CHAPTER I

Preliminaries from Manifolds

Riemannian geometry is usually developed on smooth manifolds. In this chapter we review some fundamental notions on manifolds. Since there are many books on manifolds, for proofs of many results in this chapter we refer the reader to the references cited at the end of this book. Those readers who are familiar with the fundamental notions on manifolds may start with Chapter II and consult Chapter I as needed. However, since here we systematically give some fundamental concepts and results on manifolds that will be used in this book, it will be convenient to read through this chapter.

1. Vector Spaces

1.1. We mainly deal in the following with finite-dimensional real vector spaces. Let V be an m-dimensional real vector space. If we choose a basis $\{e_i\}_{i=1}^m$, V is isomorphic to the Euclidean vector space $\boldsymbol{R}^m :=$[1] $\{(x^1, \dots, x^m) \colon x^i \in \boldsymbol{R}\}$ by assigning its components to each element of V. Now we review briefly some methods which produce new vector spaces out of given vector spaces. Fundamental concepts of linear algebra, such as linear map, subspace, quotient space, direct sum, etc., are assumed to be known. We denote by $\dim V$ the dimension of the vector space V.

(I) (dual space). $V^* := \{\alpha : V \to \boldsymbol{R};\ \alpha \text{ is a linear map}\}$ has the structure of an m-dimensional vector space and is called the *dual space* of V. For a basis $\{e_i\}$ of V we define $e^i \in V^*$ $(i = 1, \dots, m)$ by $e^i(e_j) := \delta_{ij}$ $(\delta_{ii} = 1, \delta_{ij} = 0$ for $i \neq j)$. Then $\{e^i\}_{i=1}^m$ forms a basis of V^* which is called the *dual basis* of $\{e_i\}_{i=1}^m$. We have a natural isomorphism from V onto $(V^*)^*$, if we assign to every $v \in V$ the element of $(V^*)^*$ defined as $v(w^*) := w^*(v)$, $w^* \in V^*$.

(II) (tensor product). Let V and W be vector spaces of dimension m and n, respectively. Then the space $\mathrm{Hom}(V, W) := \{\varphi : V \to W;\ \varphi \text{ is a linear map}\}$ has the structure of a vector space of dimension mn. In fact, if we take bases $\{e_i\}$ and $\{f_j\}$ of V and W, respectively, and define $\varphi_{ij} \in \mathrm{Hom}(V, W)$ $(1 \leq i \leq m, 1 \leq j \leq n)$ by $\varphi_{ij}(e_k) = \delta_{ik} f_j$, then $\{\varphi_{ij}\}$ forms a basis of $\mathrm{Hom}(V, W)$. Note that $\mathrm{Hom}(V, W)$ is isomorphic to the vector space of real $n \times m$ matrices in this way.

$\mathrm{Hom}(V^*, W)$, also denoted by $V \otimes W$, is called the *tensor product* of V and W. For $v \in V$, $w \in W$ we define $v \otimes w \in V \otimes W$ by $v \otimes w(v^*) := v^*(v)w$. Then any element of $V \otimes W$ may be expressed as a linear combination of elements of the form $v \otimes w$, and, in fact, $\{e_i \otimes f_j\}_{1 \leq i \leq m, 1 \leq j \leq n}$ forms a basis of $V \otimes W$. Note that $V \otimes W$ is isomorphic to the vector space $\{\varphi : V^* \times W^* \to \boldsymbol{R};\ \varphi \text{ is a bilinear map}\}$ by assigning to $v \otimes w$ the bilinear map: $(v^*, w^*) \in V^* \times W^* \mapsto w^*(v \otimes w(v^*)) = v^*(v) w^*(w) \in \boldsymbol{R}$. Further, we obviously have $\mathrm{Hom}(V, W) \cong V^* \otimes W$, $V \otimes \boldsymbol{R} \cong V$, where "$\cong$" denotes an isomorphism of vector spaces. We also note that linear maps $f : V \to V_1$ and

[1] The symbol "$:=$" means that its left-hand side is defined by the right-hand side.

$g : W \to W_1$ determine a linear map $f \otimes g : V \otimes V_1 \to W \otimes W_1$ defined by $(f \otimes g)(v \otimes w) := f(v) \otimes g(w)$.

(III) (tensor space). For a vector space V we define the *tensor space* of type (r, s) of V, which is denoted by

$$T_s^r(V) = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}},$$

as the vector space

$$\{\varphi : \underbrace{V^* \times \cdots \times V^*}_{r \text{ times}} \times \underbrace{V \times \cdots \times V}_{s \text{ times}} \to \boldsymbol{R};\ \varphi \text{ is a multilinear (i.e., linear with respect to each variable) map}\}.$$

Its elements are called tensors of type (r, s). Also we set $T_0^0(V) := \boldsymbol{R}$. If $x_i \in V$ $(1 \leq i \leq r)$, $y_j^* \in V^*$ $(1 \leq j \leq s)$ are given, then we get an (r, s)-tensor by the following formula:

$$x_1 \otimes \cdots \otimes x_r \otimes y_1^* \otimes \cdots \otimes y_s^*(x_1^*, \ldots, x_r^*, y_1, \ldots, y_s) := \prod_{i,j} x_i^*(x_i) y_j^*(y_j).$$

Then we easily see that $\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\}$ forms a basis of $T_s^r(V)$, and $\dim T_s^r(V) = m^{r+s}$. Thus $t \in T_s^r(V)$ may be expressed as

$$t = \sum_{i_1, \ldots, i_r, j_1, \ldots, j_s} t_{j_1 \ldots j_s}^{i_1 \ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

in terms of the components. In the present book we shall follow *Einstein's convention* that we omit the summation symbol $\sum$ when the same indices (for instance i_1, j_1, etc., in the above) appear in pairs, one upstairs and the other downstairs. For instance, the above equation is written as

$$t = t_{j_1 \ldots j_s}^{i_1 \ldots i_r} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}.$$

We note that we have canonical isomorphisms $T_s^r(V)^* \cong T_s^r(V^*)$ and $T_{s+s'}^{r+r'}(V) \cong T_s^r(V) \otimes T_{s'}^{r'}(V)$. Then $T(V) := \bigoplus_{r,s \geq 0} T_s^r(V)$ carries the structure of an algebra relative to " $\otimes$ ". Further, for $T_s^r(V)$ and fixed $1 \leq k \leq r, 1 \leq l \leq s$ we have a linear map $C = C_l^k : T_s^r(V) \to T_{s-1}^{r-1}(V)$, called the *contraction*, which is defined as[2]

$$\begin{aligned} &C_l^k(x_1 \otimes \cdots \otimes x_r \otimes y_1^* \otimes \cdots \otimes y_s^*) \\ &\quad := y_l^*(x_k) x_1 \otimes \cdots \otimes \hat{x}_k \otimes \cdots \otimes x_r \otimes y_1^* \otimes \cdots \hat{y}_l^* \otimes \cdots \otimes y_s^*. \end{aligned} \tag{1.1}$$

Following Einstein's convention, contraction may be written in terms of the components as $(C_l^k(t))_{j_1 \ldots j_{s-1}}^{i_1 \ldots i_{r-1}} = t_{j_1 \ldots m \ldots j_{s-1}}^{i_1 \ldots m \ldots i_{r-1}}$, where upstairs (resp., downstairs) m appears in the k-th (resp., l-th) position.

Now let $A : V \to W$ be a linear isomorphism. Then the transpose linear map $A^* : W^* \to V^*$ defined as $A^*(w^*)(v) := w^*(A(v))$ is also a linear isomorphism. A and A^* induce a linear isomorphism

$$\begin{aligned} \underbrace{A \otimes \cdots \otimes A}_{r \text{ times}} \otimes \underbrace{A^{*-1} \otimes \cdots \otimes A^{*-1}}_{s \text{ times}} : \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}} \\ \to \underbrace{W \otimes \cdots \otimes W}_{r \text{ times}} \otimes \underbrace{W^* \otimes \cdots \otimes W^*}_{s \text{ times}}, \end{aligned}$$

[2]In (1.1) $\hat{x}_k$, etc. means that the term x_k, etc. should be omitted in the expression.

which preserves type (i.e., maps $T^r_s(V)$ to $T^r_s(W)$) and commutes with contractions. Thus a linear isomorphism $A : V \to W$ may be extended to an algebra isomorphism $\tilde{A} : T(V) \to T(W)$ between tensor algebras. Conversely, any such tensor algebra isomorphism $\tilde{A} : T(V) \to T(W)$, that preserves type and commutes with contractions is induced from a linear isomorphism from V onto W. In fact, $A := \tilde{A} \mid T^1_0(V)$[3] is a linear isomorphism from $V = T^1_0(V)$ onto $W = T^1_0(W)$. Setting $B := \tilde{A} \mid T^1_0(V)$, we have $B = A^{*-1}$ because $(Bv^*)(Av) = C(Av \otimes Bv^*) = \tilde{A}(C(v \otimes v^*)) = v^*(v) = (A^{*-1}v^*)(Av)$ for any $v \in V$, $v^* \in V^*$. Note that $\tilde{A} \mid T^0_0(V) : \boldsymbol{R} \to \boldsymbol{R}$ is the identity map.

Next let $D : T(V) \to T(V)$ be a linear map which preserves type and commutes with contractions. D is called a *derivation* of $T(V)$ if D satisfies the *Leibniz formula*

$$D(t \otimes s) = D\,t \otimes s + t \otimes D\,s. \tag{1.2}$$

Again note that such a derivation may be induced from a linear map $A : V \to V$, where $D \mid T^0_1(V) = -A^*$ and $D \mid T^0_0(V) = 0$. The set of all derivations of V obviously has a vector space structure. Moreover, it is a Lie algebra if we define the bracket operation by $[D, D'] := D \circ D' - D' \circ D$ for derivations D, D' (see (2.8) and (2.9)).

(IV) (exterior algebra). We call the vector space

$$\Lambda^k(V) := \{\alpha : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \boldsymbol{R};\ \alpha \text{ is a skew-symmetric } k\text{-linear map}\}$$

the *k-th exterior power* of V^* and its elements *k-forms*. Here α is said to be skew-symmetric if for any permutation σ of $\{1, \ldots, k\}$ we have $\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \operatorname{sgn}\sigma \cdot \alpha(x_1, \ldots, x_k)$, where $\operatorname{sgn}\sigma$ denotes the sign of a permutation σ. For instance, we define

$$x^*_1 \wedge \cdots \wedge x^*_k(x_1, \ldots, x_k) := \det(x^*_i(x_j)) \qquad \text{for} \quad x^*_1, \ldots, x^*_k. \tag{1.3}$$

Then we easily check that $x^*_1 \wedge \cdots \wedge x^*_k \in \Lambda^k(V)$ and that $x^*_{\sigma(1)} \wedge \cdots \wedge x^*_{\sigma(k)} = \operatorname{sgn}\sigma \cdot x^*_1 \wedge \cdots \wedge x^*_k$. Then $\{e^{i_1} \wedge \cdots \wedge e^{i_k};\ i_1 < \cdots < i_k\}$ forms a basis of $\Lambda^k(V)$, and $\dim \Lambda^k(V) = \binom{m}{k}$. In particular, we have $\Lambda^0(V) = \boldsymbol{R}$, $\Lambda^1(V) = V^*$, $\Lambda^k(V) = \{0\}$ $(k > m)$. Further we define for $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$ their *exterior product* $\alpha \wedge \beta \in \Lambda^{k+l}(V)$ by

$$\begin{aligned} &\alpha\wedge\beta(x_1, \ldots, x_{k+l}) \\ &\quad := \frac{1}{k!\,l!} \sum_\sigma (\operatorname{sgn}\sigma)\alpha(x_{\sigma(1)}, \ldots, x_{\sigma(k)})\beta(x_{\sigma(k+1)}, \ldots, x_{\sigma(k+l)}). \end{aligned} \tag{1.4}$$

Note that $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$ and $\Lambda^*(V) := \bigoplus_{k=0}^m \Lambda^k(V)$ has the structure of an algebra with respect to " $\wedge$ ".

