_n - f|^p\,,$$

and for $|f_n - f| < |f|$,

$$|f_n|^p \geq |f|^p + p(f_n - f)|f|^{p-2}f + C_1(f_n - f)^2\frac{|f|^p}{f^2}.$$

Simple estimates give

$$\begin{aligned}\int_{\{|f_n - f|<|f|\}} |f_n - f|^p &\leq \int_{\{|f_n - f|<|f|\}} |f|^{p-1}|f_n - f| \\ &\leq \left(\int_{\{|f_n - f|<|f|\}} |f|^p\right)^{1/2}\left(\int_{\{|f_n - f|<|f|\}} |f|^{p-2}(f_n - f)^2\right)^{1/2} \\ &\leq \|f\|_p^{p/2}\frac{1}{C_1^{1/2}}\left(\int_X |f_n|^p - |f|^p - p(f_n - f)|f|^{p-2}f\right)^{1/2}.\end{aligned}$$

Combining the argument before, this is enough for the claim.