# 0 FOUNDATIONAL MATERIAL

In this chapter we sketch the foundational material from several complex variables, complex manifold theory, topology, and differential geometry that will be used in our study of algebraic geometry. While our treatment is for the most part self-contained, it is tacitly assumed that the reader has some familiarity with the basic objects discussed. The primary purpose of this chapter is to establish our viewpoint and to present those results needed in the form in which they will be used later on. There are, broadly speaking, four main points:

1. *The Weierstrass theorems and corollaries*, discussed in Sections 1 and 2. These give us our basic picture of the local character of analytic varieties. The theorems themselves will not be quoted directly later, but the picture—for example, the local representation of an analytic variety as a branched covering of a polydisc—is fundamental. The foundations of local analytic geometry are further discussed in Chapter 5.
2. *Sheaf theory*, discussed in Section 3, is an important tool for relating the analytic, topological, and geometric aspects of an algebraic variety. A good example is the *exponential sheaf sequence*, whose individual terms $\mathbf{Z}$, $\mathcal{O}$, and $\mathcal{O}^*$ reflect the topological, analytic, and geometric structures of the underlying variety, respectively.
3. *Intersection theory*, discussed in Section 4, is a cornerstone of classical algebraic geometry. It allows us to treat the incidence properties of algebraic varieties, a priori a geometric question, in topological terms.
4. *Hodge theory*, discussed in Sections 6 and 7. By far the most sophisticated technique introduced in this chapter, Hodge theory has, in the present context, two principal applications: first, it gives us the *Hodge decomposition* of the cohomology of a Kähler manifold; then, together with the formalism introduced in Section 5, it gives the vanishing theorems of the next chapter.

# 1. RUDIMENTS OF SEVERAL COMPLEX VARIABLES

## Cauchy's Formula and Applications

NOTATION. We will write $z=(z_1,\dots,z_n)$ for a point in $\mathbb{C}^n$, with

$$z_i = x_i + \sqrt{-1}\, y_i;$$
$$\|z\|^2 = (z,z) = \sum_{i=1}^{n} |z_i|^2.$$

For $U$ an open set in $\mathbb{C}^n$, write $C^\infty(U)$ for the set of $C^\infty$ functions defined on $U$; $C^\infty(\overline{U})$ for the set of $C^\infty$ functions defined in some neighborhood of the closure $\overline{U}$ of $U$.

The cotangent space to a point in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ is spanned by $\{dx_i, dy_i\}$; it will often be more convenient, however, to work with the complex basis

$$dz_i = dx_i + \sqrt{-1}\, dy_i, \qquad d\bar{z}_i = dx_i - \sqrt{-1}\, dy_i$$

and the dual basis in the tangent space

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right).$$

With this notation, the formula for the total differential is

$$df = \sum_i \frac{\partial f}{\partial z_i} dz_i + \sum_j \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

In one variable, we say a $C^\infty$ function $f$ on an open set $U \subset \mathbb{C}$ is *holomorphic* if $f$ satisfies the Cauchy-Riemann equations $\partial f/\partial \bar{z} = 0$. Writing $f(z) = u(z) + \sqrt{-1}\, v(z)$, this amounts to

$$\mathrm{Re}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$
$$\mathrm{Im}\left(\frac{\partial f}{\partial \bar{z}}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

We say $f$ is *analytic* if, for all $z_0 \in U$, $f$ has a local series expansion in $z - z_0$, i.e.,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in some disc $\Delta(z_0, \varepsilon) = \{z : |z - z_0| < \varepsilon\}$, where the sum converges absolutely and uniformly. The first result is that $f$ is analytic if and only if it is holomorphic; to show this, we use the

**Cauchy Integral Formula.** *For $\Delta$ a disc in $\mathbb{C}$, $f \in C^\infty(\overline{\Delta})$, $z \in \Delta$,*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{f(w)\, dw}{w - z} + \frac{1}{2\pi\sqrt{-1}} \int_{\Delta} \frac{\partial f(w)}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z},$$

*where the line integrals are taken in the counterclockwise direction* (the fact that the last integral is defined will come out in the proof).

***Proof.*** The proof is based on Stokes' formula for a differential form with singularities, a method which will be formalized in Chapter 3. Consider the differential form

$$\eta = \frac{1}{2\pi\sqrt{-1}}\frac{f(w)\,dw}{w-z};$$

we have for $z \neq w$

$$\frac{\partial}{\partial \bar{w}}\left(\frac{1}{w-z}\right) = 0$$

and so

$$d\eta = -\frac{1}{2\pi\sqrt{-1}}\frac{\partial f(w)}{\partial \bar{w}}\frac{dw\wedge d\bar{w}}{w-z}.$$

Let $\Delta_\varepsilon = \Delta(z,\varepsilon)$ be the disc of radius $\varepsilon$ around $z$. The form $\eta$ is $C^\infty$ in $\Delta - \Delta_\varepsilon$, and applying Stokes' theorem we obtain

$$\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta_\varepsilon}\frac{f(w)\,dw}{w-z} = \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z} + \frac{1}{2\pi\sqrt{-1}}\int_{\Delta-\Delta_\varepsilon}\frac{\partial f}{\partial \bar{w}}\frac{dw\wedge d\bar{w}}{w-z}.$$