Now in the same manner we may construct

$$\Lambda_k(V) := \{\xi : \underbrace{V^* \times \cdots \times V^*}_{k \text{ times}} \to \boldsymbol{R}; \xi \text{ is a skew-symmetric } k\text{-linear map}\}.$$

Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k}; i_1 < \cdots < i_k\}$ forms a basis of $\Lambda_k(V)$, and we may consider the exterior product $\xi \wedge \eta \in \Lambda_{k+l}(V)$ of $\xi \in \Lambda_k(V)$ and $\eta \in \Lambda_l(V)$ as above. Note that for $f \in \operatorname{Hom}(V)$ we may define $f_* \in \operatorname{Hom}(\Lambda_k(V))$ by $f_*(x_1 \wedge \cdots \wedge x_k) := f(x_1) \wedge \cdots \wedge f(x_k)$.

[3] $\tilde{A} \mid T^1_0(V)$ means the restriction of $\tilde{A}$ to $T^1_0(V)$.

1.2. Let V be an m-dimensional real vector space. An *inner product* g on V is defined as a map $g : V \times V \to \boldsymbol{R}$ which satisfies

(**I. 1**) g is a bilinear map;

(**I. 2**) $g(x, y) = g(y, x), \quad x, y \in V$;

(**I. 3**) $g(x, x) \geq 0$ for all $x \in V$, where equality holds if and only if $x = 0$.

We also denote $g(x, y)$ by $\langle x, y\rangle$. For instance, $\boldsymbol{R}^m$ carries the canonical inner product g_0 defined by $g_0((x^1, \ldots, x^m), (y^1, \ldots, y^m)) := \sum_{i=1}^m x^i y^i$. Now once an inner product is given on V we may define the norm $\|x\|$ of $x \in V$ by $\sqrt{\langle x, x\rangle}$. Then from the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\|\,\|y\| \tag{1.5}$$

(equality holds if and only if x and y are linearly dependent), we may define the angle $\angle(x, y)$ $(0 \leq \angle(x, y) \leq \pi)$ of $x, y\ (\neq 0) \in V$ by

$$\cos \angle(x, y) = \langle x/\|x\|,\ y/\|y\|\rangle.$$

Now a basis $\{e_i\}_{i=1}^m$ is called an *orthonormal basis* if $\langle e_i, e_j\rangle = \delta_{ij}$ $(1 \leq i,\ j \leq m)$. In the following we write simply *o.n.b.* for orthonormal basis. In this manner we may define the concepts about measure in terms of the inner product. For instance, the r-dimensional volume of the parallelotope $P(v_1, \ldots, v_r) := \{\sum_{i=1}^r t_i\, v_i;\ 0 \leq t_i \leq 1\}$ spanned by $v_1, \ldots, v_r \in V$ $(r \leq m = \dim V)$ is given by $\sqrt{\det(\langle v_i, v_j\rangle)}$.

A linear map $f : V \to V$ is called an *orthogonal transformation* (or *linear isometry*) if the equality

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \qquad (x, y \in V)$$

holds, and the set of all orthogonal transformations of V forms a group $O(V)$. In particular, the orthogonal transformation group of $(\boldsymbol{R}^m, g_0)$ is denoted by $O(m)$.

Next in terms of a given inner product we get a linear isomorphism $\flat : V \to V^*$ defined by $\flat(v)(w) := \langle v, w\rangle$. Then we may define the inner product on V^* so that $\flat : V \to V^*$ is a linear isometry. We easily see that if $\{e_i\}$ is an o.n.b. of V then its dual basis $\{e^i\}$ forms an o.n.b. of V^*.

Exercise 1. Set $g_{ij} = \langle e_i, e_j\rangle$ for a basis $\{e_i\}$ of V. Then show that $\flat(x) = g_{ij} x^j e^i$ for $x = x^i e_i$, where we follow Einstein's convention.

We may also define the inner products on $T^r_s(V)$ and on $\Lambda_k(V)$ and $\Lambda^k(V)$ from an inner product on V so that $\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\}$, and $\{e^{i_1} \wedge \cdots \wedge e^{i_k};\ i_1 < \ldots < i_k\}, \{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ are o.n.b.'s, respectively, where $\{e_i\}$ is an o.n.b. of V. For instance, we have $\langle x_1 \otimes \cdots \otimes x_k,\ y_1 \otimes \cdots \otimes y_k\rangle = \prod_{i=1}^k \langle x_i, y_i\rangle$. Let $v_1, \ldots, v_r \in V$ be linearly independent and $\{e_i\}_{i=1}^r$ an o.n.b. of the r-dimensional subspace $\langle v_1, \ldots, v_r\rangle_{\boldsymbol{R}}$ spanned by $v_1, \ldots, v_r$. Writing $v_i = a_i^j e_j$, we get

$$\begin{aligned} v_1 \wedge \cdots \wedge v_r &= a_1{}^{j_1} \cdots a_r{}^{j_r} e_{j_1} \wedge \cdots \wedge e_{j_r} \\ &= \left\{ \operatorname{sgn} \begin{pmatrix} 1 & & r \\ j_1 & & j_r \end{pmatrix} a_1{}^{j_1} \cdots a_r{}^{j_r} \right\} e_1 \wedge \cdots \wedge e_r \end{aligned}$$

and consequently

$$\|v_1 \wedge \cdots \wedge v_r\| = |\det(a_i^j)| = \sqrt{\det(\langle v_i, v_j\rangle)}$$

is equal to the volume of the parallelotope spanned by $v_1, \ldots, v_r$.

1.3. We may also consider various geometric structures on a vector space V besides the inner product. Let $\omega : V \times V \to \boldsymbol{R}$ be a skew-symmetric bilinear map, namely, a 2-form on V. ω is said to be *nondegenerate* if its null space $N_\omega := \{x \in V; \omega(x,y) = 0 \text{ for any } y \in V\}$ consists only of the 0-vector, or equivalently $\det(\omega_{ij}) \neq 0$ if we express ω as $\omega = \omega_{ij} e^i \wedge e^j, \omega_{ji} = -\omega_{ij}$.

A nondegenerate 2-form ω on V is called a *symplectic form*, and V is called a *symplectic vector space.*

Exercise 2. Show that symplectic vector spaces are of even-dimension. Further show that we may choose a basis $\{e_i, e_{n+i}\}_{1 \leq i \leq n}$ of V so that $\omega(e_i, e_j) = \omega(e_{n+i}, e_{n+j}) = 0$ and $\omega(e_i, e_{n+j}) = \delta_{ij}$ $(1 \leq i, j \leq n)$.

Now a subspace W of a symplectic vector space V is said to be *isotropic* if $\omega \mid W \times W \equiv 0$. For instance, 1-dimensional subspaces are isotropic, and the dimension of an isotropic subspace is less than or equal to $n := \dim V/2$. To see this we introduce an inner product on V and define a linear transformation $I : V \to V$ by $\langle I(x), y \rangle = \omega(x, y)$. Then I is a linear isomorphism because ω is nondegenerate. Now for an isotropic subspace W we see that $I(W)$ is orthogonal to W, and we get $2 \dim W = \dim W + \dim I(W) \leq \dim V$. In particular, we call a maximal isotropic subspace, which is of dimension n, a *Lagrangian subspace.*

Now note that $\boldsymbol{C}^n := \{(z_1, \dots, z_n);\ z_i = x_i + \sqrt{-1} y_i \in \boldsymbol{C}\}$ (or generally a complex vector space of complex dimension n) may be considered as a real vector space isomorphic to $\boldsymbol{R}^{2n} = \{(x^1, \dots, x^n, y^1, \dots, y^n)\}$. We define a linear isomorphism $J : \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}$ by $J(z^1, \dots, z^n) := \sqrt{-1}(z^1, \dots, z^n)$. Note that we have a matrix representation

$$J = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix},$$

where E_n denotes the n-th unit matrix. J is in fact an orthogonal transformation and satisfies $J^2 = -E_{2n}$. Then $\omega(u, v) := \langle J(u), v \rangle$ $(u,\ v \in \boldsymbol{R}^{2n})$ defines a symplectic form on $\boldsymbol{R}^{2n}$. We easily see that $\boldsymbol{R}^n := \{(x^1, \dots, x^n, 0, \dots, 0); x^i \in \boldsymbol{R}\}$ is a Lagrangian subspace. Moreover, for any $\varphi \in U(n) := \{\varphi \in O(2n);\ \varphi \circ J = J \circ \varphi\}$, $\varphi(\boldsymbol{R}^n)$ gives a Lagrangian subspace.

Exercise 3. Verify the above fact. Show that, conversely, any Lagrangian subspace of $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$ may be written in this form.

2. Manifolds

2.1. Let M be a Hausdorff topological space. A pair (U, φ) of an open set U of M and a homeomorphism $\varphi : U \to \boldsymbol{R}^m$ from U onto an open subset of $\boldsymbol{R}^m$ is called a (*local*) *chart* and U is called a *coordinate neighborhood.* If we have a family $\mathcal{A} := \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of charts in M with $\bigcup_{\alpha \in A} U_\alpha = M$, then we say that M is an m-dimensional *topological manifold* with an *atlas* $\mathcal{A}$. Roughly speaking, a chart (U, φ) gives a coordinate system or a map on U, and a manifold M may be described by an atlas consisting of such maps as the globe. Thus topological manifolds are locally homeomorphic to Euclidean space of fixed dimension, and we want to apply calculus of several variables, which is a powerful tool in Euclidean space. However we should note that coordinates depend on the choice of charts. We say that an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is of *class* C^∞ (or just C^∞, or smooth) if the following holds:

(2.1) Whenever $U_\alpha \cap U_\beta \neq \phi$, coordinate transformations $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are C^∞ maps between open subsets of $\boldsymbol{R}^m$.

Since $\varphi_\alpha \circ \varphi_\beta^{-1}$ is the inverse of $\varphi_\beta \circ \varphi_\alpha^{-1}$, $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a diffeomorphism and its Jacobian matrix $D(\varphi_\alpha \circ \varphi_\beta^{-1})$ is of rank m everywhere. Let u^i $(i = 1, \dots, m)$ denote the coordinates in $\boldsymbol{R}^m$. For a chart $(U_\alpha, \varphi_\alpha)$ we set $x^i_\alpha := u^i \circ \varphi_\alpha$ $(i = 1, \dots, m)$, which are called *local coordinates.* A topological manifold M with a C^∞ atlas is said to be a C^∞ *manifold.* However, note that there is a large choice of atlas on a C^∞manifold M. We say that a chart (U, φ) is *compatible* with a C^∞ atlas $\mathcal{A}$ if $\varphi \circ {\varphi_\alpha}^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are C^∞ maps whenever $U \cap U_\alpha \neq \phi$. Then all charts compatible with $\mathcal{A}$ form a maximal atlas containing $\mathcal{A}$, and their coordinate neighborhoods form a base for the topology of M.

