

and the first Čech cohomology group

$$H^1(\underline{U}, \mathcal{C}) = \frac{Z^1(\underline{U}, \mathcal{C})}{B^1(\underline{U}, \mathcal{C})}$$

is the obstruction to solving the problem in general.

Dolbeault. As before, take f_α to be a local solution in U_α and let ρ_α be a bump function, 1 in a neighborhood of $p_n \in U_\alpha$ and having compact support contained in U_α . Then

$$\varphi = \sum_{\alpha} \bar{\partial}(\rho_{\alpha} f_{\alpha})$$

is a $\bar{\partial}$ -closed $C^\infty(0,1)$ -form on S ($\varphi \equiv 0$ in a neighborhood of p_n). If $\varphi = \bar{\partial}\eta$ for $\eta \in C^\infty(S)$, then the function

$$f = \sum_{\alpha} \rho_{\alpha} f_{\alpha} - \eta$$

satisfies the conditions of the problem; thus the obstruction to solving the problem is in $H_{\bar{\partial}}^{0,1}(S)$.

Sheaves

Given X a topological space, a *sheaf* \mathcal{F} on X associates to each open set $U \subset X$ a group $\mathcal{F}(U)$, called the sections of \mathcal{F} over U , and to each pair $U \subset V$ of open sets a map $r_{V,U}: \mathcal{F}(V) \rightarrow \mathcal{F}(U)$, called the restriction map, satisfying

1. For any triple $U \subset V \subset W$ of open sets,

$$r_{W,U} = r_{V,U} \cdot r_{W,V}.$$

By virtue of this relation, we may write $\sigma|_U$ for $r_{V,U}(\sigma)$ without loss of information.

2. For any pair of open sets $U, V \subset M$ and sections $\sigma \in \mathcal{F}(U)$, $\tau \in \mathcal{F}(V)$ such that

$$\sigma|_{U \cap V} = \tau|_{U \cap V}$$

there exists a section $\rho \in \mathcal{F}(U \cup V)$ with

$$\rho|_U = \sigma, \quad \rho|_V = \tau.$$

3. If $\sigma \in \mathcal{F}(U \cup V)$ and

$$\sigma|_U = \sigma|_V = 0$$

then $\sigma = 0$.

Notation. The following are the sheaves we will be dealing with most often. In every case the restriction maps are the obvious ones, and the groups are additive unless otherwise stated.

1. On any C^∞ manifold M , we define sheaves C^∞ , C^* , \mathcal{Q}^p , \mathcal{Z}^p , \mathcal{Z} , \mathcal{Q} , \mathcal{R} , and \mathcal{C} by

$$C^\infty(U) = C^\infty \text{ functions on } U$$

$$C^*(U) = \text{multiplicative group of nonzero } C^\infty \text{ functions on } U,$$

$$\mathcal{Q}^p(U) = C^\infty \text{ } p\text{-forms on } U,$$

$$\mathcal{Z}^p(U) = \text{closed } C^\infty \text{ } p\text{-forms on } U,$$

$$\mathcal{Z}(U), \mathcal{Q}(U), \mathcal{R}(U), \mathcal{C}(U) = \text{locally constant } \mathcal{Z}\text{-, } \mathcal{Q}\text{-, } \mathcal{R}\text{-, or } \mathcal{C}\text{-valued functions on } U.$$

2. If M is a complex manifold, $V \subset M$ an analytic subvariety of M , and $E \rightarrow M$ a holomorphic vector bundle (defined below), we define the sheaves \mathcal{O} , \mathcal{O}^* , \mathcal{Q}^p , $\mathcal{Q}^{p,q}$, $\mathcal{Z}_\partial^{p,q}$, \mathcal{I}_V , $\mathcal{O}(E)$, and $\mathcal{Q}^{p,q}(E)$ by

$$\mathcal{O}(U) = \text{holomorphic functions on } U,$$

$$\mathcal{O}^*(U) = \text{multiplicative group of nonzero holomorphic functions on } U,$$

$$\mathcal{Q}^p(U) = \text{holomorphic } p\text{-forms on } U,$$

$$\mathcal{Q}^{p,q}(U) = C^\infty \text{ forms of type } (p, q) \text{ on } U,$$

$$\mathcal{Z}_\partial^{p,q}(U) = \bar{\partial}\text{-closed } C^\infty \text{ forms of type } (p, q) \text{ on } U,$$

$$\mathcal{I}_V(U) = \text{holomorphic functions on } U \text{ vanishing on } V \cap U;$$

$$\mathcal{O}(E)(U) = \text{holomorphic sections of } E \text{ over } U,$$

$$\mathcal{Q}^{p,q}(E)(U) = C^\infty \text{ } E\text{-valued } (p, q)\text{-forms over } U.$$

3. If M is again a complex manifold, a *meromorphic function* f on an open set $U \subset M$ is given locally as the quotient of two holomorphic functions—i.e., for some covering $\{U_i\}$ of U , $f|_{U_i} = g_i/h_i$, where g_i, h_i are relatively prime in $\mathcal{O}(U_i)$ and $g_i h_j = g_j h_i$ in $\mathcal{O}(U_i \cap U_j)$. This definition makes implicit use of the proposition on p. 10. A meromorphic function f is not, strictly speaking, a function even if we consider ∞ a value: at points where $g_i = h_i = 0$, it is not defined. The sheaf of meromorphic functions on M is denoted \mathfrak{M} ; the multiplicative sheaf of meromorphic functions not identically zero is denoted \mathfrak{M}^* .

