CHAPTER 2

Foundational Questions. Essential Facts Concerning Functions on a Manifold. Typical Smooth Mappings.

The present chapter is devoted to foundational questions in the theory of smooth manifolds. The proofs of the theorems will play no role whatever in the development of the basic topology and geometry of manifolds contained in succeeding chapters. Consequently in this chapter the reader may, if he wishes, acquaint himself with the definitions and statements of results only, without thereby sacrificing anything in the way of comprehension of the later material.

The subject matter of the chapter falls into two parts. In the first part "partitions of unity", so-called, are constructed, and then used in proving various "existence theorems" (which are in many concrete instances self-evident): the existence of Riemannian metrics and connexions on manifolds, the rigorous verification of the general Stokes formula, the existence of a smooth embedding of any compact manifold into a suitable Euclidean space, the approximability of continuous functions and mappings by smooth ones, and the definition of the operation of "group averaging" of a form or metric on a manifold with respect to a compact transformation group.

The second part, beginning with "Sard's theorem", is concerned with making precise ideas of the "typical" singularities of a function or mapping. This part will be found very useful in subsequent concrete topological constructions, so that the definitions and statements of results contained in it merit closer study.

§8. Partitions of Unity and Their Applications

We first introduce some notation. The space of all (real-valued) functions on a manifold M, with continuous partial derivatives of all orders, will be denoted by $C^\infty(M)$ (these will be our "smooth" functions); the supremum (i.e. least upper bound) of the values $f(x)$ taken by a function f will be denoted by $\sup f(x)$; and $\operatorname{supp} f$ will denote the support of f, i.e. the closure of the set of all points x at which $f(x) \neq 0$.

8.1. Partitions of Unity

We begin with a lemma concerning Euclidean space $\mathbb{R}^n$.

8.1.1. Lemma. *Let A, B be two non-intersecting, closed subsets of Euclidean space $\mathbb{R}^n$, with A bounded. Then there exists a C^∞-function φ on $\mathbb{R}^n$ such that $\varphi(x) \equiv 1$ on A and $\varphi(x) \equiv 0$ on B (see Figure 13). Moreover such a φ can be found satisfying $0 \leq \varphi(x) \leq 1$.*

PROOF. Let a, b be two real numbers with $0 < a < b$. It is easy to verify that the function on $\mathbb{R}^1$ defined by

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right) & \text{for } a < x < b, \\ 0 & \text{for all other } x, \end{cases}$$

is smooth (i.e. is C^∞). (Verify it!) In terms of f we define a new smooth function F by

$$F(x) = \left(\int_x^b f(t)\,dt\right) \Big/ \int_a^b f(t)\,dt.$$

It is readily seen that this smooth function F has the following properties:

$$F(x) \begin{cases} = 0 & \text{for } x \geq b, \\ = 1 & \text{for } x \leq a, \\ \text{decreases from 1 to 0} & \text{for } a \leq x \leq b. \end{cases}$$