Now let $f : M \to \boldsymbol{R}$ be a real-valued function on a C^∞ manifold M. f is said to be of class C^∞ at $p \in M$, if $f \circ {\varphi_\alpha}^{-1} : \varphi_\alpha(U_\alpha) \to \boldsymbol{R}$ is of class C^∞ at $\varphi_\alpha(p)$, where $(U_\alpha, \varphi_\alpha)$ is a chart around $p \in U_\alpha$. Note that by (2.1) this definition does not depend on the choice of charts around p. We denote by $\mathcal{F}(V)$ the set of all real-valued functions defined on an open subset $V \subset M$ and of class C^∞ everywhere. $\mathcal{F}(V)$ carries the structure of an algebra with respect to the usual addition and multiplication of functions. We also denote by $\mathcal{F}(p)$ the family of C^∞ functions defined on neighborhoods of p. Next a continuous map $\Phi : M \to N$ between C^∞ manifolds M and N is called a C^∞ *map* if $f \circ \Phi \in \mathcal{F}(M)$ whenever $f \in \mathcal{F}(N)$. If a C^∞ map $\Phi : M \to N$ is bijective and its inverse $\Phi^{-1} : N \to M$ is again C^∞, we say that Φ is a *diffeomorphism* and M is *diffeomorphic* to N. In the following, *manifolds are assumed to be of class C^∞ and connected, and to satisfy the second countability axiom* unless otherwise stated. Such manifolds are paracompact and admit partitions of unity, which will be given in the following two forms:

(2.2) For an open covering $\{V_\beta\}_{\beta \in B}$ of M we may choose $\{\rho_\beta\}_{\beta \in B} \subset \mathcal{F}(M)$ which satisfies the following:[4]

(i) $\mathrm{supp}\rho_\beta \subset V_\beta$ and $\{\mathrm{supp}\rho_\beta\}_{\beta \in B}$ is locally finite. Namely, for any $p \in M$ there exists a neighborhood W of p such that there are only finite many β's with $W \cap \mathrm{supp}\rho_\beta \neq \phi$.

(ii) $\rho_\beta \geq 0$ and $\sum_{\beta \in B} \rho_\beta = 1$ (for $p \in M$ note that $\sum_{\beta \in B} \rho_\beta(p)$ is in fact a finite sum because of (i)).

We call $\{\rho_\beta\}_{\beta \in B}$ *a partition of unity subordinate to* $\{V_\beta\}_{\beta \in B}$.

(2.3) For an open covering $\{V_\beta\}_{\beta \in B}$ of M we may choose at most countably many functions $\rho_i \in \mathcal{F}(M)$ $(i = 1, 2, \dots)$ which satisfy the following:

(i) For each i, $\mathrm{supp}\rho_i$ is contained in some V_β and compact. Further, $\{\mathrm{supp}\rho_i\}$ is locally finite (this is different from (2.2), where $\mathrm{supp}\rho_i$ is compact).

(ii) $\rho_i \geq 0$ and $\sum_{i=1}^\infty \rho_i = 1$.

2.2. Recall that smooth curves and smooth surfaces in Euclidean space may be approximated at every point by tangent lines and tangent planes, respectively, which are linear objects. To every point p of a C^∞ manifold M of dimension m, we may also assign an m-dimensional vector space T_pM, called *the tangent space to M at p.*

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[4] $\mathrm{supp}\rho_\beta :=$ closure of $\{p \in M; \rho_\beta(p) \neq 0\}$.

Let (a, b) be an open interval containing 0. A C^∞ map $c : (a, b) \to M$ with $c(0) = p$ is called a (C^∞) curve through p. We want to define the tangent space to M at p as the space of "tangent vectors $\dot{c}(0)$" to a curve c through p. Although we cannot define $\dot{c}(0)$ as in Euclidean spaces, we may consider the directional derivative $Xf := \frac{d}{dt}\mid_{t=0} f(c(t))$ of $f \in \mathcal{F}(p)$, which satisfies

$$X(af + bg) = aXf + bXg, \qquad X(fg) = f(p)Xg + g(p)Xf \qquad a, b \in \boldsymbol{R};\quad f, g \in \mathcal{F}(p). \tag{2.4}$$

Now we define this X as $\dot{c}(0)$, and call it the *tangent vector* to c at p. In general, we call $X : \mathcal{F}(p) \to \boldsymbol{R}$ satisfying (2.4) a *derivation* of $\mathcal{F}(p)$. Then the space of all derivations of $\mathcal{F}(p)$ forms a vector space if we define as $(aX + bY)f := aXf + bYf$ for derivations X, Y, and $a, b \in \boldsymbol{R}$. We denote this vector space by T_pM and call it the *tangent space* to M at p. Take a chart (U, φ, x^i). Then for $q \in U$ we define $\left(\partial/\partial x^i\right)(q) \in T_qM$ $(i = 1, \dots, m)$ by

$$\frac{\partial}{\partial x^i}(q)f := \frac{\partial}{\partial u^i} f \circ \varphi^{-1}(\varphi^{-1}(q)), \tag{2.5}$$

where $\partial/\partial u^i$ denotes partial differentiation with respect to the i-th coordinate. Then $\{\partial/\partial x^i(q)\}_{i=1}^m$ gives a basis of T_qM for each $q \in U$, which will be called the *natural basis*. In particular, T_pM is an m-dimensional vector space. Note that $\dot{c}(0)$ defines an element of T_pM, and conversely any tangent vector may be expressed in this form. Now if we take two charts $(U_\alpha, \varphi_\alpha, x_\alpha^i), (V_\beta, \varphi_\beta, x_\beta^i)$ around p, then the Jacobian matrix $D(\varphi_\beta \circ {\varphi_\alpha}^{-1}) = [\partial x_\beta^i/\partial x_\alpha^j]_{1\le i,j\le m}$ of the coordinate transformation $\varphi_\beta \circ {\varphi_\alpha}^{-1} : (x_\alpha^1, \dots, x_\alpha^1) \mapsto (x_\beta^1, \dots, x_\beta^m)$ induces the change of basis of T_pM given by

$$\frac{\partial}{\partial x_\alpha^i}(p) = \sum_j \frac{\partial x_\beta^j}{\partial x_\alpha^i}(\varphi_\alpha^{-1}(p)) \cdot \frac{\partial}{\partial x_\beta^j}(p).$$

We also write ∂_i instead of $\partial/\partial x^i$, when we fix a chart.

Now let $TM = \bigcup_{p\in M} T_pM$ be the set of tangent vectors to M and $\tau_M : TM \to M$ the map assigning p to $x \in T_pM$. Then it is an important fact that TM carries a $2m$-dimensional C^∞ manifold structure such that τ_M is a C^∞ map, and this indicates that the concept of manifold is natural and useful. In fact, for an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of M set $\tilde{U}_\alpha := {\tau_M}^{-1}(U_\alpha)$. For $X \in \tilde{U}_\alpha, \tau_M(X) = p$ we may write X in the form $X = \xi^i(\partial/\partial x_\alpha^i)(p)$ with respect to the natural basis, and we set $\psi_\alpha(X) := (x_\alpha^1(p), \dots, x_\alpha^m(p), \xi^1, \dots, \xi^m) \in \boldsymbol{R}^{2m}$. Then $\{(\tilde{U}_\alpha, \psi_\alpha)\}_{\alpha\in A}$ gives an atlas for TM. We call TM the *tangent bundle* of M.

Exercise 1. Let V be an m-dimensional vector space which is diffeomorphic to $\boldsymbol{R}^m$. For $x \in V$, define a map $\phi : V \to T_pV$ by $\phi(x)f := \frac{d}{dt}\mid_{t=0} f(p+tx)$, $f \in \mathcal{F}(p)$. Show that ϕ is a linear isomorphism. We denote the inverse of ϕ by $\iota_p : T_pV \to V$ and call it the *canonical identification*. Writing $x = x^i e_i$ with respect to a basis $\{e_i\}$, show that $\iota_p((\partial/\partial x^i)(p)) = e_i$ $(i = 1, \dots, m)$.

Now let $\Phi : M \to N$ be a C^∞ map. For $p \in M$ we may define a linear map $D\Phi(p) : T_pM \to T_{\Phi(p)}N$, which is called the *differential* of Φ at p, by

$$(D\Phi(p)(X))f := X(f \circ \Phi), \qquad f \in \mathcal{F}(\Phi(p))$$

for $X \in T_pM$. Note that this induces a C^∞ map $D\Phi : TM \to TN$. The following theorem shows that we may see the local behavior of Φ through its differential.

Theorem 2.1 (mapping theorem). *Let $\Phi : M \to N$ be a C^∞ map and r the rank of the differential $D\Phi(p)$ of Φ at $p \in M$. Set $m = \dim M, n = \dim N$.*

(1) *If $r = m\,(\leq n)$, namely, $D\Phi(p)$ is injective, then we may choose a chart $(U,\ \varphi)$ around p and a chart $(V,\ \psi)$ around $\Phi(p)$ with respect to which Φ is expressed in the following form:*

$$\psi \circ \Phi \circ \varphi^{-1}(u^1, \dots, u^m) = (u^1, \dots, u^m, 0, \dots, 0).$$

(2) *If $r = n\,(\leq m)$, namely, $D\Phi(p)$ is surjective, then we may choose a chart $(U,\ \varphi)$ around p and a chart $(V,\ \psi)$ around $\Phi(p)$ with respect to which Φ is expressed in the following form:*

$$\psi \circ \Phi \circ \varphi^{-1}(u^1, \dots, u^m) = (u^1, \dots, u^n).$$

(3) (Inverse mapping theorem). *If $r = m = n$, namely, $D\Phi(p)$ is bijective, then there exists an open neighborhood U of p such that $\Phi \mid U$ is a diffeomorphism from U onto an open set $\Phi(U)$ of N.*

In particular, if $D\Phi(p)$ is injective at every point $p \in M$, we call $\Phi : M \to N$ an *immersion*. For an injective immersion $\Phi : M \to N$ we may identify M with a subset $\Phi(M)$ of N. However, in general it is not true that $\Phi : M \to \Phi(M)$ $(\subset N)$ is a homeomorphism with respect to the relative topology. If this is true then we call an injective immersion $\Phi : M \to N$ an *embedding*. For an immersion $\Phi : M \to N$ we may choose an open neighborhood U of any point $p \in M$ so that $\Phi \mid U$ is an embedding from the mapping theorem (1). Now a subset S of M is called a *submanifold* of M if S carries a C^∞ manifold structure such that the inclusion map $\iota : S \hookrightarrow M$ is an embedding. We call $\dim M - \dim S$ the *codimension* of S. For instance, any open subset of M is a submanifold of codimension 0. When an injective immersion $\Phi : M \to N$ is given, some authors call N an (immersed) submanifold of N. By virtue of the fundamental results due to H. Whitney, any m-dimensional manifold $(m > 1)$ may be immersed into $\boldsymbol{R}^{2m-1}$ and embedded into $\boldsymbol{R}^{2m}$. Moreover, such immersion and embedding may be realized by proper maps.[5]

Next $\Phi : M \to N$ is called a *submersion* if $D\Phi(p)$ is surjective for every point p. Then from the mapping theorem (2), $\Phi^{-1}(q)$ is an $(m-n)$-dimensional submanifold of M for every $q \in \Phi(M)$, and is called the fiber over q.