A map of sheaves $\mathcal{F} \xrightarrow{\alpha} \mathcal{G}$ on M is given by a collection of homomorphisms $\{\alpha_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U)\}_{U \subset M}$ such that for $U \subset V \subset M$, α_U and α_V commute with the restriction maps. The *kernel* of the map $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ is just the sheaf $\text{Ker}(\alpha)$ given by $\text{Ker}(\alpha)(U) = \text{Ker}(\alpha_U: \mathcal{F}(U) \rightarrow \mathcal{G}(U))$; it is easy to check that this assignment does in fact define a sheaf. The *cokernel* of α is harder to define: if we set $\text{Coker}(\alpha)(U) = \mathcal{G}(U)/\alpha_U \mathcal{F}(U)$, Coker may not satisfy the conditions on p. 35. [The basic example of this is the sheaf map

$$\exp: \mathcal{O} \rightarrow \mathcal{O}^*$$

on $\mathbb{C} - \{0\}$ given by sending $f \in \mathcal{O}(U)$ to $e^{2\pi\sqrt{-1}f} \in \mathcal{O}^*(U)$. The section $z \in \mathcal{O}^*(\mathbb{C} - \{0\})$ is not in the image of $\mathcal{O}(\mathbb{C} - \{0\})$ under \exp , but its restric-

tion to any contractible open set $U \subset \mathbb{C} - \{0\}$ is in the image of $\theta(U)$.] Instead, we take a section of the cokernel sheaf $\text{Coker}(\alpha)$ over U to be given by an open cover $\{U_\alpha\}$ of U together with sections $\sigma_\alpha \in \mathcal{G}(U_\alpha)$ such that for all α, β ,

$$\sigma_\alpha|_{U_\alpha \cap U_\beta} - \sigma_\beta|_{U_\alpha \cap U_\beta} \in \alpha_{U_\alpha \cap U_\beta}(\mathcal{F}(U_\alpha \cap U_\beta));$$

we identify two such collections $\{(U_\alpha, \sigma_\alpha)\}$ and $\{(U'_\alpha, \sigma'_\alpha)\}$ if for all $p \in U$ and $U_\alpha, U'_\beta \ni p$, there exists V with $p \in V \subset (U_\alpha \cap U'_\beta)$ such that $\sigma'_\alpha|_V - \sigma_\beta|_V \in \alpha_V(\mathcal{F}(V))$.

We say that a sequence of sheaf maps

$$0 \rightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \rightarrow 0 \quad \beta \circ \alpha = 0$$

is exact if $\mathcal{E} = \text{Ker}(\beta)$ and $\mathcal{G} = \text{Coker}(\alpha)$; in this case we also say that \mathcal{E} is a subsheaf of \mathcal{F} and \mathcal{G} the quotient sheaf of \mathcal{F} by \mathcal{E} , written \mathcal{F}/\mathcal{E} . More generally, we say a sequence

$$\cdots \rightarrow \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \xrightarrow{\alpha_{n+1}} \mathcal{F}_{n+2} \rightarrow \cdots$$

is exact if $\alpha_{n+1} \circ \alpha_n = 0$ and

$$0 \rightarrow \text{Ker}(\alpha_n) \rightarrow \mathcal{F}_n \rightarrow \text{Ker}(\alpha_{n+1}) \rightarrow 0$$

is exact for each n . Note that by our definition of Coker , this does not imply that

$$0 \rightarrow \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \rightarrow 0$$

is exact for all U ; it does imply that this sequence is exact at the first two stages for all U , and that for any section $\sigma \in \mathcal{G}(U)$ and any point $p \in U$ there exists a neighborhood V of p in U such that $\sigma|_V$ is in the image of β_V .

A note: if $M \subset N$ is a subspace, \mathcal{F} a sheaf on M , we can "extend \mathcal{F} by zero" to obtain a sheaf $\tilde{\mathcal{F}}$ on N , setting

$$\tilde{\mathcal{F}}(U) = \mathcal{F}(U \cap M)$$

and letting the restriction maps be the obvious ones. Thus we may consider \mathcal{F} as a sheaf on either M or N .

Examples

1. On any complex manifold, the sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

is exact, where i is the obvious inclusion and \exp the exponential map $\exp(f) = e^{2\pi\sqrt{-1}f}$. This fundamental sequence is called the *exponential sheaf sequence*.