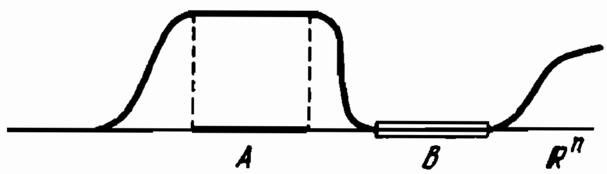

Figure 13

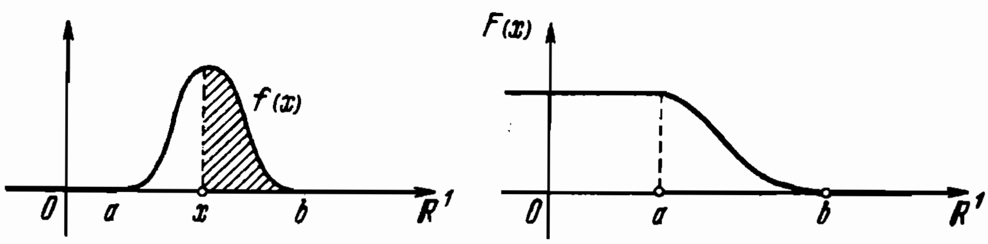

Figure 14

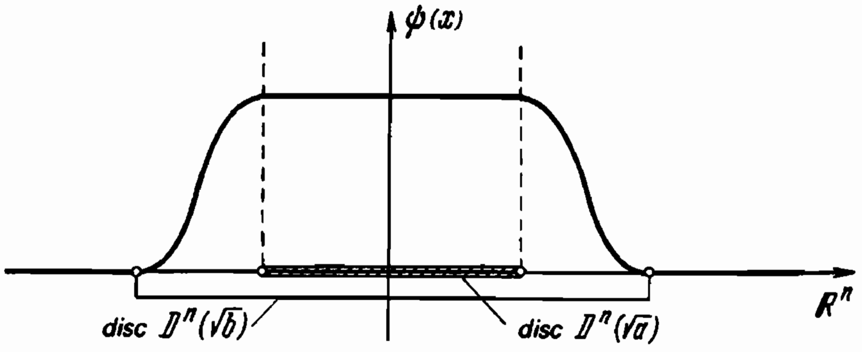

Figure 15

We next define a function ψ on $\mathbb{R}^n$, by the formula

$$\psi(x^1, \ldots, x^n) = F((x^1)^2 + \cdots + (x^n)^2) = F\left(\sum_{i=1}^{n} (x^i)^2\right).$$

It is again clear that ψ is a smooth function with the following properties (see Figure 15):

$$\psi(x) \begin{cases} = 0 & \text{for } r^2 \geq b, \\ = 1 & \text{for } r^2 \leq a, \\ \text{decreases from 1 to 0} & \text{for } a \leq r^2 \leq b. \end{cases}$$

(Here of course $r^2 = \sum_{i=1}^n (x^i)^2$.) We have thus shown that, given any two concentric spheres S and S' in $\mathbb{R}^n$, with S the larger, there exists a C^∞-function ψ which vanishes identically outside S, and is identically 1 on the ball bounded by S'.

Consider now the sets A, B (as in the lemma). Since A is compact, B closed, and $A \cap B = \varnothing$, there exists a finite collection of spheres S_i $(1 \leq i \leq m)$ such that the open balls D_i which they bound ($\partial \bar{D}_i = S_i$, where the bar denotes the closure operation), cover the set A (i.e. $A \subset \bigcup_{i=1}^m D_i$), and have the further property that $\bar{D}_i \cap B = \varnothing$ for all i. It is clear that for each i we can find a strictly smaller S_i' concentric with S_i such that the open balls D_i' which they bound still cover A (i.e. $A \subset \bigcup_{i=1}^m D_i'$). For each $i = 1, \ldots, m$, let ψ_i be a function in $C^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_i(x) \leq 1$ and

$$\psi_i(x) = \begin{cases} 1 & \text{on } D_i', \\ 0 & \text{outside } D_i, \end{cases}$$

and set $\varphi(x) = 1 - \prod_{i=1}^m (1 - \psi_i(x))$. It is then immediate that $\varphi(x) \in C^\infty(\mathbb{R}^n)$, and that $\varphi(x) \equiv 1$ on A and $\varphi(x) \equiv 0$ on B, completing the proof. □

8.1.2. Lemma. *Let C be a compact subset of a smooth manifold M, and let V be any open subset of M containing C. Then there exists a function $\varphi \in C^\infty(M)$ such that $0 \leq \varphi(x) \leq 1$ on M, $\varphi(x) \equiv 1$ on C, and $\varphi(x) \equiv 0$ outside V.*

PROOF. In the case $M = \mathbb{R}^n$ this follows from Lemma 8.1.1. For general M, let $(U_\alpha, \varphi_\alpha)$ be a chart of M, where $\varphi_\alpha: U_\alpha \to \mathbb{R}^n$ is the identification of U_α with a region $\varphi_\alpha(U_\alpha)$ of Euclidean $\mathbb{R}^n$. Let S_α be any compact subset of U_α. Since $\varphi_\alpha(U_\alpha)$ is an open subset of $\mathbb{R}^n$, there exists by Lemma 8.1.1 a smooth function f_α on $\mathbb{R}^n$ such that $f(x) \equiv 1$ on $\varphi_\alpha(S_\alpha)$ and $\operatorname{supp} f_\alpha \subset \varphi_\alpha(U_\alpha)$, i.e. $f_\alpha(x) \equiv 0$ outside $\varphi_\alpha(U_\alpha)$. Consider the function $F_\alpha(P)$ on M defined by

$$F_\alpha(P) = \begin{cases} f_\alpha(\varphi_\alpha(P)) & \text{for } P \in U_\alpha, \\ 0 & \text{for } P \notin U_\alpha. \end{cases}$$

Clearly $F_\alpha \in C^\infty(M)$, $F_\alpha(P) \equiv 1$ on S_α, and $F_\alpha(P) \equiv 0$ outside U_α.