Exercise 2. For a C^∞ curve $c : (a, b) \to M$ we define $\dot{c}(t) \in T_{c(t)}M$ by $\dot{c}(t)f = \frac{d}{dt}f(c(t))$. Then show that $\dot{c}(t) = Dc(\partial/\partial t)$, where t denotes the coordinate of $\boldsymbol{R}$.

2.3. Let M be a C^∞ manifold and suppose that to every point $p \in M$ a tangent vector $X_p \in T_pM$ is assigned. If a map $X : M \to TM$ given by $p \mapsto X_p$ is C^∞, then X is said to be a (C^∞) *vector field* on M. Note that the space $\mathcal{X}(M)$ of all vector fields on M forms a vector space (and in fact an $\mathcal{F}(M)$-module). We may define vector fields on an open set U of M in the same manner. In particular, with respect to a chart (U, φ, x^i) we get the vector fields $\partial/\partial x^i : p \mapsto (\partial/\partial x^i)(p)$ on $U\,(i = 1, \dots, m)$. Then any $X \in \mathcal{X}(U)$ may be uniquely expressed as $X = X^i\partial/\partial x^i, X^i \in \mathcal{F}(U)$. Now we consider vector fields from the following two viewpoints.

[5] This means that the inverse image of every compact subset is compact.

(**I**) A vector field X may be characterized as a *derivation* of the algebra $\mathcal{F}(M)$. Namely, if for $f \in \mathcal{F}(M)$ we define $Xf(p) := X_p f$, then $Xf \in \mathcal{F}(M)$ and X satisfies the following properties of the derivation.

$$X(af + bg) = aXf + bXg, \qquad X(fg) = fXg + gXf, \qquad a, b \in \boldsymbol{R};\ f, g \in \mathcal{F}(M). \tag{2.6}$$

Conversely, for a derivation $X : \mathcal{F}(M) \to \mathcal{F}(M)$ which saisfies (2.6) we define $X_p \in T_pM$, $p \in M$ as follows. First note that $Xf(p) = 0$ if $f \mid U \equiv 0$ on a neighborhood U of p. In fact, choose a $\varphi \in \mathcal{F}(M)$ so that $\varphi(p) = 0$ and $\varphi \mid M \setminus U \equiv 1$. Then we get $f \equiv \varphi f$, and consequently $Xf(p) = \varphi(p)Xf(p) + f(p)X\varphi(p) = 0$. Now for $f \in \mathcal{F}(M)$ we define $X_p f := X\tilde{f}(p)$, where $\tilde{f} \in \mathcal{F}(M)$ is an extension of f. Note that this does not depend on the above choice of $\tilde{f}$, and we see that $X_p \in T_pM$. Since locally we may write $X = (Xx^i)\partial/\partial x^i$, $p \mapsto X_p$ defines an element of $\mathcal{X}(M)$.

Now for $X, Y \in \mathcal{X}(M)$ we define the bracket operation by

$$[X, Y]f = X(Yf) - Y(Xf), \qquad f \in \mathcal{F}(M). \tag{2.7}$$

Then we easily see that $[X, Y] \in \mathcal{X}(M)$ and

$$[X,\ Y] = -[Y,\ X], \qquad [fX,\ Y] = f[X,\ Y] - (Yf)X, \qquad [X + Y,\ Z] = [X,\ Z] + [Y,\ Z], \tag{2.8}$$

and also (the Jacobi identity)

$$[[X,\ Y],\ Z] + [[Y,\ Z],\ X] + [[Z,\ X],\ Y] = 0. \tag{2.9}$$

Namely, $\mathcal{X}(M)$ carries the structure of a Lie algebra with respect to $[\ ,\ ]$.

(**II**) (dynamical systems viewpoint). For a vector field X on M and $p \in M$, a curve $c : (-\delta, \delta) \to M$ with $c(0) = p$ is called an *integral curve* of X through p, if $X_{c(t)} = \dot{c}(t)$ holds everywhere. Taking a chart $(U,\ \varphi,\ x^i)$ around p and writing $x^i(t) := x^i(c(t))$, $X = X^i\partial/\partial x^i$, we may get an integral curve through p of X by solving the system of ordinary differential equations

$$\frac{d}{dt}x^i = X^i \circ \varphi^{-1} \qquad (i = 1, \dots, m)$$

under the initial condition $x^i(0) = x^i(p)$ $(i = 1, \dots, m)$. Thus from the fundamental theorem of systems of differential equations we see the following: For any $p \in M$ there exist an open neighborhood U of p and an $\varepsilon > 0$ such that we have a unique integral curve $c_q(t)$ through every $q \in U$ defined for $|t| < \varepsilon$. Moreover, $c_q(t)$ depends smoothly on $(q,\ t)$.

Now taking a different viewpoint, we fix t, $\mid t \mid < \varepsilon$, and set $\varphi_t(q) := c_q(t)$. Then φ_t defines a diffeomorphism from U onto an open set $\varphi_t(U)$ of M, and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ holds where the both sides are defined. Namely, a vector field X generates a local one parameter group φ_t of local diffeomorphisms, which is also called the *flow* generated by X.

Especially for any vector field X on a compact manifold M (or more generally X with compact support), φ_t is defined above on all of M and for any $t \in \boldsymbol{R}$. Thus $\varphi_t \circ \varphi_s = \varphi_{t+s}$ everywhere, and X generates a one parmeter group of diffeomorphisms of M. If we may take such a global flow $\{\varphi_t\}_{t \in \boldsymbol{R}}$ for X, we say that X is *complete.* For instance, suppose we have $a > 0$ such that an integral curve c of X through any point $p \in M$ is defined for $\mid t \mid < a$; then X is complete.

We note that for a diffeomorphism Φ of M and $X \in \mathcal{X}(M)$ we get $D\Phi(X) \in \mathcal{X}(M)$, which is defined by $D\Phi(X)(p) := D\Phi(p)X_{\Phi^{-1}(p)}$. Then it is easy to show that $D\Phi([X,Y]) = [D\Phi(X), D\Phi(Y)]$.

Exercise 3. Let $\{\varphi_t\}$ be the flow generated by a vector field X. For $Y \in \mathcal{X}(M)$, show that $[X,Y]_p = \frac{d}{dt}\,|_{t=0}\; D\varphi_{-t}(Y_{\varphi_t(p)})$. Next let $\{\psi_s\}$ be the flow generated by Y. If X and Y are complete, show that we have $[X,Y] \equiv 0$ if and only if $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$ for all $s, t \in \boldsymbol{R}$.

Since $[X,Y]$ may be expressed in terms of differentiation using the flow $\{\varphi_t\}$ of X, we also denote $[X,Y]$ by $\mathcal{L}_X Y$ and call it the *Lie derivative* of Y by X. In the same way we may consider the Lie derivative of various geometric objects, e.g., tensor fields, by X using $\{\varphi_t\}$ (see §3.1).

Now we state the Frobenius theorem in terms of vector fields; this theorem plays a fundamental role in the geometry of manifolds. If to every point p of a C^∞ manifold M a k-dimensional subspace D_p of T_pM is assigned, we say that a k-dimensional *distribution* $\mathcal{D}$ is given on M. When for every point $p \in M$ there exist an open neighborhood U of p and $X_1, \ldots, X_k \in \mathcal{X}(U)$ such that $\{X_i(q)\}_{i=1}^k$ forms a basis of D_q at every $q \in U$, we call $\mathcal{D}$ a C^∞ distribution (or *subbundle* of TM). For instance, a vector field X which vanishes nowhere defines a 1-dimensional C^∞ distribution on M. Just like integral curves of X, a submanifold N of M containing a point p is called an *integral manifold* of $\mathcal{D}$ through p if $T_qN = D_q$ for every $q \in N$. Now when does there exist an integral manifold of $\mathcal{D}$ through every point of M?

Theorem 2.2 (Frobenius theorem). *Let $\mathcal{D}$ be a k-dimensional C^∞ distribution on M. We call $\mathcal{D}$ involutive, if for any vector fields X and Y that take values in $\mathcal{D}$ (i.e., $X_p, Y_p \in D_p$, $p \in M$), $[X,Y]$ takes value in $\mathcal{D}$. $\mathcal{D}$ is said to be completely integrable if for any $p \in M$ there exists an integral manifold N of $\mathcal{D}$ through p.*

Then any completely integrable distribution $\mathcal{D}$ is involutive, and the converse is also true. More precisely, if $\mathcal{D}$ is involutive, then for any $p \in M$ we have a chart (U, φ, x^i) around p with $\varphi(p) = 0$ and $\varphi(U) = \{(u^1, \ldots, u^m); |u^i| < a\}$ $(a > 0)$ such that the submanifold $\{q \in U; x^{k+i}(q) = \xi^{k+i}\ (i = 1, \ldots, m-k)\}$ in U is an integral manifold of $\mathcal{D}$ for any $\xi^{k+1}, \ldots, \xi^m$ $(\in \boldsymbol{R})$ with $|\xi^{k+i}| < a$.

We call integral manifolds of the above form *slices*. We also say that a k-dimensional completely integrable distribution defines a *foliation* of codimension $m - k$ on M.

Remark. For an involutive distribution $\mathcal{D}$ on M there exists a unique maximal connected integral manifold of $\mathcal{D}$ through any point $p \in M$ which is in general an immersed submanifold of M. Here maximal means that it is not a proper subset of another integral manifold (see [War-3] for more detail).

2.4. Let $f : M \to \boldsymbol{R}$ be a C^∞ function. Then the behavior of levels $f^{-1}(t)$ as t varies is also affected by the manifold structure. Regarding $Df(p) : T_pM \to T_{f(p)}\boldsymbol{R} \simeq \boldsymbol{R}$ as an element of $(T_pM)^*$, a point $p \in M$ with $Df(p) = 0$ is called a *critical point* of f, and $f(p)$ is called the *critical value*. If $f^{-1}(t)$ $(\neq \phi)$ does not contain critical points we say that t is a *regular value* of f. In this case $f^{-1}(t)$ is a hypersurface of M (i.e., submanifold of codimension 1), as is seen by Theorem 2.1 (2). On the other hand, for a critical value t_0, $f^{-1}(t_0)$ may be rather complicated and the topology of $f^{-1}(t)$ may change when t passes throuh a critical value t_0. This may be explicitly analized when critical points satisfy the following nondegeneracy condition. For a critical point p of f we may define the symmetric bilinear form $D^2f(p)$ as $D^2f(p)(u,v) := X(Yf)(p)$, where X, Y are vector fields on M with $X_p = u$, $Y_p = v$. Then we easily see that $D^2f(p)$ is symmetric with respect to X, Y and does not depend on the choice of X, Y. We call $D^2f(p)$ the *Hessian* of f at a critical point p. A critical point p is said to be *nondegenerate* if $D^2f(p)$ is nondegenerate, i.e., if its null space $\{u \in T_pM;\ D^2f(u,v) = 0 \text{ for any } v \in T_pM\} = \{0\}$. Next, we call $D^2f(p)$ *negative definite* on a subspace W of T_pM if $D^2f(p)(w,w) < 0$ for all nonzero $w \in W$, and we define the *index* of a critical point p as the dimension of a maximal negative definite subspace of $D^2f(p)$. If we consider the symmetric $m \times m$ matrix $[(\partial^2(f \circ \varphi^{-1})/\partial u^i \partial u^j)(\varphi(p))]_{1 \le i,j \le m}$ taking a chart $(U,\ \varphi,\ x^i)$ around the critical point p, then p is nondegenerate if and only if this matrix is regular, and the index is equal to the number of its negative eigenvalues counted with multiplicities.