Setting $w - z = re^{i\theta}$,

$$\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta_\varepsilon}\frac{f(w)\,dw}{w-z} = \frac{1}{2\pi}\int_0^{2\pi} f(z+\varepsilon e^{i\theta})\,d\theta,$$

which tends to $f(z)$ as $\varepsilon \to 0$; moreover,

$$dw\wedge d\bar{w} = -2\sqrt{-1}\,dx\wedge dy = -2\sqrt{-1}\,r\,dr\wedge d\theta$$

so

$$\left|\frac{\partial f(w)}{\partial \bar{w}}\frac{dw\wedge d\bar{w}}{w-z}\right| = 2\left|\frac{\partial f}{\partial \bar{w}}\,dr\wedge d\theta\right| < c\,|dr\wedge d\theta|.$$

Thus $(\partial f/\partial \bar{w})(dw\wedge d\bar{w})/(w-z)$ is absolutely integrable over $\Delta$, and

$$\int_{\Delta_\varepsilon}\frac{\partial f}{\partial \bar{w}}\frac{dw\wedge d\bar{w}}{w-z} \to 0$$

as $\varepsilon \to 0$; the result follows. Q.E.D.

Now we can prove the

**Proposition.** *For* U *an open set in* $\mathbb{C}$ *and* $f \in C^\infty(U)$, f *is holomorphic if and only if* f *is analytic.*

***Proof.*** Suppose first that $\partial f/\partial\bar{z}=0$. Then for $z_0\in U$, $\varepsilon$ sufficiently small, and $z$ in the disc $\Delta=\Delta(z_0,\varepsilon)$ of radius $\varepsilon$ around $z_0$,

$$\begin{aligned} f(z) &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)\left(1-\dfrac{z-z_0}{w-z_0}\right)} \\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}}\right)(z-z_0)^n; \end{aligned}$$

so, setting

$$a_n=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{(w-z_0)^{n+1}},$$

we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta$, where the sum converges absolutely and uniformly in any smaller disc.

Suppose conversely that $f(z)$ has a power series expansion

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

for $z\in\Delta=\Delta(z_0,\varepsilon)$. Since $(\partial/\partial\bar{z})(z-z_0)^n=0$, the partial sums of the expansion satisfy Cauchy's formula without the area integral, and by the uniform convergence of the sum in a neighborhood of $z_0$ the same is true of $f$, i.e.,

$$f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{f(w)\,dw}{w-z}.$$

We can then differentiate under the integral sign to obtain

$$\frac{\partial}{\partial\bar{z}}f(z)=\frac{1}{2\pi\sqrt{-1}}\int_{\partial\Delta}\frac{\partial}{\partial\bar{z}}\left(\frac{f(w)}{w-z}\right)dw=0,$$

since for $z\neq w$

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{w-z}\right)=0. \qquad \text{Q.E.D.}$$

We prove a final result in one variable, that given a $C^\infty$ function $g$ on a disc $\Delta$ the equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

can always be solved on a slightly smaller disc; this is the

**$\bar{\partial}$-Poincaré Lemma in One Variable.** *Given* $g(z) \in C^\infty(\bar{\Delta})$, *the function*

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{g(w)}{w-z}\, dw \wedge d\bar{w}$$

*is defined and* $C^\infty$ *in* $\Delta$ *and satisfies*

$$\frac{\partial f}{\partial \bar{z}} = g.$$

***Proof.*** For $z_0 \in \Delta$ choose $\varepsilon$ such that the disc $\Delta(z_0, 2\varepsilon) \subset \Delta$ and write

$$g(z) = g_1(z) + g_2(z),$$

where $g_1(z)$ vanishes outside $\Delta(z_0, 2\varepsilon)$ and $g_2(z)$ vanishes inside $\Delta(z_0, \varepsilon)$. The integral

$$f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_2(w) \frac{dw \wedge d\bar{w}}{w-z}$$

is well-defined and $C^\infty$ for $z \in \Delta(z_0, \varepsilon)$; there we have

$$\frac{\partial}{\partial \bar{z}} f_2(z) = \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial}{\partial \bar{z}} \left( \frac{g_2(w)}{w-z} \right) dw \wedge d\bar{w} = 0.$$

Since $g_1(z)$ has compact support, we can write

$$\begin{aligned} \frac{1}{2\pi\sqrt{-1}} \int_\Delta g_1(w) \frac{dw \wedge d\bar{w}}{w-z} &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(w) \frac{dw \wedge d\bar{w}}{w-z} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{C}} g_1(u+z) \frac{du \wedge d\bar{u}}{u}, \end{aligned}$$

where $u = w - z$. Changing to polar coordinates $u = re^{i\theta}$ this integral becomes

$$f_1(z) = -\frac{1}{\pi} \int_{\mathbf{C}} g_1(z + re^{i\theta}) e^{-i\theta}\, dr \wedge d\theta,$$

which is clearly defined and $C^\infty$ in $z$. Then

$$\begin{aligned} \frac{\partial f_1(z)}{\partial \bar{z}} &= -\frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial g_1}{\partial \bar{z}} (z + re^{i\theta}) e^{-i\theta}\, dr \wedge d\theta \\ &= \frac{1}{2\pi\sqrt{-1}} \int_\Delta \frac{\partial g_1}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w-z}; \end{aligned}$$

but $g_1$ vanishes on $\partial\Delta$, and so by the Cauchy formula

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{\partial}{\partial \bar{z}} f_1(z) = g_1(z) = g(z). \qquad \text{Q.E.D.}$$

**Several Variables**

In the formula

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i} dz_i + \sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

for the total differential of a function $f$ on $\mathbb{C}^n$, we denote the first term $\partial f$ and the second term $\bar{\partial} f$; $\partial$ and $\bar{\partial}$ are differential operators invariant under a complex linear change of coordinates. A $C^\infty$ function $f$ on an open set $U \subset \mathbb{C}^n$ is called *holomorphic* if $\bar{\partial} f = 0$; this is equivalent to $f(z_1, \ldots, z_n)$ being holomorphic in each variable $z_i$ separately.