2. If M is a complex manifold, $V \subset M$ a complex submanifold, the sheaf \mathcal{O}_V may, by extension by zero, be considered a sheaf on M . The sequence

$$0 \rightarrow \mathcal{I}_V \xrightarrow{i} \mathcal{O}_M \xrightarrow{r} \mathcal{O}_V \rightarrow 0,$$

where i is inclusion and r restriction, is then exact.

3. By the ordinary Poincaré lemma, the sequence

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{C}^\infty \xrightarrow{d} \mathcal{Q}^1 \xrightarrow{d} \mathcal{Q}^2 \rightarrow \dots$$

is exact on any real manifold.

4. By the $\bar{\partial}$ -Poincaré lemma, the sequence

$$0 \rightarrow \Omega^p \rightarrow \mathcal{Q}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{Q}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{Q}^{p,2} \rightarrow \dots$$

is exact on any complex manifold.

5. If M is a Riemann surface and we let \mathcal{PP} be the quotient sheaf of the sheaf \mathcal{M} by the subsheaf $\mathcal{O} \xrightarrow{i} \mathcal{M}$, then for $U \subset M$ open,

$$\mathcal{PP}(U) = \{(p_n, f_n)\} : \begin{cases} \{p_n\} \subset U \text{ discrete,} \\ f_n \in \mathcal{M}_{p_n} / \mathcal{O}_{p_n}; \end{cases}$$

i.e., giving a section of \mathcal{PP} over U is the same as specifying the data of a Mittag-Leffler problem for U .

Cohomology of Sheaves

Let \mathcal{F} be a sheaf on M , and $\underline{U} = \{U_\alpha\}$ a locally finite open cover. We define

$$\begin{aligned} C^0(\underline{U}, \mathcal{F}) &= \prod_{\alpha} \mathcal{F}(U_\alpha), \\ C^1(\underline{U}, \mathcal{F}) &= \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta), \\ &\vdots \end{aligned}$$

$$C^p(\underline{U}, \mathcal{F}) = \prod_{\alpha_0 \neq \alpha_1 \neq \dots \neq \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \dots \cap U_{\alpha_p})$$

An element $\sigma = \{\sigma_I \in \mathcal{F}(\cap U_{i_k})\}_{I=p+1}$ of $C^p(\underline{U}, \mathcal{F})$ is called a p -cochain of \mathcal{F} . We define a *coboundary operator*

$$\delta: C^p(\underline{U}, \mathcal{F}) \rightarrow C^{p+1}(\underline{U}, \mathcal{F})$$

by the formula

$$(\delta\sigma)_{i_0, \dots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \dots, \hat{i}_j, \dots, i_{p+1}} \Big|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}$$

In particular, if $\sigma = \{\sigma_U\} \in C^0(\underline{U}, \mathcal{F})$,

$$(\delta\sigma)_{U,V} = -\sigma_U + \sigma_V;$$

and if $\sigma = \{\sigma_{U,V}\} \in C^1(\underline{U}, \mathcal{F})$,

$$(\delta\sigma)_{U,V,W} = \sigma_{UV} + \sigma_{VW} - \sigma_{UW}$$

(omitting the restriction).

A p -cochain $\sigma \in C^p(\underline{U}, \mathcal{F})$ is called a *cocycle* if $\delta\sigma = 0$. Note that any cocycle σ must satisfy the skew-symmetry condition

$$\sigma_{i_0, \dots, i_p} = -\sigma_{i_0, \dots, i_{q-1}, i_q+1, i_q, i_q+2, \dots, i_p}.$$

σ is called a *coboundary* if $\sigma = \delta\tau$ for some $\tau \in C^{p-1}(\underline{U}, \mathcal{F})$. It is easy to see that $\delta^2 = 0$ —i.e., a coboundary is a cocycle—and we set

$$Z^p(\underline{U}, \mathcal{F}) = \text{Ker } \delta \subset C^p(\underline{U}, \mathcal{F})$$

and

$$H^p(\underline{U}, \mathcal{F}) = \frac{Z^p(\underline{U}, \mathcal{F})}{\delta C^{p-1}(\underline{U}, \mathcal{F})}.$$

Now, given two coverings $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ and $\underline{U}' = \{U'_\beta\}_{\beta \in I'}$ of M , we say that \underline{U}' is a *refinement* of \underline{U} if for every $\beta \in I'$ there exists $\alpha \in I$ such that $U'_\beta \subset U_\alpha$; we write $\underline{U}' < \underline{U}$. If $\underline{U}' < \underline{U}$, we can choose a map $\varphi: I' \rightarrow I$ such that $U'_\beta \subset U_{\varphi\beta}$ for all β ; then we have a map

$$\rho_\varphi: C^p(\underline{U}, \mathcal{F}) \rightarrow C^p(\underline{U}', \mathcal{F})$$

given by

$$(\rho_\varphi \sigma)_{\beta_0, \dots, \beta_p} = \sigma_{\varphi\beta_0, \dots, \varphi\beta_p}|_{U_{\beta_0} \cap \dots \cap U_{\beta_p}}.$$