Now it is possible to find a canonical form for f around a nondegenerate critical point p. In fact, the *Morse lemma* asserts that we may find a chart $(U,\ \varphi)$ around p so that f may be expressed as

$$f \circ \varphi^{-1}(u_1, \cdots, u_m) = f(p) - \sum_{i=1}^{k} {u_i}^2 + \sum_{i=k+1}^{m} {u_i}^2, \tag{2.10}$$

where k denotes the index of p. Thus nondegenerate critical points are isolated and the index controls the behavior of f around p. A C^∞ function which admits only nondegenerate critical points is called a *Morse function*. It is known that Morse functions are in fact generic and any C^∞ function on M may be approximated by Morse functions (with respect to the C^∞ topology).

Now for $f : M \to \boldsymbol{R}$, we set $M^{a-} := \{p \in M; f(p) < a\}$, $M^a := \{p \in M; f(p) \le a\}$. Then the behavior of M^a as a increases is described by the following two fundamental results in Morse theory (see e.g., Milnor [M-1][6]).

Theorem 2.3. *Let $f^{-1}([a,b])$ be compact and contain no critical points of f. Then $f^{-1}([a,b])$ is diffeomorphic to $f^{-1}(a) \times [a,b]$, and M^a is diffeomorphic to M^b. Moreover, the inclusion map $\iota : M^a \hookrightarrow M^b$ gives a homotopy equivalence. (In fact, diffeomorphism is given by the flow of the vector field $\nabla f / \| \nabla f \|^2$, where ∇f denotes the gradient vector of f with respect to a Riemannian metric on M defined in Chapter* II, §1.3).

Theorem 2.4. *Suppose that $f^{-1}([a,b])$ is compact and contains only one critical point p of index k, which is nondegenerate and in $f^{-1}((a,b))$. Then we may take a k-cell e^k (i.e., an embedded closed k-dimensional disk in M) in $f^{-1}([a,b])$ such that $e^k \cap f^{-1}(a) = \partial e^k$, and there exists a deformation retraction from $f^{-1}([a,b])$ onto $f^{-1}(a) \cup e^k$. Namely, we have a homotopy $H : f^{-1}([a,b]) \times [0,1] \to f^{-1}([a,b])$ with $H(q,0) = q, H(q,1) \in f^{-1}(a) \cup e^k\ (q \in f^{-1}([a,b]))$ and $H(q,t) = q\ (q \in f^{-1}(a) \cup e^k,\ 0 \le t \le 1)$.*

Let $f : M \to \boldsymbol{R}$ be a Morse function such that M^a are compact for all $a \in \boldsymbol{R}$. Then, combining the above theorems, we see that M carries a homotopy type of a CW-complex obtained by attaching k-cells for every critical point of f with index k.

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[6]See the Bibliography at the end of this book.

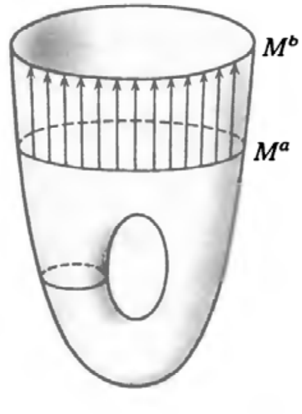

FIGURE 1

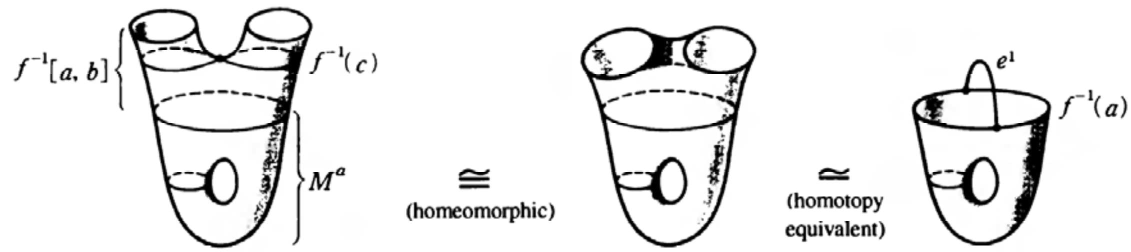

FIGURE 2

Remark 2.5. Suppose that we have a curve c in $f^{-1}([a,b])$ joining two points in $f^{-1}(a)$ in Theorem 2.4. If the index k of the critical point p is greater than 1, then c is homotopic to a curve in $f^{-1}(a)$ fixing the end points. In fact, first deform c to a curve c_1 in $f^{-1}(a) \cup e^k$ fixing the end points via the above deformation retraction. Since $k \geq 2$, we may deform c_1 slightly so that c_1 does not pass through the center of e^k. Then we may deform the part of c_1 which is contained in e^k along radial segments from the center to a curve in $\partial e^k \subset f^{-1}(a)$. Thus for a Morse function f, with Theorem 2.3 we see that any curve in $f^{-1}([a,b])$ joining two points in $f^{-1}(a)$ may be deformed to a curve in $f^{-1}(a)$ fixing the end points, if a is a regular value and the indices of critical points of f in in $f^{-1}([a,b])$ are greater than or equal to 2.

2.5. If a group G has the structure of a C^∞ manifold such that the map $G \times G \to G$ defined by $(a,b) \mapsto ab^{-1}$ is of class C^∞, G is called a *Lie group*. Then for $a \in G$ we have diffeomorphisms of G defined by $L_a : x \mapsto ax, R_a : x \mapsto xa$, which are called the *left translation* and *right translation* by a, respectively. A vector field $X \in \mathcal{X}(G)$ is said to be *left invariant* if $DL_a X = X$ for all $a \in G$. Denoting by $\mathfrak{g}$ the vector space of all left invariant vector fields on G, we may easily see that $[X,Y] \in \mathfrak{g}$, if $X, Y \in \mathfrak{g}$. Namely, $\mathfrak{g}$ carries the structure of a Lie algebra as a subalgebra of $\mathcal{X}(G)$. For any vector x in the tangent space T_eG to G at the identity e, we define the vector field X on G by $X_a := DL_a(e)x$. Then X is in fact of class C^∞ and left invariant. Therefore, a map assigning $X_e \in T_eG$ to $X \in \mathfrak{g}$ gives a linear isomorphism, and we have $\dim \mathfrak{g} = \dim G$. $\mathfrak{g}$ is called the *Lie algebra* of a Lie group G. Sometimes we define the bracket $[x,y]$ on T_eG by $[x,y] = [X,Y]_e$ and identify $\mathfrak{g}$ with T_eG.

Now we give some examples of Lie groups. $\boldsymbol{R}^m$ is an m-dimensional (abelian) Lie group with respect to addition. A discrete subgroup Γ of rank m of $\boldsymbol{R}^m$ is called a *lattice*. Γ may be written as $\Gamma = \{\sum n_i e_i : n_i \in \boldsymbol{Z}\}$ with respect to a basis $\{e_i\}_{i=1}^m$ of $\boldsymbol{R}^m$. Now the quotient group $T^m := \boldsymbol{R}^m/\Gamma$ is a compact abelian Lie group, called an m-dimensional *torus*. The Lie algebras of $\boldsymbol{R}^m$ and T^m are given by $\boldsymbol{R}^m$ with the trivial bracket operation (i.e., $[x, y] \equiv 0$).

Now let $\mathcal{M}_n(\boldsymbol{R})$ (resp., $\mathcal{M}_n(\boldsymbol{C})$) denote the vector space of all real (resp., complex) square matrices of degree n, which carries the structure of a Lie algebra relative to the bracket operation $[A, B] := AB - BA$. Note that $\dim \mathcal{M}_n(\mathbf{R}) = n^2$ and $\dim \mathcal{M}_n(\mathcal{C}) = 2n^2$. In the following we shall give some examples of Lie groups consisting of matrices. We denote by E_n the identity matrix of degree n, and the determinant, trace and transpose of a square matrix A will be denoted by $\det A$, $\operatorname{trace} A$ and ${}^tA$, respectively. For a complex matrx A, $\bar{A}$ stands for its conjugate matrix.

(2.11) $GL(n, \boldsymbol{R}) := \{A \in \mathcal{M}_n(\boldsymbol{R});\ \det A \neq 0\}$ has the structure of a C^∞ manifold as an open subset of $\mathcal{M}_n(\boldsymbol{R})$ and is a (nonconnected) Lie group of dimension n^2, whose Lie algebra $\mathfrak{gl}(\mathfrak{n}, \boldsymbol{R})$ is isomorphic to $\mathcal{M}_n(\boldsymbol{R})$. Similarly, $GL(n, \boldsymbol{C})$ is a (connected) Lie group of dimension $2n^2$ whose Lie algebra $\mathfrak{gl}(\mathfrak{n}, \boldsymbol{C})$ is isomorphic to $\mathcal{M}_n(\boldsymbol{C})$. They are called the *general linear* groups.

(2.12) Let $O(n) := \{A \in \mathcal{M}_n(\boldsymbol{R});\ {}^tAA = E_n\}$ be the group of orthogonal matrices of degree n. Then $O(n)$ is a (nonconnected) Lie group of dimension $n(n-1)/2$ with Lie algebra $\mathfrak{o}(n) := \{A \in \mathcal{M}_n(\boldsymbol{R}); {}^tA + A = 0\}$. $SO(n) := \{A \in O(n); \det A = 1\}$ is a (connected) Lie group and is in fact the identity component of $O(n)$. They are called the *orthogonal* and the *special orthogonal* groups, respectively.

(2.13) $U(n) := \{A \in \mathcal{M}_n(\boldsymbol{C});\ {}^tA\bar{A} = E_n\}$ is an n^2-dimensional (connected) Lie group with Lie algebra $\mathfrak{u}(n) := \{A \in \mathcal{M}_n(\boldsymbol{C});\ {}^tA + \bar{A} = 0\}$. $SU(n) := \{A \in U(n);\ \det A = 1\}$ is a (connected) Lie group of dimension $n^2 - 1$, and its Lie algebra is given by $\mathfrak{su}(n) := \{A \in \mathfrak{u}(n);\ \operatorname{trace} A = 0\}$. They are called the *unitary* and the *special unitary* group, respectively. We note that $U(n)$ is isomorphic to the one given in §1.3.