As in the one-variable case, a function $f$ is holomorphic if and only if it has local power series expansions in the variables $z_i$. This is clear in one direction: by the same argument as before, a convergent power series defines a holomorphic function. We check the converse in the case $n=2$; the computation for general $n$ is only notationally more difficult. For $f$ holomorphic in the open set $U \subset \mathbb{C}^2$, $z_0 \in U$, we can fix $\Delta$ the disc of radius $r$ around $z_0 \in U$ and apply the one-variable Cauchy formula twice to obtain, for $(z_1, z_2) \in \Delta$,

$$\begin{aligned} f(z_1, z_2) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \frac{f(z_1, w_2)\, dw_2}{w_2 - z_2} \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_2 - z_{0_2}| = r} \left[ \frac{1}{2\pi\sqrt{-1}} \int_{|w_1 - z_{0_1}| = r} \frac{f(w_1, w_2)\, dw_1}{w_1 - z_1} \right] \frac{dw_2}{w_2 - z_2} \\ &= \left( \frac{1}{2\pi\sqrt{-1}} \right)^2 \int\!\!\int_{|w_i - z_{0_i}| = r} \frac{f(w_1, w_2)\, dw_1\, dw_2}{(w_1 - z_1)(w_2 - z_2)}. \end{aligned}$$

Using the series expansion

$$\frac{1}{(w_1 - z_1)(w_2 - z_2)} = \sum_{m,n=0}^{\infty} \frac{(z_1 - z_{0_1})^m (z_2 - z_{0_2})^n}{(w_1 - z_{0_1})^{m+1} (w_2 - z_{0_2})^{n+1}},$$

we find that $f$ has a local series expansion

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} (z_1 - z_{0_1})^m (z_2 - z_{0_2})^n. \qquad \text{Q.E.D.}$$

Many results in several variables carry directly over from the one-variable theory, such as the identity theorem: *If* f *and* g *are holomorphic on a connected open set* U *and* f = g *on a nonempty open subset of* U, *then* f = g, and the maximum principle: *the absolute value of a holomorphic function* f *on an open set* U *has no maximum in* U. There are, however, some striking differences between the one- and many-variable cases. For example, let $U$ be the polydisc $\Delta(r)=\{(z_1,z_2):|z_1|,|z_2|<r\}$, and let $V\subset U$ be the smaller polydisc $\Delta(r')$ for any $r'<r$. Then we have

**Hartogs' Theorem.** *Any holomorphic function* f *in a neighborhood of* U − V *extends to a holomorphic function on* U.

***Proof.*** In each vertical slice $z_1=$ constant, the region $U-V$ looks either like the annulus $r'<|z_2|<r$ or like the disc $|z_2|<r$. We try to extend $f$ in each slice by Cauchy's formula, setting

$$F(z_1,z_2)=\frac{1}{2\pi\sqrt{-1}}\int_{|w_2|=r}\frac{f(z_1,w_2)\,dw_2}{w_2-z_2}.$$

$F$ is defined throughout $U$; it is clearly holomorphic in $z_2$, and since $(\partial/\partial\bar{z}_1)f=0$, it is holomorphic in $z_1$ as well. Moreover, in the open subset $|z_1|>r'$ of $U-V$, $F(z_1,z_2)=f(z_1,z_2)$ by Cauchy's formula; thus $F|_{U-V}=f$.

Q.E.D.

Hartogs' theorem applies to many pairs of sets $V\subset U\subset\mathbb{C}^n$; it is commonly applied in the form

*A holomorphic function on the complement of a point in an open set* $\mathrm{U}\subset\mathbb{C}^n$ $(\mathrm{n}>1)$ *extends to a holomorphic function in all of* U.

## Weierstrass Theorems and Corollaries

In one variable, every analytic function has a unique local representation

$$f(z)=(z-z_0)^n u(z),\qquad u(z_0)\neq 0,$$

from which we see in particular that the zero locus of $f$ is discrete. Similarly, the Weierstrass theorems give local representations of holomorphic functions in several variables, from which we get a picture of the local geometry of their zero sets.

Suppose we are given a function $f(z_1,\ldots,z_{n-1},w)$ holomorphic in some neighborhood of the origin in $\mathbb{C}^n$, with $f(0,\ldots,0)=0$. Assume that $f$ does not vanish identically on the $w$-axis, i.e., the power series expansion for $f$ around the origin contains a term $a\cdot w^d$ with $a\neq 0$ and $d\geq 1$; clearly this will be the case for most choices of coordinate system.

