

Theorem 7.28. Let Ω be a $C^{1,1}$ domain in \mathbb{R}^n and $u \in W^{k,p}(\Omega)$. Then for any $\varepsilon > 0$, $0 < |\beta| < k$,

$$(7.61) \quad \|D^\beta u\|_{p;\Omega} \leq \varepsilon \|u\|_{k,p;\Omega} + C\varepsilon^{|\beta|/(|\beta|-k)} \|u\|_{p;\Omega}$$

where $C = C(k, \Omega)$.

Alternative derivations of interpolation inequalities are treated in Problems 2.15, 7.18 and 7.19. The density, extension, imbedding, and interpolation results of Theorems 7.25, 7.26 and 7.28 are all valid under less restrictive hypotheses on the domains Ω ; (see [AD]).

Notes

For related material on Sobolev spaces the reader is referred to the books [AD], [FR], [MY 5] and [NE]. We have followed the custom of referring to the spaces of this chapter as Sobolev spaces although various notions of spaces of weakly differentiable functions were used prior to Sobolev's work [SO 1]; (in this regard see [MY 1] and [MY 5]). The process of mollification or regularization appeared in Friedrich's work [FD 1]. The density theorem, Theorem 7.9, is due to Meyers and Serrin [MS 2]. The Sobolev inequalities, Theorem 7.10, were essentially proved by Sobolev [SO 1, 2]; we have followed the proof of Nirenberg [NI 3] for the case $p < n$. The Hölder estimates, Theorems 7.17 and 7.19 were derived by Morrey [MY 1]. Theorem 7.21 is due to John and Nirenberg [JN]; our proof is taken from [TR 2] where also the estimate Theorem 7.15 appeared. The compactness result, Theorem 7.22, is due to Rellich [RE] in the case $p = 2$ and to Kondrachov [KN] for the general case.

Problems

7.1. Let Ω be a bounded domain in \mathbb{R}^n . If u is a measurable function on Ω such that $|u|^p \in L^1(\Omega)$ for some $p \in \mathbb{R}$, we define

$$\Phi_p(u) = \left[\frac{1}{|\Omega|} \int_{\Omega} |u|^p dx \right]^{1/p}.$$

Show that: (i) $\lim_{p \rightarrow \infty} \Phi_p(u) = \sup_{\Omega} |u|$;

(ii) $\lim_{p \rightarrow -\infty} \Phi_p(u) = \inf_{\Omega} |u|$;

(iii) $\lim_{p \rightarrow 0} \Phi_p(u) = \exp \left[\frac{1}{|\Omega|} \int_{\Omega} \log |u| dx \right]$.

7.2. Show that a function u is weakly differentiable in a domain Ω if and only if it is weakly differentiable in a neighborhood of every point in Ω .

7.3. Let α, β be multi-indices and u be a locally integrable function on a domain Ω . Show that provided any two of the weak derivatives $D^{\alpha+\beta}u, D^\alpha(D^\beta u), D^\beta(D^\alpha u)$ exist, they all exist and coincide a.e. (Ω).

7.4. Derive the product formula (7.18). (Hint: consider first the case, $u \in W^1(\Omega), v \in C^1(\Omega)$).

7.5. Derive the formula (7.19), and show that it remains valid if we assume only $\psi \in C^{0,1}(\Omega), \psi^{-1} \in C^{0,1}(\bar{\Omega})$.

7.6. Let Ω be a domain in \mathbb{R}^n containing the origin. Show that the function γ given by $\gamma(x) = |x|^{-\alpha}$ belongs to $W^k(\Omega)$ provided $k + \alpha < n$.

7.7. Let Ω be a domain in \mathbb{R}^n . Show that a function $u \in C^{0,1}(\Omega)$ if and only if u is weakly differentiable with locally bounded weak derivatives.

7.8. Let Ω be a domain in \mathbb{R}^n . Show that a function u is weakly differentiable in Ω if and only if it is equivalent to a function \bar{u} that is absolutely continuous on almost all line segments in Ω parallel to the coordinate axes and whose partial derivatives, (which consequently exist a.e. (Ω)), are locally integrable in Ω . (See [MY 5], p. 66). Derive from this characterization the product formula and chain rule for weak differentiation.