We are now ready to turn our attention to the compact subset C of M contained in the open subset V (as in the lemma). In view of the compactness of C we can find a finite collection of (possibly new) local co-ordinate neighbourhoods $U_1, \ldots, U_N$ and compact subsets $S_1, \ldots, S_N$, such that

$$S_\alpha \subset U_\alpha, \qquad C \subset \bigcup_{\alpha=1}^{N} S_\alpha, \qquad \bigcup_{\alpha=1}^{N} U_\alpha \subset V.$$

By what we have just shown, for each $\alpha = 1, \ldots, N$ there exists a function $F_\alpha \in C^\infty(M)$ such that $F_\alpha \equiv 1$ on S_α and $F_\alpha \equiv 0$ outside u_α. The function $F = 1 - \prod_{\alpha=1}^{N} (1 - F_\alpha)$ then belongs to $C^\infty(M)$, is identically 1 on C, and vanishes outside $\bigcup_{\alpha=1}^{N} U_\alpha$, so that certainly $F(P) \equiv 0$ outside V. □

8.1.3. Theorem (Existence of "Partitions of Unity"). *Let M be a compact, smooth manifold and let $\{U_\alpha\}$ $(1 \le \alpha \le N)$ be an arbitrary finite covering of M by local co-ordinate regions (for instance by open balls). Then there exists a family of functions $\varphi_\alpha \in C^\infty(M)$ with the following properties:*

(i) $\operatorname{supp} \varphi_\alpha \subset U_\alpha$ *for all* α;
(ii) $0 \le \varphi_\alpha(x) \le 1$ *for all* $x \in M$;
(iii) $\sum_\alpha \varphi_\alpha(x) \equiv 1$ *for all* $x \in M$.

PROOF. There always exists a "constricted" family of open sets V_α, $1 \le \alpha \le N$, such that $\bar{V}_\alpha \subset U_\alpha$ and $\{V_\alpha\}$ still covers M. By Lemma 8.1.2 applied to each pair U_α, V_α, there exists a function $\psi_\alpha \in C^\infty(M)$ such that $0 \le \psi_\alpha(x) \le 1$ on M, $\psi_\alpha(x) \equiv 1$ on $\bar{V}_\alpha$, and $\psi_\alpha(x) \equiv 0$ outside U_α. It is immediate that the function $\psi = \sum_{\alpha=1}^{N} \psi_\alpha$ belongs to $C^\infty(M)$ and is positive on M, i.e. $\psi(x) > 0$ for all $x \in M$. If we take $\varphi_\alpha = \psi_\alpha/\psi$, then these φ_α satisfy the requirements of the theorem. This completes the proof. □

The family of functions φ_α is called a *partition of unity subordinate to the covering* $\{U_\alpha\}$.

Remark. The assumption that the manifold M be compact is not essential. It is readily seen that the proof of the existence of partitions of unity carries over to manifolds having suitable "locally finite" coverings (such a covering being one for which there is a neighbourhood of each point intersecting only finitely many regions of the covering). Recall that a Hausdorff topological space is called *paracompact* if every open covering has a locally finite refinement which covers the space. Thus the above proof of the existence of partitions of unity works more generally for any manifold which is paracompact.

8.2. The Simplest Applications of Partitions of Unity. Integrals Over a Manifold and the General Stokes Formula

The theorem on the existence of partitions of unity has useful consequences; we shall now consider some of these. For the sake of simplicity we shall assume throughout that the manifolds we deal with are compact.