For suitable $r$, $\delta$, and $\varepsilon>0$, then, $|f(0,w)|>\delta>0$ for $|w|=r$, and consequently $|f(z,w)|>\delta/2$ for $|w|=r$, $\|z\|\leq\varepsilon$. Now if $w=b_1,\dots,b_d$ are the roots of $f(z,w)=0$ for $|w|<r$, by the residue theorem

$$b_1^q+b_2^q+\cdots+b_d^q=\frac{1}{2\pi\sqrt{-1}}\int_{|w|=r}\frac{w^q(\partial f/\partial w)(z,w)}{f(z,w)}dw;$$

so the power sums $\Sigma b_i(z)^q$ are analytic functions of $z$ for $\|z\|<\varepsilon$. Let $\sigma_1(z),\dots,\sigma_d(z)$ be the elementary symmetric polynomials in $b_1,\dots,b_d$; $\sigma_1,\dots,\sigma_d$ can be expressed as polynomials in the power sums $\Sigma b_i(z)^q$. Thus the function

$$g(z,w)=w^d-\sigma_1(z)w^{d-1}+\cdots+(-1)^d\sigma_d(z)$$

is holomorphic in $\|z\|<\varepsilon, |w|<r$, and vanishes on exactly the same set as $f$. The quotient

$$h(z,w)=\frac{f(z,w)}{g(z,w)}$$

is defined and holomorphic in $\|z\|<\varepsilon$, $|w|<r$, at least outside the zero set of $f$ and $g$. Moreover, for fixed $z$, $h(z,w)$ has only removable singularities in the disc $|w|<r$, so $h$ can be extended to a function in all of $\|z\|<\varepsilon$, $|w|<r$ and analytic in $w$ for each fixed $z$, as well as in the complement of the zero locus. Writing

$$h(z,w)=\frac{1}{2\pi\sqrt{-1}}\int_{|u|=r}\frac{h(z,u)\,du}{u-w},$$

we see that $h$ is holomorphic in $z$ as well.

Definition. A *Weierstrass polynomial* in $w$ is a polynomial of the form

$$w^d+a_1(z)w^{d-1}+\cdots+a_d(z),\qquad a_i(0)=0.$$

We have proved the existence part of the

**Weierstrass Preparation Theorem.** *If* f *is holomorphic around the origin in* $\mathbb{C}^n$ *and is not identically zero on the* w-*axis, then in some neighborhood of the origin* f *can be written uniquely as*

$$f=g\cdot h,$$

*where* g *is a Weierstrass polynomial of degree* d *in* w *and* $h(0)\neq 0$.

The uniqueness is clear, since the coefficients of any Weierstrass polynomial $g$ vanishing exactly where $f$ does are given as polynomials in the integrals

$$\int_{|w|=r}\frac{w^q(\partial f/\partial w)(z,w)\,dw}{f(z,w)}.$$

We see from the Weierstrass theorem that the zero locus of a function $f$, holomorphic in a neighborhood of the origin in $\mathbf{C}^n$, is for most choices of coordinate system $z_1,\ldots,z_{n-1},w$ the zero locus of a Weierstrass polynomial

$$g(z,w) = w^d + a_1(z)w^{d-1} + \cdots + a_d(z).$$

Now, the roots $b_i(z)$ of the polynomial $g(z,\cdot)$ are, away from those values of $z$ for which $g(z,\cdot)$ has a multiple root, locally single-valued holomorphic functions of $z$. Since the discriminant of $g(z,\cdot)$ is an analytic function of $z$,

*The zero locus of an analytic function* $f(z_1,\ldots,z_{n-1},w)$, *not vanishing identically on the* $w$*-axis, projects locally onto the hyperplane* $(w=0)$ *as a finite-sheeted cover branched over the zero locus of an analytic function.*

As a corollary of the preparation theorem, we have the

**Riemann Extension Theorem.** *Suppose* $f(z,w)$ *is holomorphic in a disc* $\Delta\subset\mathbf{C}^n$ *and* $g(z,w)$ *is holomorphic in* $\bar{\Delta}-\{f=0\}$ *and bounded. Then* $g$ *extends to a holomorphic function on* $\Delta$.

***Proof*** (in a neighborhood of 0). Assume that the line $z=0$ is not contained in $\{f=0\}$. As before, we can find $r$, $\varepsilon$, and $\delta>0$ such that $|f(0,w)|>\delta>0$ for $|w|=r$ and $\varepsilon$ such that $|f(z,w)|>\delta/2$ for $\|z\|<\varepsilon$, $|w|=r$; $f$ then has zeros only in the interior of the discs $z=z_0$, $|w|<r$. By the one-variable Riemann extension theorem, we can extend $g$ to a function $\tilde{g}$ in $|z|<\varepsilon$, $|w|<r$, holomorphic away from $\{f=0\}$ and holomorphic in $w$ everywhere. As before, we write

$$\tilde{g}(z,w) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=r}\frac{\tilde{g}(z,u)\,du}{u-w}$$

to see that $\tilde{g}$ is holomorphic in $z$ as well. Q.E.D.

We recall some facts and definitions from elementary algebra:

Let $R$ be an *integral domain*, i.e., a ring such that for $u,v\in R$, $u\cdot v=0\Rightarrow u=0$ or $v=0$. An element $u\in R$ is a *unit* if there exists $v\in R$ such that $u\cdot v=1$; $u$ is *irreducible* if for $v,w\in R$, $u=v\cdot w$ implies $v$ is a unit or $w$ is a unit. $R$ is a *unique factorization domain* (*UFD*) if every $u\in R$ can be written as a product of irreducible elements $u_1,\ldots,u_l$, the $u_i$'s unique up to multiplication by units. The main facts we shall use are

1. $R$ is a UFD$\Rightarrow R[t]$ is a UFD (Gauss' lemma).
2. If $R$ is a UFD and $u,v\in R[t]$ are relatively prime, then there exist relatively prime elements $\alpha,\beta\in R[t]$, $\gamma\neq 0\in R$, such that

$$\alpha u+\beta v=\gamma.$$

$\gamma$ is called the *resultant* of $u$ and $v$.