Evidently $\delta \circ \rho_\varphi = \rho_\varphi \circ \delta$, and so ρ_φ induces a homomorphism

$$\rho: H^p(\underline{U}, \mathcal{F}) \rightarrow H^p(\underline{U}', \mathcal{F}),$$

which is independent of the choice of φ . (The reader may wish to check that the chain maps ρ_φ and ρ_ψ associated to two inclusion associations φ and ψ are *chain homotopic* and thus induce the same map on cohomology.) We define the p^{th} Čech cohomology group of \mathcal{F} on M to be the direct limit of the $H^p(\underline{U}, \mathcal{F})$'s as \underline{U} becomes finer and finer:

$$H^p(M, \mathcal{F}) = \varinjlim_{\underline{U}} H^p(\underline{U}, \mathcal{F}).$$

Where there is a possibility of confusion, we will denote Čech cohomology groups by \check{H} . Clearly, for any covering \underline{U}

$$H^0(M, \mathcal{F}) = H^0(\underline{U}, \mathcal{F}) = \mathcal{F}(M).$$

Note that if $M \subset N$ is a closed subspace, \mathcal{F} any sheaf on M , then extending \mathcal{F} by zero to a sheaf on N , we have

$$H^*(M, \mathcal{F}) = H^*(N, \mathcal{F}).$$

The definition of $H^*(M, \mathcal{F})$ as a direct limit is, in practice, more or less impossible to work with. What is needed is a simple sufficient condition on a cover \underline{U} for

$$H^*(\underline{U}, \mathcal{F}) = H^*(M, \mathcal{F}),$$

and this is provided by the

Leray Theorem. *If the covering \underline{U} is acyclic for the sheaf \mathcal{F} in the sense that*

$$H^q(U_{i_1} \cap \cdots \cap U_{i_p}, \mathcal{F}) = 0, \quad q > 0, \quad \text{any } i_1 \cdots i_p,$$

then $H^(\underline{U}, \mathcal{F}) \cong H^*(M, \mathcal{F})$.*

We will prove the Leray theorem in those cases where it will be used.

The most basic property of sheaf cohomology is: Given an exact sequence

$$0 \rightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \rightarrow 0$$

of sheaves on M , we have maps

$$C^p(\underline{U}, \mathcal{E}) \xrightarrow{\alpha} C^p(\underline{U}, \mathcal{F}), \quad C^p(\underline{U}, \mathcal{F}) \xrightarrow{\beta} C^p(\underline{U}, \mathcal{G})$$

that commute with δ and hence induce maps

$$H^p(M, \mathcal{E}) \xrightarrow{\alpha^*} H^p(M, \mathcal{F}), \quad H^p(M, \mathcal{F}) \xrightarrow{\beta^*} H^p(M, \mathcal{G}).$$

We next define the coboundary map $\delta^*: H^p(M, \mathcal{G}) \rightarrow H^{p+1}(M, \mathcal{E})$: given $\sigma \in C^p(\underline{U}, \mathcal{G})$ with $\delta\sigma = 0$, we can always pass to a refinement \underline{U}' of \underline{U} and find $\tau \in C^p(\underline{U}', \mathcal{F})$ such that $\beta(\tau) = \rho\sigma$. Then $\beta\delta\tau = \delta\beta\tau = \delta\rho\sigma = 0$, so by passing to a further refinement \underline{U}'' we can find $\mu \in C^{p+1}(\underline{U}'', \mathcal{E})$ such that $\alpha\mu = \delta\tau$; $\alpha\delta\mu = \delta\alpha\mu = \delta^2\tau = 0$ and since α is injective this means $\delta\mu = 0$. Thus $\mu \in Z^{p+1}(\underline{U}'', \mathcal{E})$ and we take $\delta^*\sigma = M \in H^{p+1}(M, \mathcal{E})$.

Basic Fact. *The sequence*

$$\begin{aligned} 0 &\rightarrow H^0(M, \mathcal{E}) \rightarrow H^0(M, \mathcal{F}) \rightarrow H^0(M, \mathcal{G}) \\ &\rightarrow H^1(M, \mathcal{E}) \rightarrow H^1(M, \mathcal{F}) \rightarrow H^1(M, \mathcal{G}) \rightarrow \cdots \\ &\vdots \\ &\rightarrow H^p(M, \mathcal{E}) \rightarrow H^p(M, \mathcal{F}) \rightarrow H^p(M, \mathcal{G}) \rightarrow \cdots \end{aligned}$$

is exact.