8.2.1. Corollary. *On any compact manifold a Riemannian metric can be defined.*

PROOF. Let $\{U_\alpha\}$, $1 \le \alpha \le N$, be any finite covering of a compact manifold M by open balls U_α with local co-ordinates x^i_α. In each U_α take any Riemannian metric $(g^{(\alpha)}_{ab})$ (e.g. $g^{(\alpha)}_{ab} = \delta_{ab}$); we then need somehow to combine the $g^{(\alpha)}_{ab}$ to obtain a metric on M. This is done by defining

$$g_{ab} = \sum_{\alpha=1}^{N} g^{(\alpha)}_{ab}(x)\, \psi_\alpha(x),$$

where $\{\psi_\alpha\}$ is a partition of unity subordinate to the covering $\{U_\alpha\}$. Clearly the g_{ab} are smooth. Since $\psi_\alpha(x) \ge 0$ for all x, and since the set of Riemannian metrics on any space forms a "convex cone" (i.e. for any Riemannian metrics g_1, g_2 and any positive reals c, d, the linear combination $cg_1 + dg_2$ is again a Riemannian metric), it follows that (g_{ab}) is indeed a Riemannian metric. □

It follows immediately that

8.2.2. Corollary. *On any compact manifold there exists a Riemannian connexion.*

The existence of partitions of unity is similarly exploited in defining the integral of an exterior form ω of degree $n = \dim M$ over a manifold M. As before let $\{U_\alpha\}$, $\alpha = 1, \ldots, N$, be a finite covering of the (compact) manifold M by charts U_α with local co-ordinates $x^1_\alpha, \ldots, x^n_\alpha$. In terms of these local co-ordinates the form $\omega^{(n)}$ can in each U_α be written as

$$\omega^{(n)}(x) = a_{1\ldots n}(x)\, dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha,$$

and the integral of $\omega^{(n)}$ over the region U_α is, as usual, just the multiple integral:

$$\int_{U_\alpha} \omega^{(n)} = \int_{U_\alpha} a_{1\ldots n}(x)\, dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha.$$

To define the integral over the whole of $M = M^n$ we need to piece these integrals together. With this in view, we take a partition of unity $\{\psi_\alpha\}$ subordinate to $\{U_\alpha\}$. The desired integral is then defined by:

$$\int_{M^n} \omega^{(n)} = \int_{M^n} \left(\sum_{\alpha=1}^{N} \psi_\alpha(x) \right) \omega^{(n)}(x) = \sum_{\alpha=1}^{N} \int_{U_\alpha} \psi_\alpha(x)\, \omega^{(n)}(x).$$

(Recall that $\psi_\alpha(x) \equiv 0$ outside U_α.) The verification that this definition is independent of the particular finite covering $\{U_\alpha\}$ and the partition of unity, presents no essential difficulty, and we omit the details.

As our next application of the existence of partitions of unity we give a rigorous proof of the general Stokes formula. Let $D \subset \mathbb{R}^n$ be a bounded region with smooth boundary ∂D, given in terms of Euclidean co-ordinates $x^1, \ldots, x^n$ by an equation $f(x^1, \ldots, x^n) \geq 0$, where $\operatorname{grad} f|_{\partial D} \neq 0$; thus the boundary of D is a smooth, non-singular hypersurface in $\mathbb{R}^n$. An orientation of $\mathbb{R}^n$ determines the order of the co-ordinates $x^1, \ldots, x^n$ (up to an even permutation), since the orientation is prescribed by the frame (i.e. ordered basis for the tangent space) $(e_1, \ldots, e_n)$ consisting of the standard basis vectors in the natural order, which frame moves smoothly from point to point in $\mathbb{R}^n$. For each point P in ∂D, denote by $n(P)$ the outward normal to ∂D. In some neighbourhood of each point P of ∂D we can define smooth local co-ordinates $y^1, \ldots, y^{n-1}$, which can be ordered so as to define an orientation of ∂D; recall that this orientation is said to be *induced by the orientation on D* if at each point of ∂D the frame $(\partial/\partial y^1, \ldots, \partial/\partial y^{n-1}, n(P))$ is obtained from the frame $(e_1, \ldots, e_n)$ by means of a linear transformation with positive determinant.