Let $\mathcal{O}_{n,z}$ denote the ring of holomorphic functions defined in some neighborhood of $z \in \mathbb{C}^n$; write $\mathcal{O}_n$ for $\mathcal{O}_{n,0}$. $\mathcal{O}_n$ is an integral domain by the identity theorem, and moreover is a *local ring* whose *maximal ideal* $m$ is $\{f\colon f(0)=0\}$. $f \in \mathcal{O}_n$ is a unit if and only if $f(0) \neq 0$. The first result is

**Proposition.** *$\mathcal{O}_n$ is a UFD.*

***Proof.*** We proceed by induction. Assume $\mathcal{O}_{n-1}$ is a UFD and let $f \in \mathcal{O}_n$. We may assume $f$ is regular with respect to $w = z_n$; i.e., $f(0,\ldots,0,w) \not\equiv 0$. Write

$$f = g \cdot u,$$

where $u$ is a unit in $\mathcal{O}_n$ and $g \in \mathcal{O}_{n-1}[w]$ is a Weierstrass polynomial. $\mathcal{O}_{n-1}[w]$ is a UFD by Gauss' lemma, and so we can write $g$ as a product of irreducible elements $g_1,\ldots,g_m \in \mathcal{O}_{n-1}[w]$

$$(*) \qquad f = g_1 \cdots g_m \cdot u,$$

where the factors $g_i$ are uniquely determined up to multiplication by units. Now suppose we write $f$ as a product of irreducible elements $f_1,\ldots,f_k \in \mathcal{O}_n$. Each $f_i$ must be regular with respect to $w$, and we can write

$$f_i = g_i' \cdot u_i$$

with $u_i$ a unit, $g_i'$ a Weierstrass polynomial, necessarily irreducible in $\mathcal{O}_{n-1}[w]$. We have

$$f = g \cdot u = \Pi g_i' \cdot \Pi u_i',$$

with $g$ and $\Pi g_i'$ both Weierstrass polynomials; by the Weierstrass preparation theorem

$$g = \Pi g_i',$$

and since $\mathcal{O}_{n-1}[w]$ is a UFD, it follows that the $g_i'$ are the same, up to units, as the $g_i$. Thus the expression $(*)$ represents a unique factorization of $f$ in $\mathcal{O}_n$. Q.E.D.

**Proposition.** *If* f *and* g *are relatively prime in $\mathcal{O}_{n,0}$, then for $\|z\| < \varepsilon$,* f *and* g *are relatively prime in $\mathcal{O}_{n,z}$.*

***Proof.*** We may assume that $f$ and $g$ are regular with respect to $z_n$ and are both Weierstrass polynomials; for each fixed $z' \in \mathbb{C}^{n-1}$ sufficiently small we have $f(z',z_n) \not\equiv 0$ in $z_n$. Now we can write

$$\alpha f + \beta g = \gamma$$

with $\alpha, \beta \in \mathcal{O}_{n-1}[w]$, $\gamma \in \mathcal{O}_{n-1}$; the equation holds in some neighborhood of $0 \in \mathbb{C}^n$.

If for some small $z_0 \in \mathbb{C}^n$, $f(z_0) = g(z_0) = 0$ and $f$ and $g$ have a common factor $h(z',z_n)$ in $\mathcal{O}_{n,z_0}$ with $h(z_0)=0$, then

$$\begin{aligned} h \mid f, h \mid g &\Rightarrow h \mid \gamma \\ &\Rightarrow h \in \mathcal{O}_{n-1}. \end{aligned}$$

But then $h(z_{0_1},\ldots,z_{0_{n-1}},z_n)$ vanishes identically in $z_n$, contradicting our assumption that $f(z_{0_1},\ldots,z_{0_{n-1}},z_n)\not\equiv 0$. Q.E.D.

We now prove the

**Weierstrass Division Theorem.** *Let* $g(z,w)\in\mathcal{O}_{n-1}[w]$ *be a Weierstrass polynomial of degree* $k$ *in* $w$. *Then for any* $f\in\mathcal{O}_n$, *we can write*

$$f = g\cdot h + r$$

*with* $r(z,w)$ *a polynomial of degree* $<k$ *in* $w$.

***Proof.*** For $\varepsilon,\delta>0$ sufficiently small, define for $\|z\|<\varepsilon$, $|w|<\delta$,

$$h(z,w) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}\,\frac{du}{u-w}.$$

$h$ is clearly holomorphic, and hence so is $r=f-gh$. We have

$$\begin{aligned} r(z,w) &= f(z,w)-g(z,w)\cdot h(z,w)\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\left[f(z,u)-g(z,w)\frac{f(z,u)}{g(z,u)}\right]\frac{du}{u-w}\\ &= \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}\,\frac{g(z,u)-g(z,w)}{u-w}\,du. \end{aligned}$$

But $(u-w)$ divides $[g(z,u)-g(z,w)]$ as polynomials in $w$; thus

$$p(z,u,w) = \frac{g(z,u)-g(z,w)}{u-w}$$

is a polynomial in $w$ of degree $<k$. Since the factor $w$ appears only in $p$ in the expression for $r(z,w)$, we see that $r(z,w)$ is a polynomial of degree $<k$ in $w$. Explicitly, if

$$p(z,u,w) = p_1(z,u)\cdot w^{k-1}+\cdots+p_k(z,u),$$

then

$$r(z,w) = a_1(z)\cdot w^{k-1}+\cdots+a_k(z),$$

where

$$a_i(z) = \frac{1}{2\pi\sqrt{-1}}\int_{|u|=\delta}\frac{f(z,u)}{g(z,u)}p_i(z,u)\,du. \qquad \text{Q.E.D.}$$

**Corollary (Weak Nullstellensatz).** *If* $f(z,w)\in\mathcal{O}_n$ *is irreducible and* $h\in\mathcal{O}_n$ *vanishes on the set* $f(z,w)=0$, *then* $f$ *divides* $h$ *in* $\mathcal{O}_n$.