For most exact sequences $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ that actually arise naturally—and certainly for all sheaves with which we shall deal in this book—it is the case that there exist arbitrarily fine coverings \underline{U} such that for every

open set $U = U_{i_0} \cap \dots \cap U_{i_r}$, the sequence

$$0 \rightarrow \mathcal{E}(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{G}(U) \rightarrow 0$$

is exact. Thus, we can find arbitrarily fine coverings \underline{U} of M for which the cochain groups form an exact sequence

$$0 \rightarrow C^p(\underline{U}, \mathcal{E}) \rightarrow C^p(\underline{U}, \mathcal{F}) \rightarrow C^p(\underline{U}, \mathcal{G}) \rightarrow 0.$$

In this case, our basic fact is easy to verify: for example, to see that

$$H^p(\underline{U}, \mathcal{F}) \xrightarrow{\beta^*} H^p(\underline{U}, \mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(\underline{U}, \mathcal{E})$$

is exact, let $\sigma \in C^p(\underline{U}, \mathcal{G})$ with $\delta\sigma = 0$ and $\delta^*\sigma = 0$ in $H^{p+1}(\underline{U}, \mathcal{E})$. Then there exists $\tau \in C^p(\underline{U}, \mathcal{F})$ such that $\beta\tau = \sigma$ and $\mu \in C^{p+1}(\underline{U}, \mathcal{F})$ such that $\alpha\mu = \delta\tau$; by definition $\mu = \delta^*\sigma$ in $H^{p+1}(\underline{U}, \mathcal{E})$, so $\mu = \delta\nu$ for some $\nu \in C^p(\underline{U}, \mathcal{E})$. Then $\tau - \alpha\nu$ is a cocycle in $C^p(\underline{U}, \mathcal{F})$ with $\beta(\tau - \alpha\nu) = \beta\tau = \sigma$, showing $\sigma \in \beta^*(H^p(\underline{U}, \mathcal{F}))$. Conversely, it is clear that $\delta^*\beta^* = 0$. The remaining stages are similar but easier.

The most common application of the *exact cohomology sequence* associated to a sheaf sequence

$$0 \rightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \rightarrow 0$$

is to answer the question: given a global section σ of \mathcal{G} , when is σ the image under β of a global section of \mathcal{F} ? The answer, according to the exact cohomology sequence, is that this is the case exactly when $\delta^*\sigma = 0$ in $H^1(M, \mathcal{E})$.

For example, we consider again the exact sequence

$$0 \rightarrow \mathcal{O} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} \mathcal{P} \rightarrow 0$$

on a Riemann surface M . The data of the Mittag-Leffler problem consist of a global section $g \in \mathcal{P}(M) = H^0(M, \mathcal{P})$; the question is whether $g = \beta^*f$ for some global meromorphic function f . If $\{f_U\}$ are the local solutions of the problem, we have seen that

$$(\delta^*g)_{U,V} = f_V - f_U$$

and that $g = \alpha^*f$ if and only if $\delta^*g = 0$ in $H^1(M, \mathcal{O})$.

There are, roughly speaking, three kinds of sheaves we will encounter:

1. *Holomorphic sheaves*—such as \mathcal{O} , \mathcal{I}_V , $\mathcal{O}(E)$, and Ω^p —whose sections are given locally by n -tuples of holomorphic functions. These contain for us the most information and are the principal objects of interest.
2. *C^∞ sheaves*, such as $\mathcal{Q}^{p,q}$, whose local sections can be expressed as n -tuples of C^∞ functions. These are generally used in an auxiliary manner.
3. *Constant sheaves*, such as $\mathbb{Z}, \mathbb{R}, \mathbb{C}$. These, as we will see, contain topological information about the underlying manifold.

There are a couple of observations to be made about the latter two classes of sheaves:

1. $H^p(M, \mathcal{Q}^{r,s}) = 0$ for $p > 0$.

Proof. Given any locally finite cover $\underline{U} = \{U_\alpha\}_{\alpha \in I}$ of M , we can find a partition of unity subordinate to \underline{U} , i.e., C^∞ functions ρ_α on M such that $\sum \rho_\alpha \equiv 1$ and $\text{support}(\rho_\alpha) \subset U_\alpha$. Now given $\sigma \in Z^p(U, \mathcal{Q}^{r,s})$, we define $\tau \in C^{p-1}(\underline{U}, \mathcal{Q}^{r,s})$ by setting

$$\tau_{\alpha_0 \dots \alpha_{p-1}} = \sum_{\beta \in I} \rho_\beta \sigma_{\beta, \alpha_0, \dots, \alpha_{p-1}},$$

where the section $\rho_\beta \sigma_{\beta, \alpha_0, \dots, \alpha_{p-1}}$ extends to $U_{\alpha_0} \cap \dots \cap U_{\alpha_{p-1}}$ by zero; one verifies that $\delta\tau = \sigma$. In the case $p = 1$, explicitly:

$$\begin{aligned} \sigma &= \{\sigma_{UV} \in \mathcal{Q}^{r,s}(U \cap V)\}; \\ \sigma_{UV} + \sigma_{VW} + \sigma_{WU} &= 0 \quad \text{in } U \cap V \cap W. \end{aligned}$$