8.2.3. Theorem. *Let ω be an exterior differential form of degree $n-1$ on the region D of $\mathbb{R}^n$. Then*

$$\int_D d\omega = \int_{\partial D} i^*(\omega),$$

where $i \colon \partial D \to D$ is the embedding, $i^*(\omega)$ is the restriction of the form ω to the boundary ∂D of D (see §22.1 *of Part* I), *and the orientation on ∂D is that induced by the orientation on D.*

(Note that the orders of the co-ordinates $x^1, \ldots, x^n$ and $y^1, \ldots, y^{n-1}$, which are determined (up to even permutations) by the orientation, must be stipulated in calculating integrals of forms, since the order determines the sign of the integral.)

PROOF. Let $\{U_\alpha\}$, $1 \leq \alpha \leq N$, be a finite covering of the region D by open balls, and let $h_\alpha \colon B^n \to \mathbb{R}^n$; $h_\alpha(B^n) = U_\alpha$, be fixed co-ordinate maps, where B^n is the unit open ball in $\mathbb{R}^n$ (with fixed co-ordinates $x^1, \ldots, x^n$). Thus h_α assigns co-ordinates to the chart U_α. By choosing the U_α sufficiently small and arranging the co-ordinatization appropriately, we may assume (by virtue of the Implicit Function Theorem) that every intersection $\partial D \cap U_\alpha$ which is non-empty is given by the equation $x_\alpha^n=0$, where $x_\alpha^1,\dots,x_\alpha^n$ are the local co-ordinates on U_α.

Now let $\{\varphi_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$; thus $\{\varphi_\alpha\}$ has the following properties:

(i) $\operatorname{supp}\varphi_\alpha \subset U_\alpha$ for all α;
(ii) $\varphi_\alpha(x)\geq 0$ for all $x\in\bigcup_\alpha U_\alpha$;
(iii) $\sum_\alpha \varphi_\alpha(x)\equiv 1$ for all $x\in\bigcup_\alpha U_\alpha$.

From (iii), and since the φ_α are scalars, we have in view of the linearity of integrals that

$$\int_{\partial D} i^*(\omega)=\sum_\alpha\int_{\partial D} i^*(\varphi_\alpha\omega),$$

$$\int_D d\omega=\sum_\alpha\int_D d(\varphi_\alpha\omega).$$

Hence it suffices to show that for each α $(1\leq\alpha\leq N)$,

$$\int_{\partial D} i^*(\varphi_\alpha\omega)=\int_D d(\varphi_\alpha\omega). \tag{1}$$

If in terms of the local co-ordinates $x_\alpha^1,\dots,x_\alpha^n$ on U_α, we write

$$\varphi_\alpha\omega=\tilde{\omega}_\alpha=\sum_{k=1}^n(-1)^{k-1}a_k(x)\,dx_\alpha^1\wedge\cdots\wedge\widehat{dx_\alpha^k}\wedge\cdots\wedge dx_\alpha^n \tag{2}$$

(where $a_k(x)\in C^\infty(D)$, and the hatted symbol is understood as omitted), then (see §25.2 of Part I)

$$d\tilde{\omega}_\alpha=\left(\sum_{k=1}^n\frac{\partial a_k(x)}{\partial x_\alpha^k}\right)dx_\alpha^1\wedge\cdots\wedge dx_\alpha^n. \tag{3}$$

First case: $U_\alpha\cap\partial D=\varnothing$. Since $\operatorname{supp}\varphi_\alpha\subset U_\alpha$, it follows that $\operatorname{supp}(\varphi_\alpha\omega)\subset U_\alpha$; hence if $U_\alpha\cap\partial D=\varnothing$, then $\varphi_\alpha(x)\equiv 0$ on ∂D, whence $\int_{\partial D} i^*(\varphi_\alpha\omega)=0$. We therefore wish to show that also $\int_D d(\varphi_\alpha\omega)=0$.