***Proof.*** First, we may assume $f$ is a Weierstrass polynomial of degree $k$ in $w$. Since $f$ is irreducible, $f$ and $\partial f/\partial w$ are relatively prime in $\mathcal{O}_{n-1}[w]$ ($\deg_w f>\deg_w \partial f/\partial w$); thus we can write

$$\alpha\cdot f+\beta\cdot\frac{\partial f}{\partial w} = \gamma, \qquad \gamma\in\mathcal{O}_{n-1}, \quad \gamma\not\equiv 0.$$

If, for a given $z_0$, $f(z_0,w)\in\mathbb{C}[w]$ has a multiple root $u$, we have

$$f(z_0,u)=\frac{\partial f}{\partial w}(z_0,u)=0$$
$$\Rightarrow \gamma(z_0)=0;$$

thus: $f(z,w)$ *has* $k$ *distinct roots in* $w$ *for* $\gamma(z)\neq 0$.

Now by the division theorem, we can write

$$h=f\cdot g+r,\qquad r\in\mathcal{O}_{n-1}[w],\quad \deg r<k.$$

But for any $z_0$ outside the locus $(\gamma=0)$, $f(z_0,w)$ and hence $h(z_0,w)$ have at least $k$ distinct roots in $w$. Since degree $r<k$, this implies $r(z_0,w)=0\in\mathbb{C}[w]$; it follows that $r\equiv 0$ and $h=f\cdot g$. Q.E.D.

### Analytic Varieties

The main purpose of the results given above is to describe the basic local properties of analytic varieties in $\mathbb{C}^n$. We say a subset $V$ of an open set $U\subset\mathbb{C}^n$ is an *analytic variety* in $U$ if, for any $p\in U$, there exists a neighborhood $U'$ of $p$ in $U$ such that $V\cap U'$ is the common zero locus of a finite collection of holomorphic functions $f_1,\ldots,f_k$ on $U'$. In particular, $V$ is called an *analytic hypersurface* if $V$ is locally the zero locus of a single nonzero holomorphic function $f$.

An analytic variety $V\subset U\subset\mathbb{C}^n$ is said to be *irreducible* if $V$ cannot be written as the union of two analytic varieties $V_1,V_2\subset U$ with $V_1,V_2\neq V$; it is said to be *irreducible at* $p\in V$ if $V\cap U'$ is irreducible for small neighborhoods $U'$ of $p$ in $U$. Note first that if $f\in\mathcal{O}_n$ is irreducible in the ring $\mathcal{O}_n$, then the analytic hypersurface $V=\{f(z)=0\}$ given by $f$ in a neighborhood of 0 is irreducible at 0: if $V=V_1\cup V_2$, with $V_1,V_2$ analytic varieties $\neq V$, then there exist $f_1,f_2\in\mathcal{O}_n$ with $f_1$ (respectively $f_2$) vanishing identically on $V_1$ (respectively $V_2$) but not on $V_2$ (respectively $V_1$). By the Nullstellensatz, $f$ must divide the product $f_1\cdot f_2$; since $f$ is irreducible, it follows that $f$ must divide either $f_1$ or $f_2$, i.e., either $V_1\supset V$ or $V_2\supset V$, a contradiction. In addition to the basic picture of an analytic hypersurface (p. 9) we see that

1. Suppose $V\subset U\subset\mathbb{C}^n$ is an analytic hypersurface, given by $V=\{f(z)=0\}$ in a neighborhood of $0\in V$. Since $\mathcal{O}_n$ is a UFD, we can write

$$f=f_1\cdots f_n$$

with $f_i$ irreducible in $\mathcal{O}_n$; if we set $V_i=\{f_i(z)=0\}$ then we have

$$V=V_1\cup\cdots\cup V_k$$

with $V_i$ irreducible at 0. Thus if $p$ is any point on any analytic hypersurface $V\subset U\subset\mathbb{C}^n$, *V can be expressed uniquely in some neighborhood* U′ *of* p *as the union of a finite number of analytic hypersurfaces irreducible at* p.

2. Let $W \subset U \subset \mathbf{C}^n$ be an analytic variety given in a neighborhood $\Delta$ of $0 \in W$ as the zero locus of two functions $f, g \in \mathcal{O}_n$. If $W$ contains no analytic hypersurface through 0, then $f$ and $g$ are necessarily relatively prime in $\mathcal{O}_n$; if $W$ does not contain the line $\{z'=0\}$, then by taking linear combinations we may assume that neither $\{f(z)=0\}$ or $\{g(z)=0\}$ contains $\{z'=0\}$, and hence that $f$ and $g$ are Weierstrass polynomials in $z_n$. Let

$$\gamma = \alpha f + \beta g \neq 0 \in \mathcal{O}_{n-1}$$

be the resultant of $f$ and $g$. We claim that the image of $W$ under the projection map $\pi: \mathbf{C}^n \to \mathbf{C}^{n-1}$ is just the locus of $\gamma$. To see this, write