(2.14) $SL(n, \boldsymbol{R}) := \{A \in \mathcal{M}_n(\mathbf{R});\ \det A = 1\}$ is a (connected) Lie group of dimension $n^2 - 1$ with Lie algebra $\mathfrak{sl}(n, \boldsymbol{R}) := \{A \in \mathcal{M}_n(\mathbf{R});\ \operatorname{trace} A = 0\}$ and is called the *special linear group*. $SL(n, \boldsymbol{C})$ and $\mathfrak{sl}(n, \boldsymbol{C})$ are defined similarly.

(2.15) We put

$$K = \begin{bmatrix} E_n & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{M}_{n+1}(\boldsymbol{R}).$$

Then $O(n, 1) := \{A \in GL(n+1, \boldsymbol{R});\ {}^tAKA = K\}$, which consists of linear transformations leaving the Lorentz inner product $(x^1)^2 + \cdots + (x^n)^2 - (x^{n+1})^2$ invariant, is a (nonconnected) Lie group of dimension $n(n+1)/2$. Note that its Lie algebra $\mathfrak{o}(n, 1)$ is given by $\{U \in \mathcal{M}_{n+1}(\boldsymbol{R});\ {}^tUK + KU = 0\}$.

Now let $\mathfrak{g}$ be the Lie algebra of a Lie group G. We denote by φ_t the flow generated by $X \in \mathfrak{g}$. Since X is left invariant, if $\varphi_t(e)$ is defined for $|\ t\ | \leq \varepsilon$ then $\varphi_t(a) = a\varphi_t(e)$ is also defined for $|\ t\ | \leq \varepsilon$. Namely, X is complete and $t \mapsto \varphi_t(e)$ is a homomorphism from $\boldsymbol{R}$ to G, which is called a *one parameter subgroup* of G. If we put $\exp X := \varphi_1(e)$, then we get a C^∞ map $\exp\colon \mathfrak{g} \to G$, which is called the *exponential map* of G. Note that $\exp tX = \varphi_t(e)$, because $s \mapsto \varphi_{st}(e)$ is an integral curve of tX. Thus, regarding $T_0\mathfrak{g} \cong \mathfrak{g}$ at the zero-vector 0 of $\mathfrak{g}$ and $T_eG \cong \mathfrak{g}$, we see that $D\exp(0)$ is the identity map. Then, by the inverse mapping theorem, exp gives a diffeomorphism from an open neighborhood of 0 in $\mathfrak{g}$ onto an open neighborhood of the identity e of G.

Exercise 4. Show that we have $\exp A = \sum_{k=0}^{\infty} A^k/k!$ for the examples (2.11)–(2.15).

Exercise 5. Show that the flow generated by $X \in \mathfrak{g}$ is given by $t \mapsto R_{\exp tX}$.

Now a homomorphism from a Lie group G to a general linear group $GL(V)$ is called a *representation* of G over a vector space V. For $a \in G$, $L_a \circ R_a{}^{-1}$: $h \in G \mapsto aha^{-1} \in G$ is a C^∞ group isomorphism of G, and its differential $\mathrm{Ad}_{\mathfrak{g}}a := D(L_a \circ R_a{}^{-1})(e)$ at e gives a Lie algebra isomorphism of $\mathfrak{g} = T_eG$. Then $a \in G \mapsto \mathrm{Ad}_{\mathfrak{g}}a \in GL(\mathfrak{g})$ gives a representation of G, which is called the *adjoint representation.* Note that we have

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\mathfrak{g}}(\exp tX)Y = [X, Y] \qquad (:= \mathrm{ad}\ X(Y)). \tag{2.16}$$

In fact, this follows from

$$\begin{aligned}
[X,Y]_e &= \left.\frac{d}{dt}\right|_{t=0} DR_{\exp(-tX)}Y_{\exp tX} = \left.\frac{d}{dt}\right|_{t=0} (DR_{\exp(-tX)}DL_{\exp tX}\, Y_e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\mathfrak{g}}(\exp tX)Y_e.
\end{aligned}$$

We write just $\mathrm{Ad}\, a$ instead of $\mathrm{Ad}_{\mathfrak{g}}a$, when there is no fear of confusion.

Exercise 6. Show that $\exp(\mathrm{Ad}\, a(X)) = a \cdot \exp X \cdot a^{-1}$.

Now let M be a C^∞ manifold and G a Lie group. If we have a C^∞ map $\mu : G \times M \to M$ such that $\mu(ab, p) = \mu(a, \mu(b, p))$ and $\mu(e, p) = p$ for all $a, b \in G$ and $p \in M$, we call G a *Lie transformation group* acting on M. Denoting $\mu(a, p)$ also by $a \cdot p$ for $a \in G$, we get a diffeomorphism $a : p \mapsto a \cdot p$ of M. In fact, note that a^{-1} gives the inverse map of a. In particular, we say that G acts *transitively* on M if for any $p, q \in M$ there exists an $a \in G$ such that $a \cdot p = q$. We give an example of Lie transformation group. Let H be a closed subgroup of G. Then H is an (embedded) submanifold of G and is a Lie group with respect to this manifold structure. Moreover, the coset space G/H has a C^∞ manifold structure such that the canonical projection $\pi : G \to G/H$ is a surjective submersion (see e.g., [Hel], [Ma], [War-3]). If we define $\mu : G \times G/H \to G/H$ as $\mu(a, bH) := abH$, we get a Lie transformation group G acting on G/H transitively. In this case we also denote the action of $a \in G$ by L_a.

Conversely, let G be a Lie transformation group acting on M. We set $H_p := \{a \in G;\ a \cdot p = p\}$, which is a closed subgroup of G and is called the *isotropy group* of G at p. If G acts transitively on M, then it is known that G/H is diffeomorphic to M, where a diffeomorphism is given by $aH_p \mapsto a \cdot p$. The manifolds of the form G/H are called *homogeneous spaces*, which give many examples of manifolds and may be studied in detail using the theory of Lie groups and Lie algebras.

Exercise 7. Show that $SO(m+1)/SO(m)$ is diffeomorphic to the sphere $S^m := \{x \in \boldsymbol{R}^{m+1};\ \| x \| = 1\}$, and $U(n+1)/U(n)$ is diffeomorphic to S^{2n+1}.

Exercise 8. Let G be a Lie transformation group acting on M and $\tilde{X}$ an element of the Lie algebra of G. Define the vector field $X := \mu_*\tilde{X}$ on M by $X_p := \frac{d}{dt}\mid_{t=0} \exp t\tilde{X} \cdot p$, and show that $[\mu_*\tilde{X}, \mu_*\tilde{Y}]_M = -\mu_*[\tilde{X}, \tilde{Y}]$, where $[\ ,\ ]_M$ denotes the bracket of vector fields on M.

3. Vector Bundles and Linear Connection

3.1. Recall that the tangent bundle TM of a C^∞ manifold M carries a C^∞ manifold structure such that $\tau_M : TM \to M$ is a C^∞ map. Checking the manifold structure of TM, we see that $\tau_M : TM \to M$ has the structure of a vector bundle, defined as follows.

Definition 3.1. $\tau : E \to M$ is called a k-dimensional (real) *vector bundle* if the following two conditions are satisfied:

(1) E, M are C^∞ manifolds and $\tau : E \to M$ is a surjective C^∞ map. For every $p \in M$, $\tau^{-1}(p)$ is a k-dimensional (real) vector space.

(2) For every $p \in M$, there exist an open neighborhood U of p and a diffeomorphism $\Phi_U : \tau^{-1}(U) \to U \times \boldsymbol{R}^k$ with the following properties:

(i) $pr_1 \circ \Phi_U = \tau \mid \tau^{-1}(U)$. In particular, τ is a submersion.

(ii) For any $q \in U$, $\Phi_q^U := pr_2 \circ \Phi_U \mid \tau^{-1}(q) : \tau^{-1}(q) \to \boldsymbol{R}^k$ is a linear isomorhism, where $pr_1 : U \times \boldsymbol{R}^k \to U, pr_2 : U \times \boldsymbol{R}^k \to \boldsymbol{R}^k$ denote the canonical projections.

We call (U, Φ_U) a *chart* of the vector bundle $\tau : E \to M$. E, M and τ are called the *total space*, *base* and *projection* of the bundle, respectively. $\tau^{-1}(p)$, $p \in M$, is called the *fiber* over p, and is also denoted by $F_p(\tau)$.

As examples of vector bundles we have tangent bundles, and the product bundle $M \times \boldsymbol{R}^k$ with the projection $pr_1 : M \times \boldsymbol{R}^k \to M$. Now for vector bundles $\tau : E \to M, \sigma : F \to M$, we call τ a *subbundle* of σ if $E \subset F, \sigma \mid E = \tau$ and $\tau^{-1}(p)$ are subspaces of $\sigma^{-1}(p)$ for all $p \in M$. Next for k-dimensional vector bundles $\sigma : F \to N$ and $\tau : E \to M$, a C^∞ map $\Phi : F \to E$ is called a *bundle map* if Φ maps each fiber $\sigma^{-1}(q)$, $q \in N$, linear isomorphically onto some fiber $\tau^{-1}(\varphi(q))$, $\varphi(q) \in M$. Then $\varphi : N \to M$ is in fact a C^∞ map. In particular, if $M = N$ and there exists a bundle map Φ which is a diffeomorphism with $\varphi = \mathrm{id}_M$, then σ and τ are said to be isomorphic as vector bundles. Vector bundles that are isomorphic to product bundles are called *trivial*. Now we will construct some new vector bundles from given vector bundles as in §1.1.

(I) (induced bundle). Let $\tau : E \to M$ be a k-dimensional vector bundle, and let a C^∞ map $\varphi : N \to M$ be given. Then we have a k-dimensional vector bundle $\varphi^*\tau$ over N which is constructed as follows: First set $E_1 := \{(q, v) \in N \times E;\ \varphi(q) = \tau(v)\}$. We define $\tau_1 : E_1 \to N$ and $\Phi_1 : E_1 \to E$ by $\tau_1(q, v) := q$ and $\Phi_1(q, v) := v$, respectively. Obviously we have $\varphi \circ \tau_1 = \tau \circ \Phi_1$. Choose a coordinate neighborhood V of $q \in N$ and a chart (U, Φ_U) of τ around $\varphi(q)$ such that $\varphi(V) \subset U$. Then we have $E_1 \cap (V \times \tau^{-1}(U)) = \{(r, (\Phi^U_{\varphi(r)})^{-1}(x));\ r \in V, x \in \boldsymbol{R}^k\}$, and we may introduce a C^∞ manifold structure on E_1 such that $E_1 \cap (V \times \tau^{-1}(U))$ is diffeomorphic to $V \times \boldsymbol{R}^k$ and E_1 is a submanifold of $N \times E$. Furthermore, $\tau_1^{-1}(q)$ has the structure of a k-dimensional vector space by $t_1(q, v_1) + t_2(q, v_2) := (q, t_1v_1 + t_2v_2)$, and $\Phi_1 : E_1 \to E$ is a C^∞ map. Also note that $\Phi_1 \mid \tau_1^{-1}(q) \to \tau^{-1}(\varphi(q))$ is a linear isomorphism for any $q \in N$. Thus if we define $\Phi_{1,V} : \tau_1^{-1}(V) \to V \times \boldsymbol{R}^k$ by $\Phi_{1,V}((r, v)) := (r, \Phi^U_{\varphi(r)}(v))$, then $\tau_1 : E_1 \to N$ is a k-dimensional vector bundle with charts $\{(V, \Phi_{1,V})\}$. We call τ_1 the *induced bundle* of τ via $\varphi : N \to M$, and denote it by $\varphi^*\tau$.