Set $\tau_U = \sum_V \rho_V \sigma_{VU}$; then

$$\begin{aligned} (\delta\tau)_{UV} &= -\tau_U + \tau_V \\ &= -\sum_W \rho_W \sigma_{WU} + \sum_W \rho_W \sigma_{WV} \\ &= \sum_W \rho_W \sigma_{UV} = \sigma_{UV}. \end{aligned}$$

In general, sheaves that admit partitions of unity [more precisely, for any $U = \bigcup U_\alpha$, maps $\eta_\alpha: \mathcal{F}(U_\alpha) \rightarrow \mathcal{F}(U)$ such that the support of $(\eta_\alpha \sigma)$ is contained in U_α and $\sum \eta_\alpha(\sigma|_{U_\alpha}) = \sigma$ for $\sigma \in \mathcal{F}(U)$] are called *fine*, and the same argument shows that their higher cohomology groups vanish.

2. For K a simplicial complex with underlying topological space M ,

$$H^*(K, \mathbb{Z}) \cong \check{H}^*(M, \mathbb{Z}),$$

that is, the Čech cohomology of the constant sheaf \mathbb{Z} on M is isomorphic to the simplicial cohomology of the complex K . To see this, we associate to every vertex v_α in K the open set $\text{St}(v_\alpha)$, called the *star* of v_α , which is the interior of the union of all simplices in K having v_α as a vertex. $\underline{U} = \{U_\alpha = \text{St}(v_\alpha)\}$ is an open covering of M . $\bigcap_{i=0}^p \text{St}(v_{\alpha_i})$ is nonempty and connected if $v_{\alpha_0} \dots v_{\alpha_p}$ are the vertices of a p -simplex in our decomposition; otherwise it is empty. Thus a p -cochain σ of the sheaf \mathbb{Z} associates to every $(\alpha_0 \dots \alpha_p)$ an element

$$\sigma_{\alpha_0 \dots \alpha_p} \in \mathbb{Z}(\bigcap \text{St}(v_{\alpha_i})) = \begin{cases} \mathbb{Z} & \text{if } v_{\alpha_i} \text{ span a } p\text{-simplex,} \\ 0 & \text{otherwise.} \end{cases}$$

Given $\sigma \in C^p(\underline{U}, \mathbb{Z})$, we are led to define a simplicial p -cochain σ' by setting, for $\Delta = \langle v_{\alpha_0} \dots v_{\alpha_p} \rangle$ a p -simplex with vertices $v_{\alpha_0} \dots v_{\alpha_p}$,

$$\sigma'(\Delta) = \sigma_{\alpha_0 \dots \alpha_p}.$$

$\sigma \mapsto \sigma'$ gives an isomorphism of Abelian groups

$$C^p(\underline{U}, \mathbb{Z}) \longrightarrow C^p(K, \mathbb{Z}),$$

and

$$\begin{aligned} \delta\sigma'(\langle \alpha_0 \cdots \alpha_{p+1} \rangle) &= \sum_i (-1)^{i+1} \sigma'(\langle \alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1} \rangle) \\ &= (\delta\sigma)', \end{aligned}$$

so that we have an isomorphism of chain complexes $C^*(\underline{U}, \mathbb{Z}) \rightarrow C^*(K, \mathbb{Z})$, hence an isomorphism $H^*(\underline{U}, \mathbb{Z}) \rightarrow H^*(K, \mathbb{Z})$. Since we can subdivide the complex K to make the cover \underline{U} of M arbitrarily fine without changing $H^*(K, \mathbb{Z})$, we finally obtain

$$\check{H}^*(M, \mathbb{Z}) \cong H^*(\underline{U}, \mathbb{Z}) \cong H^*(K, \mathbb{Z}).$$

The de Rham Theorem

Let M be a real C^∞ manifold. We say that a singular p -chain σ on M , given as a formal linear combination $\sum a_i f_i$ of maps $\Delta \xrightarrow{f_i} M$ of the standard p -simplex $\Delta \subset \mathbb{R}^p$ to M , is *piecewise smooth* if the maps f_i extend to C^∞ maps of a neighborhood of Δ to M . Let $C_p^{\text{ps}}(M, \mathbb{Z})$ denote the space of piecewise smooth integral p -chains. Clearly the boundary of a piecewise smooth chain is again piecewise smooth, so $C_*^{\text{ps}}(M, \mathbb{Z})$ forms a subcomplex of $C_*(M, \mathbb{Z})$ and we can set

$$\begin{aligned} Z_p^{\text{ps}}(M, \mathbb{Z}) &= \text{Ker } \partial: C_p^{\text{ps}}(M, \mathbb{Z}) \rightarrow C_{p-1}^{\text{ps}}(M, \mathbb{Z}) \\ H_p^{\text{ps}}(M, \mathbb{Z}) &= \frac{Z_p^{\text{ps}}(M, \mathbb{Z})}{\partial C_{p+1}^{\text{ps}}(M, \mathbb{Z})}. \end{aligned}$$

By a foundational result from differential topology, the inclusion map $C_*^{\text{ps}}(M, \mathbb{Z}) \rightarrow C_*(M, \mathbb{Z})$ induces an isomorphism

$$H_p^{\text{ps}}(M, \mathbb{Z}) \cong H_p(M, \mathbb{Z});$$

in other words, every homology class in $H_p(M, \mathbb{Z})$ can be represented by a piecewise smooth p -cycle, and if a piecewise smooth p -cycle σ is homologous to 0 in the usual sense, there exists a piecewise smooth $(p+1)$ -chain τ with $\partial\tau = \sigma$.