Since $U_\alpha\cap\partial D=\varnothing$, we must have either $U_\alpha\subset D$ or $U_\alpha\subset\mathbb{R}^n-D$. In the latter case certainly $\int_D d(\varphi_\alpha\omega)=0$, so we may suppose $U_\alpha\subset D$. Our problem is then to show that (see (3))

$$\int_{U_\alpha}\left(\sum_{k=1}^n\frac{\partial a_k}{\partial x_\alpha^k}\right)dx_\alpha^1\wedge\cdots\wedge dx_\alpha^n=0.$$

Via the co-ordinate function h_α we may identify U_α with the unit open ball $B^n\subset\mathbb{R}^n$. With this understood, we extend the region of definition of the integrand in the integral

$$\int_{U_\alpha=B^n}\left(\sum_{k=1}^n\frac{\partial a_k}{\partial x_\alpha^k}\right)dx_\alpha^1\wedge\cdots\wedge dx_\alpha^n,$$

to the whole of $\mathbb{R}^n$ by defining it to be zero outside B^n. (Recall that $\operatorname{supp} a_k \subset U_\alpha = B^n$.) Let C^n be the cube of side $2R$ in $\mathbb{R}^n$ defined by

$$C^n = \{(x^1, \ldots, x^n) \,\|\, |x^k| \leq R, 1 \leq k \leq n\},$$

large enough to contain B^n. Then

$$\int_{B^n} \left(\sum_{k=1}^n \frac{\partial a_k}{\partial x_\alpha^k} \right) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \sum_{k=1}^n \int_{C^n} \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$
$$= \sum_{k=1}^n \int_{C^{n-1}} (-1)^{k-1} \left(\int_{-R}^R \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^k \right) dx_\alpha^1 \wedge \cdots \wedge \widehat{dx_\alpha^k} \wedge \cdots \wedge dx_\alpha^n.$$

(Here C^{n-1} denotes the appropriate $(n-1)$-dimensional cube.) Up to sign, the kth term of this sum can be evaluated as follows:

$$\int_{C^{n-1}} \left(\int_{-R}^R \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^k \right) dx_\alpha^1 \wedge \cdots \wedge \widehat{dx_\alpha^k} \wedge \cdots \wedge dx_\alpha^n$$
$$= \pm \int_{C^{n-1}} \{a_k(x_\alpha^1, \ldots, x_\alpha^{k-1}, R, x_\alpha^{k+1}, \ldots, x_\alpha^n)$$
$$- a_k(x_\alpha^1, \ldots, x_\alpha^{k-1}, -R, x_\alpha^{k+1}, \ldots, x_\alpha^n)\} \, dx_\alpha^1 \wedge \cdots \wedge \widehat{dx_\alpha^k} \wedge \cdots \wedge dx_\alpha^n$$
$$= 0,$$

since $a_k(x_\alpha^1, \ldots, \pm R, \ldots, x_\alpha^n) = 0$.

Second case: $U_\alpha \cap \partial D \neq \varnothing$. We wish to establish (1). In view of the supports of the integrands it suffices to verify that

$$\int_{\partial D \cap U_\alpha} i^*(\tilde{\omega}_\alpha) = \int_{U_\alpha} d\tilde{\omega}_\alpha. \tag{4}$$

From (2) and our initial provision that $\partial D \cap U_\alpha$ be given by the equation $x_\alpha^n = 0$, it follows that

$$i^*(\tilde{\omega}_\alpha) = (-1)^{n-1} a_n \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^{n-1}.$$

Thus the equality we seek to establish, namely (4), becomes

$$\int_{\partial D \cap U_\alpha} (-1)^{n-1} a_n \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^{n-1} = \sum_{k=1}^n \int_{U_\alpha} \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n. \tag{5}$$

As in the first case we now identify U_α with the unit open ball B^n, and extend the domain of the a_k to all of $\mathbb{R}^n$ by defining them to be zero outside B^n. Then with the cube C^n as before, the right-hand side of (5) becomes

$$\sum_{k=1}^n \int_{C^n} \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n. \tag{6}$$

For $k \neq n$, certainly $\partial a_k/\partial x_\alpha^k$ is a continuous function of x_α^k, so that by the "Fundamental Theorem of Calculus"

$$\int_{C^n} \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = \int_{C^{n-1}} \left(\int_{-R}^{R} \frac{\partial a_k}{\partial x_\alpha^k} dx_\alpha^k \right) dx_\alpha^1 \wedge \cdots \wedge \widehat{dx_\alpha^k} \wedge \cdots \wedge dx_\alpha^n = 0,$$

since $a_k(x_\alpha^1, \dots, \pm R, \dots, x_\alpha^n) = 0$. On the other hand the nth summand in (6) is

$$\int_{C^n} \frac{\partial a_n}{\partial x_\alpha^n} dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n = (-1)^{n-1} \int_{C^{n-1}} \left(\int_{-R}^{R} \frac{\partial a_n}{\partial x_\alpha^n} dx_\alpha^n \right) dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^{n-1} \tag{7}$$