Note that Φ_1 is a bundle map. Conversely, if a bundle map $\Phi : F \to E$ from a vector bundle $\sigma : F \to N$ to a vector bundle $\tau : E \to M$ indudes a C^∞ map $\varphi : N \to M$, then σ is isomorphic to the induced bundle $\varphi^*\tau$. Further, if a C^∞ curve $c : [a,b] \to M$ (or, generally, a submanifold $\iota : N \hookrightarrow M$) is given, we may consider the induced bundle $c^*\tau_M$ (resp., $\iota^*\tau_M$) of the tangent bundle τ_M.

(II) (Whitney sum). For vector bundles τ and σ we may define their *direct product* $\tau \times \sigma$, which is a vector bundle with the total space $E \times F$, base space $M \times N$, projection $\tau \times \sigma : E \times F \to M \times N$ and charts $(U \times V, \Phi_U \times \Phi_V)$, where each fiber $(\tau \times \sigma)^{-1}(p,q)$ is a vector space $\tau^{-1}(p) \times \sigma^{-1}(q) \cong \tau^{-1}(p) \oplus \sigma^{-1}(q)$.

Now for vector bundles $\tau : E \to M$, $\sigma : F \to M$ over the same base M we may consider the vector bundle $\tau \oplus \sigma := \Delta^*(\tau \times \sigma)$, where $\Delta : M \to M \times M$ stands for the diagonal map, defined as $\Delta(p) := (p,p)$. We call this vector bundle $\tau \oplus \sigma$ the *Whitney sum* of τ and σ. Note that the fiber $F_p(\tau \oplus \sigma)$ over any $p \in M$ is naturally isomorphic to the direct sum $F_p(\tau) \oplus F_p(\sigma)$.

Exercise 1. Let σ_1, σ_2 be subbundles of τ such that each fiber $F_p(\tau)$ is the direct sum of $F_p(\sigma_1)$ and $F_p(\sigma_2)$. Show that τ is isomorphic to $\sigma_1 \oplus \sigma_2$.

(III) (tensor product, exterior power). Let $\tau_i : E_i \to M$ $(i = 1,2)$ be vector bundles over M. For each $p \in M$ we take the tensor product $F_p(\tau_1) \otimes F_p(\tau_2)$ of vector spaces $F_p(\tau_1) = \tau_1^{-1}(p)$, $F_p(\tau_2) = \tau_2^{-1}(p)$ and set $E := \bigcup_{p \in M} F_p(\tau_1) \otimes F_p(\tau_2)$. We define the map $\tau : E \to M$ by assigning p to elements of $F_p(\tau_1) \otimes F_p(\tau_2)$. Take charts $(U, \Phi_{i.U})$ $(i = 1,2)$ of τ_i so that they have a common coordinate neighborhood U. Now define $\Phi_U : \tau^{-1}(U) \to U \times (\boldsymbol{R}^{k_1} \otimes \boldsymbol{R}^{k_2})$ by

$$\Phi_U(v_1 \otimes v_2) := (p, \Phi^U_{1.p}(v_1) \otimes \Phi^U_{2,p}(v_2))$$

for $v_1 \otimes v_2 \in F_p(\tau_1) \otimes F_p(\tau_2)$. Then $E \to M$ carries a vector bundle structure such that (U, Φ_U) form a system of charts. This vector bundle is called the *tensor product* of τ_1 and τ_2, and denoted by $\tau_1 \otimes \tau_2$.

We may define similarly the vector bundle $\mathrm{Hom}(\tau_1, \tau_2)$ whose fibers are given by $\mathrm{Hom}(F_p(\tau_1), F_p(\tau_2))$. Note that in this case a chart (U, Φ_U) is given by $\Phi_U(f) = (p, \Phi^U_{2,p} \circ f \circ (\Phi^U_{1,p})^{-1})$ for $f \in \mathrm{Hom}(F_p(\tau_1), F_p(\tau_2))$. In particular, taking a trivial 1-dimensional vector bundle ϵ over M, we call $\tau^* := \mathrm{Hom}(\tau, \epsilon)$ the *dual vector bundle* of τ. For instance, the dual vector bundle of the tangent bundle $\tau_M : TM \to M$ is called the *cotangent bundle* of M and denoted by $\tau^*_M : T^*M \to M$. We denote by $\{dx^i\}_{i=1}^m$ the basis of T^*_pM dual to the natural basis $\{\frac{\partial}{\partial x^i}\}_{i=1}^m$ of T_pM. Further, for a k-dimensional vector bundle $\tau : E \to M$, we may define in a similar manner its *tensor bundle*

$$T^r_s(\tau) := \underbrace{\tau \otimes \cdots \otimes \tau}_{r \text{ times}} \otimes \underbrace{\tau^* \otimes \cdots \otimes \tau^*}_{s \text{ times}}$$

and its k-th *exterior powers*

$$\Lambda^k(\tau) := \underbrace{\tau^* \wedge \cdots \wedge \tau^*}_{k \text{ times}}, \ \Lambda_k(\tau) := \underbrace{\tau \wedge \cdots \wedge \tau}_{k \text{ times}}.$$

In particular, the tensor bundles and exterior powers of the tangent bundle τ_M of a C^∞ manifold M are called simply the tensor bundles and the exterior powers of M, and are denoted by $T^r_s(M)$ and $\Lambda^k(M)$, $\Lambda_k(M)$, respectively.

Now recall that vector fields play an important role in the theory of smooth manifolds. A vector field X on M may be considered as a C^∞ map $X : M \to TM$ which satisfies $\tau_M \circ X = \mathrm{id}_M$. For a general vector bundle $\tau : E \to M$, a C^∞ map $\xi : M \to E$ with $\tau \circ \xi = \mathrm{id}_M$ is called a *section* of τ. Note that the space $C^\infty(\tau)$ of sections of τ carries the structure of an $\mathcal{F}(M)$-module. In particular, we call sections of the tensor bundle $T^r_s(M)$ (resp., k-th exterior power $\Lambda^k(M)$) of M *tensor fields* of type (r,s) (resp., *differential k-forms*) on M. Now a tensor field T of type (r,s) is characterized as a map

$$T : \underbrace{\mathcal{X}^*(M) \times \cdots \times \mathcal{X}^*(M)}_{\text{r times}} \times \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{\text{s times}} \to \mathcal{F}(M)$$

that satisfies the condition

(3.1) $\qquad T$ is $\mathcal{F}(M)$-linear with respect to each variable,

where $\mathcal{X}^*(M)$ denotes the $\mathcal{F}(M)$-module of all differential 1-forms on M. In fact, let T be a tensor field of type (r,s), and for $\alpha_i \in \mathcal{X}^*(M)$ and $X_j \in \mathcal{X}(M)$ define

$$T(\alpha_1, \dots, \alpha_r, X_1, \cdots, X_s)(p) = T_p(\alpha_1(p), \cdots, \alpha_r(p), X_1(p), \dots, X_s(p)).$$

Then we may easily check (3.1). The converse may be verified by the same argument given in §2.3 (I). Similarly, a differential k-form ω may also be characterized as a skew-symmetric k-linear map $\omega : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to \mathcal{F}(M)$ of $\mathcal{F}(M)$-modules.

We denote by $\mathcal{T}^r_s(M)$ and $\Lambda^k(M)$ the $\mathcal{F}(M)$-modules of tensor fields of type (r,s) and differential k-forms on M, respectively.

Now we mention the *Lie derivative* $\mathcal{L}_X T$ of a tensor field T with respect to a vector field X. Let φ_t be the flow of local diffeomorphisms of M generated by X. Then, for $p \in M$, $D\varphi_t^{-1} = D\varphi_{-t} : T_{\varphi_t(p)}M \to T_pM$ is a linear isomorphism and may be extended to an algebra isomorphism $D\tilde{\varphi}_t$ from the tensor space $T(T_{\varphi_t(p)}M)$ onto $T(T_pM)$, which preserves type and commutes with contractions. For $T \in \mathcal{T}^r_s(M)$ we define

$$(\mathcal{L}_X T)(p) := \frac{d}{dt}\bigg|_{t=0} (D\tilde{\varphi}_t(T_{\varphi_t(p)})).$$

Then $\mathcal{L}_X$ preserves type, commutes with contractions, and satisfies the Leibniz formula $\mathcal{L}_X(T \otimes S) = \mathcal{L}_X T \otimes S + T \otimes \mathcal{L}_X S$. In particular, for $f \in \mathcal{T}^0_0(M)$ and $Y \in \mathcal{T}^1_0(M)$ we get $\mathcal{L}_X f = Xf$ and $\mathcal{L}_X(Y) = [X,Y]$. Further, for $\omega \in \mathcal{T}^0_1(M)$ we have

$$(\mathcal{L}_X\omega)(Y) = C(\mathcal{L}_X(\omega \otimes Y) - \omega \otimes \mathcal{L}_X Y) = X(\omega(Y)) - \omega([X,Y]),$$

and so on.

Now for differential forms the *exterior differentiation* $d : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is defined for $\omega \in \Lambda^k(M)$ and $X_0, \dots, X_k \in \mathcal{X}(M)$ by

$$\begin{aligned} d\omega(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k). \end{aligned} \tag{3.2}$$

Then d is $\boldsymbol{R}$-linear and satisfies $d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma$ for $\omega \in \Lambda^k(M)$. Further, d possesses the fundamental property d^2 $(:= d \circ d) = 0$. A differential form ω with $d\omega = 0$ is called a *closed form*, and a differential form ω in the form $\omega = d\sigma$ is called an *exact* form (see Appendix 5 for properties of differential forms).

We remark that we may consider various geometric structures on differentiable manifolds through tensor fields and differential forms. For instance, if there exists a closed differential 2-form α on M such that α_p is nondegenerate at any $p \in M$, then we call α a *symplectic form* and M a *symplectic manifold.* Note that, if M is a symplectic manifold, then T_pM ($p \in M$) are symplectic vector spaces and dim M is even. For instance, the cotangent bundle T^*M of M carries a natural symplectic form (see the Remark in Chapter II, §4.2 (III)).