Now let $\varphi \in A^p(M)$ be a C^∞ p -form and $\sigma = \sum a_i f_i$ a piecewise smooth p -chain; we set

$$\begin{aligned} \langle \varphi, \sigma \rangle &= \int_\sigma \varphi \\ &= \sum_i a_i \int_\Delta f_i^* \varphi. \end{aligned}$$

If φ is a closed form, then for σ the boundary of a $(p+1)$ -chain τ , by Stokes' theorem

$$\int_\sigma \varphi = \int_\tau d\varphi = 0,$$

so that φ defines a real-valued singular p -cocycle. Again by Stokes' theorem, we have for σ a cycle

$$\int_{\sigma} \varphi = \int_{\sigma} \varphi + d\eta$$

for any $\eta \in A^{p-1}(M)$; thus there is a map

$$H_{\text{DR}}^*(M) \rightarrow H_{\text{sing}}^*(M, \mathbb{R}).$$

The *de Rham theorem* says that this map is in fact an isomorphism.

De Rham's theorem was originally proved essentially by defining relative de Rham groups and showing that the resulting homology theory satisfied the axioms of Eilenberg and Steenrod. We will give here the shorter sheaf-theoretic argument that, while not so geometric, can be merely rephrased to give a proof of the Dolbeault theorem later.

First, since any differentiable manifold M can be realized as the underlying topological space of a simplicial complex K , we have

$$H_{\text{sing}}^*(M, \mathbb{R}) \cong H^*(K, \mathbb{R}) \cong \check{H}^*(M, \mathbb{R}).$$

Next by the ordinary Poincaré lemma, the sequence of sheaves

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{Q}^0 \xrightarrow{d} \mathcal{Q}^1 \xrightarrow{d} \mathcal{Q}^2 \rightarrow \dots$$

on M is exact; in other words, the sequences

$$\begin{aligned} 0 \rightarrow \mathbb{R} \rightarrow \mathcal{Q}^0 \xrightarrow{d} \mathcal{Z}^1 \rightarrow 0 \\ \vdots \\ 0 \rightarrow \mathcal{Z}_d^p \rightarrow \mathcal{Q}^p \xrightarrow{d} \mathcal{Z}^{p+1} \rightarrow 0 \end{aligned}$$

are all exact. Now we have seen that

$$H^q(M, \mathcal{Q}^p) = 0$$

for $q > 0$ and all p ; by the exact cohomology sequences associated to the short exact sheaf sequences above,

$$\begin{aligned} \check{H}^p(M, \mathbb{R}) &\cong H^{p-1}(M, \mathcal{Z}^1) \\ &\cong H^{p-2}(M, \mathcal{Z}^2) \\ &\vdots \\ &\cong H^1(M, \mathcal{Z}^{p-1}) \\ &\cong \frac{H^0(M, \mathcal{Z}^p)}{\delta H^0(M, \mathcal{Q}^{p-1})} \\ &= \frac{Z^p(M)}{dA^{p-1}(M)} \\ &= H_{\text{DR}}^p(M). \end{aligned}$$

Q.E.D.

Note that the de Rham isomorphism is functorial: if $f: M \rightarrow N$ is a differentiable map of C^∞ manifolds, φ a closed p -form on N representing $[\varphi] \in H_{\text{sing}}^p(N, \mathbb{R})$ under the de Rham map and $\sigma = \sum a_i f_i$ a piecewise smooth p -cycle on M ,

$$\begin{aligned} \langle f^* \varphi, \sigma \rangle &= \sum_i a_i \int_{\Delta} f_i^* f^* \varphi \\ &= \langle \varphi, f_* \sigma \rangle \end{aligned}$$

i.e., $f^*[\varphi] = [f^* \varphi]$.