$$\alpha = hg + r$$

with the degree of $r$ strictly less than the degree of $g$. Then

$$\gamma = rf + (\beta + hf)g.$$

Now, if for some $z$ in $\mathbf{C}^{n-1}$, $\gamma$ vanishes at $z$ but $f$ and $g$ have no common zeros along the line $\pi^{-1}(z)$, it follows that $r$ vanishes at all the zeros of $g$ in $\pi^{-1}(z)$; since $\deg(r) < \deg(g)$, this implies that $r$, and hence $\beta + hf$, vanish identically on $\pi^{-1}(z)$. Thus $r$ and $\beta + hf$ both are zero on the inverse image of any component of the zero locus of $\gamma$ other than $\pi(W)$; but $r$ and $\beta + hf$ are relatively prime and so have no common components. We see then that *$\pi(W)$ is an analytic hypersurface in a neighborhood of the origin in $\mathbf{C}^{n-1}$*, and, reiterating our basic description of analytic hypersurfaces, that *projection of W onto a suitably chosen (n−2)-plane $\mathbf{C}^{n-2} \subset \mathbf{C}^n$ expresses W locally as a finite-sheeted branched cover of a neighborhood of the origin in $\mathbf{C}^{n-2}$.*

3. Last, let $V \subset U \subset \mathbf{C}^n$ be an analytic variety irreducible at $0 \in V$ such that for arbitrarily small neighborhoods $\Delta$ of 0 in $\mathbf{C}^n$, $\pi(V \cap \Delta)$ contains a neighborhood of 0 in $\mathbf{C}^{n-1}$. Write

$$V = \{f_1(z) = \cdots = f_k(z) = 0\}$$

near 0. Then the functions $f_i \in \mathcal{O}_n$ must all have a common factor in $\mathcal{O}_n$, since otherwise $V$ would be contained in the common locus of two relatively prime functions, and by assertion 2, $\pi(V \cap \Delta)$ would be a proper analytic subvariety of $\mathbf{C}^{n-1}$. If we let $g(z)$ be the greatest common divisor of the $f_i$'s, then we can write

$$V = \{g(z)=0\} \cup \left\{ \frac{f_1(z)}{g(z)} = \cdots = \frac{f_k(z)}{g(z)} = 0 \right\}.$$

Since $V$ is irreducible at 0 and since the locus $\{f_i(z)/g(z)=0, \text{ all } i\}$ cannot contain $\{g(z)=0\}$, we must have

$$V = \{g(z)=0\},$$

i.e., $V$ is an analytic hypersurface near 0.

The results 1, 2, and 3 above, together with our basic picture of an analytic hypersurface, give us a picture of the local behavior of those analytic varieties cut out locally by one or two holomorphic functions. In fact, the same picture is in almost all respects valid for general analytic varieties, but to prove this requires some relatively sophisticated techniques from the theory of several complex variables. Since the primary focus of the material in this book is on the codimension 1 case, we will for the time being simply state here without proof the analogous results for general analytic varieties:

1. If $V \subset U \subset \mathbf{C}^n$ is any analytic variety and $p \in V$, then in some neighborhood of $p$, $V$ can be uniquely written as the union of analytic varieties $V_i$ irreducible at $p$ with $V_i \not\subset V_j$.
2. Any analytic variety can be expressed locally by a projection map as a finite-sheeted cover of a polydisc $\Delta$ branched over an analytic hypersurface of $\Delta$.
3. If $V \subset \mathbf{C}^n$ does not contain the line $z_1 = \cdots = z_{n-1} = 0$, then the image of a neighborhood of 0 in $V$ under the projection map $\pi:(z_1,\ldots,z_n) \to (z_1,\ldots,z_{n-1})$ is an analytic subvariety in a neighborhood of $0 \in \mathbf{C}^{n-1}$.

The difficulties in proving these results are more technical than conceptual. For example, to prove assertion 3, note that if $V$ is given near $0 \in \mathbf{C}^n$ by functions $f_1,\ldots,f_k$, then $\pi(V)$ is defined in a neighborhood of $0 \in \mathbf{C}^{n-1}$ by the resultants of all pairs of relatively prime linear combinations of the $f_i$. The problem then is to show that the zero locus of an arbitrary collection of holomorphic functions in a polydisc is in fact given by a finite number of holomorphic functions in a slightly smaller polydisc. Granted assertions 3 and 1, 2 is not hard to prove by a sequence of projections.

All of these facts will follow from the proper mapping theorem, which we shall state in the next section and prove in Chapter 3.

Finally, several more foundational results in several complex variables will be proved by the method of residues in Chapter 5.

## 2. COMPLEX MANIFOLDS

### Complex Manifolds

DEFINITION. A *complex manifold* $M$ is a differentiable manifold admitting an open cover $\{U_\alpha\}$ and coordinate maps $\varphi_\alpha : U_\alpha \to \mathbf{C}^n$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset \mathbf{C}^n$ for all $\alpha, \beta$.

A function on an open set $U \subset M$ is *holomorphic* if, for all $\alpha$, $f \cdot \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U \cap U_\alpha) \subset \mathbf{C}^n$. Likewise, a collection $z = (z_1,\ldots,z_n)$ of functions on $U \subset M$ is said to be a *holomorphic coordinate system* if $\varphi_\alpha \circ z^{-1}$ and $z \circ \varphi_\alpha^{-1}$ are holomorphic on $z(U \cap U_\alpha)$ and $\varphi_\alpha(U \cap U_\alpha)$, respectively, for each $\alpha$; a map $f: M \to N$ of complex manifolds is *holomorphic* if it is given in terms of local holomorphic coordinates on $N$ by holomorphic functions.