7.9. Show that the norms (7.22) and (7.23) are equivalent norms on $W^{k,p}(\Omega)$.

7.10. Prove that the space $W^{k,p}(\Omega)$ is complete under either of the norms (7.22), (7.23).

7.11. Let Ω be a domain whose boundary can be locally represented as the graph of a Lipschitz continuous function. Show that $C^\infty(\bar{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $1 \leq p < \infty, k \geq 1$, and compare this result with the density result in Theorem 7.25.

7.12. Let Ω be a $C^{0,1}$ domain. For any function $u \in W^{1,p}(\Omega)$ and $1 \leq p < n$, derive the Sobolev-Poincaré inequality

$$\|u - u_\Omega\|_{np/(n-p); \Omega} \leq C \|Du\|_{p; \Omega}$$

(where C is independent of u) by a contradiction argument based on the compactness result of Theorem 7.26.

7.13. Deduce from Theorem 7.19 the corresponding global result. Namely let $u \in W^1(\Omega), \partial\Omega \in C^{0,1}$ and suppose there exist positive constants K, α ($\alpha < 1$) such that

$$\int_{B_R} |Du| \, dx \leq KR^{n-1+\alpha} \quad \text{for all balls } B_R \subset \mathbb{R}^n.$$

Then $u \in C^{0,\alpha}(\bar{\Omega})$ and

$$[u]_{\alpha;\Omega} \leq CK$$

where $C = C(n, \alpha, \Omega)$.

7.14. Let Ω be a bounded domain for which an imbedding

$$W^{1,p}(\Omega) \rightarrow L^p(\Omega), \quad 1 \leq p < \infty,$$

is valid. Show that the imbedding

$$W^{1,p}(\Omega) \rightarrow L^q(\Omega)$$

is compact for any $q < p^*$.

7.15. Let Ω be a domain in \mathbb{R}^n . The *total variation* of a function $u \in L^1(\Omega)$ is defined by

$$\int_{\Omega} |Du| = \sup \left\{ \int_{\Omega} u \operatorname{div} \mathbf{v} \mid \mathbf{v} \in C_0^1(\Omega), |\mathbf{v}| \leq 1 \right\}.$$

Show that the space $BV(\Omega)$ of functions of finite total variation is a Banach space under the norm

$$\|u\|_{BV(\Omega)} = \|u\|_1 + \int_{\Omega} |Du|,$$

and that $W^{1,1}(\Omega)$ is a closed subspace.

7.16. Let $u \in BV(\Omega)$. By invoking the regularization of u and appropriately modifying the proof of Theorem 7.9, show that there exists a sequence $\{u_m\} \subset C^\infty(\Omega) \cap W^{1,1}(\Omega)$ such that $u_m \rightarrow u$ in $L^1(\Omega)$ and

$$\int_{\Omega} |Du_m| \rightarrow \int_{\Omega} |Du|.$$

7.17 Let Ω be a bounded domain for which the Sobolev imbedding

$$W^{1,1}(\Omega) \rightarrow L^{n/(n-1)}(\Omega)$$

is valid. Show that also

$$BV(\Omega) \rightarrow L^{n/(n-1)}(\Omega)$$

and furthermore that the imbedding

$$BV(\Omega) \rightarrow L^q(\Omega)$$

is compact for any $q < n/(n-1)$.

7.18. Derive Theorem 7.27 for $p \geq 2$ from Green's first identity (2.10); (see Problem 2.15).

7.19. Let Ω be a $C^{0,1}$ domain. Derive the interpolation inequality (7.61) in the weaker form

$$\|D^\beta u\|_{p;\Omega} \leq \varepsilon \|u\|_{k,p;\Omega} + C_\varepsilon \|u\|_{p;\Omega},$$

(C_ε independent of u), by means of a contradiction argument based on the compactness result of Theorem 7.26.

7.20. Using regularization, show that locally integrable solutions of Laplace's equation (in the sense of Problem 2.8) are smooth and hence deduce the validity of the interior estimates in Chapter 4 for such solutions of Poisson's equation.

7.21. Using Morrey's inequality (7.42), prove that functions in the Sobolev space $W^{1,p}(\Omega)$, where $p > n$, are classically differentiable almost everywhere in Ω .