The Dolbeault Theorem

We saw in the beginning of this section that the obstruction to solving the Mittag-Leffler problem on a Riemann surface S can be taken to lie in either $H^1(S, \mathcal{O})$ or $H_{\bar{\partial}}^{0,1}(S)$. In fact, this represents a special case of the

Dolbeault Theorem. For M a complex manifold,

$$H^q(M, \Omega^p) \cong H_{\bar{\partial}}^{p,q}(M).$$

Proof. By the $\bar{\partial}$ -Poincaré lemma the sequences

$$\begin{aligned} 0 \rightarrow \Omega^p \rightarrow \mathcal{Q}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{X}_{\bar{\partial}}^{p,1} \rightarrow 0 \\ \vdots \\ 0 \rightarrow \mathcal{X}_{\bar{\partial}}^{p,q} \rightarrow \mathcal{Q}^{p,q} \xrightarrow{\bar{\partial}} \mathcal{X}_{\bar{\partial}}^{p,q+1} \rightarrow 0 \end{aligned}$$

are exact for all p, q . Since

$$H^r(M, \mathcal{Q}^{p,q}) = 0$$

for $r > 0$, all p, q , the long exact cohomology sequences associated to these sheaf sequences give us

$$\begin{aligned} H^q(M, \Omega^p) &\cong H^{q-1}(M, \mathcal{X}_{\bar{\partial}}^{p,1}) \\ &\cong H^{q-2}(M, \mathcal{X}_{\bar{\partial}}^{p,2}) \\ &\vdots \\ &\cong H^1(M, \mathcal{X}_{\bar{\partial}}^{p,q-1}) \\ &\cong \frac{H^0(M, \mathcal{X}_{\bar{\partial}}^{p,q})}{\bar{\partial} H^0(M, \mathcal{Q}^{p,q-1})} \\ &= H_{\bar{\partial}}^{p,q}(M). \end{aligned}$$

Q.E.D.

As an application we will prove a special case of Leray's theorem: for a locally finite cover $\underline{U} = \{U_\alpha\}$ of M that is *acyclic* for the structure sheaf \mathcal{O} , i.e., has the property

$$H^p(U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}, \mathcal{O}) = 0 \quad \text{for } p > 0,$$

we have

$$H^*(\underline{U}, \mathcal{O}) \cong H^*(M, \mathcal{O}).$$

Proof. We have, by hypothesis,

$$\mathcal{L}_\partial^{0,r}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}) = \bar{\partial} \mathcal{Q}^{0,r-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r});$$

i.e., we have exact sequences of cochain groups

$$0 \rightarrow C^p(\underline{U}, \mathcal{L}_\partial^{0,r-1}) \rightarrow C^p(\underline{U}, \mathcal{Q}^{0,r-1}) \rightarrow C^p(\underline{U}, \mathcal{L}_\partial^{0,r}) \rightarrow 0,$$

which by the usual algebraic reasoning gives exact sequences

$$\begin{aligned} \cdots \rightarrow H^p(\underline{U}, \mathcal{Q}^{0,r-1}) &\rightarrow H^p(\underline{U}, \mathcal{L}_\partial^{0,r}) \rightarrow H^{p+1}(\underline{U}, \mathcal{L}_\partial^{0,r-1}) \\ &\rightarrow H^{p+1}(\underline{U}, \mathcal{Q}^{0,r-1}) \rightarrow \cdots \end{aligned}$$

Since $H^p(\underline{U}, \mathcal{Q}^{0,r}) = 0$ for $p > 0$ by the partition of unity argument, we find

$$\begin{aligned} H^q(\underline{U}, \mathcal{O}) &\cong H^{q-1}(\underline{U}, \mathcal{L}_\partial^{0,1}) \\ &\cong H^{q-2}(\underline{U}, \mathcal{L}_\partial^{0,2}) \\ &\vdots \\ &\cong H^1(\underline{U}, \mathcal{L}_\partial^{0,q-1}) \\ &\cong \frac{H^0(\underline{U}, \mathcal{L}_\partial^{0,q})}{\bar{\partial} H^0(\underline{U}, \mathcal{Q}^{0,q-1})} \\ &= H_\partial^{0,q}(M) \cong H^q(M, \mathcal{O}). \end{aligned} \quad \text{Q.E.D.}$$

The same argument works as well for the sheaves Ω^p .

Computations

1. The first observation is that if M is an n -dimensional complex manifold, then

$$H^q(M, \mathcal{O}) \cong H_\partial^{0,q}(M) = 0 \quad \text{for } q > n.$$

2. By the $\bar{\partial}$ -Poincaré lemma,

$$H^q(\mathbb{C}^n, \mathcal{O}) = 0 \quad \text{for } q > 0$$

and more generally

$$H^q((\mathbb{C})^k \times (\mathbb{C}^*)^l, \mathcal{O}) = 0 \quad \text{for } q > 0.$$

Since \mathbb{C}^n is contractible, moreover, we see that

$$H^q(\mathbb{C}^n, \mathbb{Z}) = 0 \quad \text{for } q > 0.$$

Now, from the long exact cohomology sequence associated to the exponential sheaf sequence on \mathbb{C}^n ,

$$H^q(\mathbb{C}^n, \mathcal{O}) \rightarrow H^q(\mathbb{C}^n, \mathcal{O}^*) \rightarrow H^{q+1}(\mathbb{C}^n, \mathbb{Z})$$