Now as a function of x^n alone (i.e. for any particular fixed values of $x^1, \dots, x^{n-1}$) a_n is continuous on each of the intervals $-R \leq x^n < 0$ and $0 < x^n \leq R$ (with a possible jump discontinuity at $x^n = 0$); hence it follows by integrating over each of these intervals and adding that

$$\int_{-R}^{R} \frac{\partial a_n}{\partial x_\alpha^n} dx^n = a_n|_{\partial D}.$$

Substituting from this in the right-hand side of (7) we get finally

$$\int_{B^n} d\tilde{\omega}_\alpha = \int_{C^{n-1}} (-1)^{n-1} a_n \, dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^{n-1},$$

as required. This completes the proof in the second case, and thereby the proof of the theorem. □

Remark. The fact that the orientation on ∂D was taken to be that induced by the given orientation of D, was used in applying the "Fundamental Theorem of Calculus" in the form $\int_a^b df(x) = f(b) - f(a)$, with $b > a$, which inequality was determined by the direction of the outward normal $n(P)$ to ∂D; if we had used instead the inward normal we would have obtained the negative of the integral in question. For fixed $x_\alpha^1, \dots, x_\alpha^{n-1}$, the function $a_n(x_\alpha^n)$ has graph something like that shown in Figure 16.

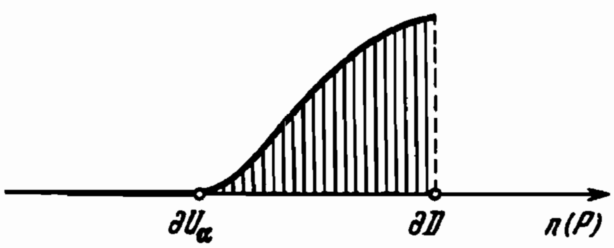

Figure 16

EXERCISE

Prove the general Stokes formula for compact manifolds M with boundary (see §1.3):

$$\int_M d\omega = \int_{\partial M} \omega.$$

(Here the orientation on the boundary ∂M of M is again chosen to be that induced by the given orientation of M.)

8.3. Invariant Metrics

We shall now show that the existence of partitions of unity allows the construction of a Riemannian metric on a manifold, invariant under the action of a given compact group of transformations.

We begin with the case of a finite group acting on a smooth, closed (i.e. compact and without boundary) manifold.

8.3.1. Theorem. *Given a smooth closed manifold M and a finite group G of transformations of M, there exists a Riemannian metric on M invariant under G.*

PROOF. We have already shown (Corollary 8.2.1) that, as a consequence of the existence of partitions of unity, there exists a Riemannian metric $g_{ab}(x)$ say, on M. Denote by $\langle\ ,\ \rangle_x$ the scalar product on T_x (the tangent space to M at each point x), defined by the metric $g_{ab}(x)$, and denote by N the order of the finite group G. We define a new scalar product $(\ ,\ )_x$ (and thereby a new Riemannian metric on M), by means of the procedure of "group averaging" of the old metric, with respect to the group G:

$$(\xi, \eta)_x = \frac{1}{N} \sum_{g \in G} \langle g_*(\xi), g_*(\eta)\rangle_{g(x)},$$

Here ζ, η are arbitrary vectors in T_x, and g_* is the map of tangent spaces induced by g. It is clear that this new metric is invariant under the action of G, i.e. that

$$(g_*(\xi), g_*(\eta))_{g(x)} = (\xi, \eta)_x,$$

for all $x \in M$, $\xi, \eta \in T_x$, $g \in G$. This completes the proof. □

An analogous procedure allows the construction of a Riemannian metric on M invariant under a (suitably restricted) Lie group of transformations of M. Thus let G be a compact, connected Lie group of transformations of M, and let $t^1, \dots, t^m$ be local co-ordinates in a neighbourhood of the identity of G. These co-ordinates yield (via, for instance, right translations, i.e. right multiplications by group elements) local co-ordinates in some neighbourhood of every point of G. In view of the smoothness of multiplication on G, this collection of co-ordinatized neighbourhoods forms an atlas on the