3.2. Let $\tau : E \to M$ be a vetor bundle and $C^\infty(E)$ the $\mathcal{F}(M)$-module of sections of τ. Now if to vector fields $X \in \mathcal{X}(M)$ and sections $\xi \in C^\infty(E)$ there correspond $\nabla_X \xi \in C^\infty(E)$ which satisfy

(3.3)

$$\begin{cases} \nabla_{fX+gY}\xi = f\nabla_X\xi + g\nabla_Y\xi, & \xi \in C^\infty(E),\ X, Y \in \mathcal{X}(M), \\ & f, g \in \mathcal{F}(M); \\ \nabla_X(\xi+\eta) = \nabla_X\xi + \nabla_X\eta, & \xi, \eta \in C^\infty(E),\ X \in \mathcal{X}(M); \\ \nabla_X(f\xi) = (Xf)\xi + f\nabla_X\xi, & \xi \in C^\infty(E), f \in \mathcal{F}(M), \end{cases}$$

we say that a *linear connection* is given on E, and $\nabla_X\xi$ is called the *covariant derivative* of ξ via X. We note that $(\nabla_X\xi)(p)$ is determined by X_p and the values of ξ on a neighborhood U of p. In fact, if ξ vanishes on U, take an $f \in \mathcal{F}(M)$ such that $f(p) = 0$ and $f \mid M \setminus U \equiv 1$. Clearly we have $\xi \equiv f\xi$. Then we get

$$(\nabla_X\xi)(p) = (\nabla_X(f\xi))(p) = (X_pf)\xi(p) + f(p)(\nabla_X\xi)(p) = 0.$$

It is also easy to check the same assertion for X. Namely, $\nabla_X\xi(p)$ is determined by the values of X, ξ on a neighborhood of p. Now we take a chart (U, φ, x^i) of M around p and write $X = X^i\partial/\partial x^i$. From (3.3) we have

$$(\nabla_X\xi)(p) = \sum X^i(p)(\nabla_{\partial/\partial x^i}\xi)(p).$$

This means that $\nabla_X\xi(p)$ (also written as $\nabla_{X_p}\xi$) is determined by X_p and the values of ξ on a neighborhood of p.

Now for $X, Y \in \mathcal{X}(M)$ we set

$$R(X,Y)\xi := \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \tag{3.4}$$

Then R satisfies (3.1), and $(R(X,Y)\xi)(p)$ is determined by $X(p), Y(p)$ and $\xi(p)$. We call R the *curvature tensor* of the linear connection.

Next we consider the induced bundle $\varphi^*\tau$ of τ, induced by a C^∞ map $\varphi : N \to M$. φ induces an $\mathcal{F}(M)$-linear map $C^\infty(\tau) \in \xi \mapsto \varphi^*\xi := \xi \circ \varphi \in C^\infty(\varphi^*\tau)$. Then from a linear connection ∇ on τ we have a linear connection ∇^* on $\varphi^*\tau$ determined by

$$\nabla^*_{Y_q}\varphi^*\xi = \Phi_1(\nabla_{D\varphi(q)Y_q}\xi), \qquad Y \in \mathcal{X}(N),\ q \in N.$$

We call ∇^* (also written $\varphi^*\nabla$) the connection induced from ∇.

Recall that $C^\infty(\tau_M) = \mathcal{X}(M)$ for the tangent bundle τ_M, and we may consider for a linear connection ∇ on τ_M

$$T(X,Y) := \nabla_XY - \nabla_YX - [X,Y] \quad (\in \mathcal{X}(M)), \tag{3.5}$$

which is $\mathcal{F}(M)$-linear with respect X, Y, and therefore defines a tensor field of type (1.2) on M. We call T the *torsion tensor* of ∇. Finally, we note that a covariant differentiation ∇_X on τ_M may be extended to a covariant differentiation on the tensor bundle $\mathcal{T}(M)$ which preservse type and commutes with contractions as in case of the Lie derivatives (see Chapter II, Proposition 1.3 for more details).

Problems for Chapter I

1. Let $\{e_i\}_{i=1}^m, \{f_j\}_{j=1}^m$ be bases of an m-dimensional real vector space V. Let $[a_j^i]$ be the matrix of the change of bases given by $f_j = a_j^i e_i$, and $[b_k^j]$ the matrix given by $f^j = b_k^j e^k$, where $\{e^i\}, \{f^j\}$ denote the bases dual to $\{e_i\}, \{f_j\}$, respectively. Note that we have $b_k^i a_j^k = \delta_j^i$. Now for a tensor $t \in T_s^r(V)$ we denote by $t_{j_1 \dots j_s}^{i_1 \dots i_r}$ and $\bar{t}_{j_1 \dots j_s}^{i_1 \dots i_r}$ the components of t with respect to $\{e_i\}$ and $\{f_j\}$, respectively. Then show that

$$(*) \qquad \bar{t}_{j_1 \dots j_s}^{i_1 \dots i_r} = t_{l_1 \dots l_s}^{k_1 \dots k_r} b_{k_1}^{i_1} \cdots b_{k_r}^{i_r} a_{j_1}^{l_1} \cdots a_{j_s}^{l_s}.$$

Conversely, suppose that for any basis $\{e_i\}$ of V we have an m^{r+s}-tuple $t_{j_i \dots j_s}^{i_1 \dots i_r}$ of real numbers which satisfy $(*)$ for the change of bases. Then show that these determine a tensor $t \in T_s^r(V)$.

2. Let A be an orthogonal matrix of degree m; that is, ${}^tAA = E_m$. Then show the following.

(1) Suppose m is odd and $\det A = 1$. Then A admits a nonzero fixed point $x \in \mathrm{R}^m$, i.e., $Ax = x\,(x \neq 0)$.

(2) Suppose m is even and $\det A = -1$. Then again A admits a nonzero fixed point x.

3. (1) Let $\Phi : M \to N$ be a C^∞ map and $q \in \Phi(M)$. Suppose that for any $p \in \Phi^{-1}(q)$ we have $\operatorname{rank} D\Phi(p) = n$ $(:= \dim N)$. Then show that $\Phi^{-1}(q)$ is a submanifold of M of codimension n. In particular, for a submersion $\Phi : M \to N$, $\Phi^{-1}(q)$ is a submanifold of M of codimension n for any $q \in \Phi(M)$.

(2) Show that the sphere $S^m(r) := \{(x^1, \dots, x^{m+1}) \in \boldsymbol{R}^{m+1}; \sum (x^i)^2 = r^2\}$ $(r > 0)$ of radius r carries the structure of an m-dimensional C^∞ manifold.

4. Show that $O(n), SO(n), U(n), SU(n)$ carry the Lie group structures, and determine their dimensions.

5. (1) Set

$$\omega = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix} \in \mathcal{M}_{2n}(\boldsymbol{C}).$$

Then show that $Sp(n) := \{A \in U(2n);\ {}^tA\omega A = \omega\}$ is a Lie group (called the *symplectic group*) whose Lie algebra is given by $\mathfrak{sp}(n) := \{A \in \mathfrak{u}(2n);\ {}^tA\omega + \omega A = 0\}$. Also show that

$$\mathfrak{sp}(n) = \left\{ \begin{bmatrix} A & B \\ -\bar{B} & -{}^tA \end{bmatrix} ;\ A \in \mathfrak{u}(n),\ B \text{ is a symmetric complex } n \times n\text{-matrix} \right\}$$

and determine $\dim Sp(n)$.

(2) Show that $Sp(n)/Sp(n-1)$ is diffeomorphic to the sphere S^{4n-1}. What is the fundamental group of $Sp(n)$?

6. Let $\Lambda(V)$ be the space of all Lagrangian subspaces of a symplectic vector space (V^{2n}, ω). Then show that $\Lambda(V)$ may be identified with $U(n)/O(n)$ and carries the structure of a C^∞ manifold of dimension $n(n+1)/2$.

7. Show that the m-dimensional real projective space $\boldsymbol{R}P^m$, which is obtained from $S^m = \{x \in \boldsymbol{R}^{m+1}; \| x \| = 1\}$ by identifying x and $-x$, carries the structure of an m-dimensional C^∞ manifold. Show that $SO(3)$ is diffeomorphic to $\boldsymbol{R}P^3$.

8. (1) Define a map Φ from the torus $T^2 = S^1 \times S^1$ to $\boldsymbol{R}^3$ by

$$\Phi(\theta, \phi) := ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta).$$

Show that Φ is an embedding and illustrate the image of Φ.

(2) Define a map Φ from S^2 to $\boldsymbol{R}^6$ by

$$\Phi(x, y, z) := (x^2, y^2, z^2, \sqrt{2}yz, \sqrt{2}zx, \sqrt{2}xy).$$

Show that Φ is an immersion and induces an embedding from $\boldsymbol{R}P^2$ to $\boldsymbol{R}^6$.

9. A C^∞ manifold M is said to be *orientable* if we may choose an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ such that the Jacobians $\det D(\varphi_\beta \circ \varphi_\alpha^{-1})$ of all coordinates transformations $\varphi_\beta \circ \varphi_\alpha^{-1}$ are positive. We say that such charts determine a positive orientation. Show that the tangent bundle TM of any C^∞ manifold M is orientable.

10. Suppose that to each fiber $F_p(\tau) = \tau^{-1}(p)$ of a vector bundle $\tau : E \to M$ an inner product g_p is assigned so that $p \mapsto g_p(\xi_p, \eta_p)$ belong to $\mathcal{F}(M)$ for any $\xi, \eta \in C^\infty(\tau)$. Then we call g a *fiber metric* of τ. Show the following.

(1) Let σ be a subbundle of τ and $F_p(\sigma)^\perp$ the orthogonal complement of $F_p(\sigma)$ in $F_p(\tau)$. Then $\bigcup_{p \in M} F_p(\sigma)^\perp$ carries the structure of a vector bundle $\sigma^\perp$ such that $\tau = \sigma \oplus \sigma^\perp$.

(2) τ is canonically isomorphic to the dual bundle $\tau^* = \mathrm{Hom}(\tau, \varepsilon)$. Show that a fiber metric g of τ may be extended to fiber metrics of the tensor bundles $T_s^r(\tau)$.

11. Let M be a submanifold of $\boldsymbol{R}^n$ and set $E := \{(p, u) \in M \times \boldsymbol{R}^n; u \perp T_pM\}$, where $u \perp T_pM$ means that u is orthogonal to T_pM. Let ν_M be the restriction of the projection $M \times \boldsymbol{R}^n \to M$ to E. Show that ν_M carries the structure of a vector bundle, which is called the *normal bundle* of M. Show also that $\iota^*\tau_{\boldsymbol{R}^n} \cong \tau_M \oplus \nu_M$, where ι denotes the embedding of M into $\boldsymbol{R}^n$. Finally, show that if M is an oriented hypersurface of $\boldsymbol{R}^n$ then ν_M is a trivial line bundle.

Notes on the References

For linear algebra, the calculus of functions of several variables, and fundamental results on ordinary differential equations, which constitute the background for the theory of differentiable manifolds and geometry of manifolds, we refer to, e.g., [Hir-Sm], [Fl], [Sp-1].

§1. For tensor products and exterior products of vector spaces, see, e.g., [War-3], [Fla], [St], [Ko-No-I].

§2. The notion of differentiable manifolds was established by Weyl and Whitney ([Whi]). Now there are many textbooks on differentiable manifolds. See, e.g., [Abr-Mar], [B-Go], [dR-2], [Hir], [Ko-No I], [Na], [Ma], [Si-Th], [St], [War-3], where proofs of results not presented in this book may be found. In particular, see [Hir] for the Whitney embedding theorem. For the proof of the Frobenius theorem and maximal integral manifolds, see [Ma], [War-3]. For Morse theory, Milnor's classic

[M-1] is still a very nice introduction (see also, e.g., [Hir]). For Lie groups and homogeneous spaces, we refer to [Hel], [Ise-Ta], [Ma], [War-3].

§3. For vector bundles and linear connections see [M-St], [Ko-No I], [Po]. In recent years symplectic geometry has been playing an important role in many fields of mathematics including Riemannian geometry. For an introduction to symplectic geometry see, e.g., [Abr-Mar], [Ar-2], [Dui-1], [Aud-Laf].