*Examples*

1. A one-dimensional complex manifold is called a *Riemann surface*.

2. Let $\mathbb{P}^n$ denote the set of lines through the origin in $\mathbb{C}^{n+1}$. A line $l \subset \mathbb{C}^{n+1}$ is determined by any $Z \neq 0 \in l$, so we can write

$$\mathbb{P}^n = \frac{\{[Z] \neq 0 \in \mathbb{C}^{n+1}\}}{[Z] \sim [\lambda Z]}.$$

On the subset $U_i = \{[Z]: Z_i \neq 0\} \subset \mathbb{P}^n$ of lines not contained in the hyperplane $(Z_i = 0)$, there is a bijective map $\varphi_i$ to $\mathbb{C}^n$ given by

$$\varphi_i([Z_0, \ldots, Z_n]) = \left(\frac{Z_0}{Z_i}, \ldots, \frac{\hat{Z}_i}{Z_i}, \ldots, \frac{Z_n}{Z_i}\right).$$

On $(z_j \neq 0) = \varphi_i(U_j \cap U_i) \subset \mathbb{C}^n$,

$$\varphi_j \circ \varphi_i^{-1}(z_1, \ldots, z_n) = \left(\frac{z_1}{z_j}, \ldots, \frac{\hat{z}_j}{z_j}, \ldots, \frac{1}{z_j}, \ldots, \frac{z_n}{z_j}\right)$$

is clearly holomorphic; thus $\mathbb{P}^n$ has the structure of a complex manifold, called *complex projective space*. The "coordinates" $Z = [Z_0, \ldots, Z_n]$ are called *homogeneous coordinates* on $\mathbb{P}^n$; the coordinates given by the maps $\varphi_i$ are called *Euclidean coordinates*. $\mathbb{P}^n$ is compact, since we have a continuous surjective map from the unit sphere in $\mathbb{C}^{n+1}$ to $\mathbb{P}^n$. Note that $\mathbb{P}^1$ is just the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Any inclusion $\mathbb{C}^{k+1} \to \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{P}^k \to \mathbb{P}^n$; the image of such a map is called a *linear subspace* of $\mathbb{P}^n$. The image of a hyperplane in $\mathbb{C}^{n+1}$ is again called a *hyperplane*, the image of a 2-plane $\mathbb{C}^2 \subset \mathbb{C}^{n+1}$ is a line, and in general the image of a $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ is called a $k$-plane. We may speak of linear relations among points in $\mathbb{P}^n$ in these terms: for example, the *span* of a collection $\{p_i\}$ of points in $\mathbb{P}^n$ is taken to be the image in $\mathbb{P}^n$ of the subspace in $\mathbb{C}^{n+1}$ spanned by the lines $\pi^{-1}(p_i)$; $k$ points are said to be *linearly independent* if their corresponding lines in $\mathbb{C}^{n+1}$ are, that is, if their span in $\mathbb{P}^n$ is a $(k-1)$-plane.

Note that the set of hyperplanes in $\mathbb{P}^n$ corresponds to the set $\mathbb{C}^{n+1*} - \{0\}$ of nonzero linear functionals on $\mathbb{C}^{n+1}$ modulo scalar multiplication; it is thus itself a projective space, called the *dual projective* space and denoted $\mathbb{P}^{n*}$.

It is sometimes convenient to picture $\mathbb{P}^n$ as the compactification of $\mathbb{C}^n$ obtained by adding on the hyperplane $H$ at infinity. In coordinates the inclusion $\mathbb{C}^n \to \mathbb{P}^n$ is $(z_1, \ldots, z_n) \to [1, z_1, \ldots, z_n]$; $H$ has equation $(Z_0 = 0)$, and the identification $H \cong \mathbb{P}^{n-1}$ comes by considering the hyperplane at infinity as the directions in which we can go to infinity in $\mathbb{C}^n$.

3. Let $\Lambda = \mathbb{Z}^k \subset \mathbb{C}^n$ be a discrete lattice. Then the quotient group $\mathbb{C}^n/\Lambda$ has the structure of a complex manifold induced by the projection map $\pi: \mathbb{C}^n \to \mathbb{C}^n/\Lambda$. It is compact if and only if $k = 2n$; in this case $\mathbb{C}^n/\Lambda$ is called a *complex torus*.

In general, if $\pi: M \to N$ is a topological covering space and $N$ is a complex manifold, then $\pi$ gives $M$ the structure of a complex manifold as well; if $M$ is a complex manifold and the deck transformations of $M$ are holomorphic, then $N$ inherits the structure of a complex manifold from $M$.

Another example of this construction is the *Hopf surface*, defined to be the quotient of $\mathbb{C}^2 - \{0\}$ by the group of automorphisms generated by $z \mapsto 2z$. The Hopf surface is the simplest example of a compact complex manifold that cannot be imbedded in projective space of any dimension.

Let $M$ be a complex manifold, $p \in M$ any point, and $z = (z_1, \ldots, z_n)$ a holomorphic coordinate system around $p$. There are three different notions of a tangent space to $M$ at $p$, which we now describe:

1. $T_{\mathbb{R},p}(M)$ is the usual *real tangent space* to $M$ at $p$, where we consider $M$ as a real manifold of dimension $2n$. $T_{\mathbb{R},p}(M)$ can be realized as the space of $\mathbb{R}$-linear derivations on the ring of real-valued $C^\infty$ functions in a neighborhood of $p$; if we write $z_i = x_i + iy_i$,

$$T_{\mathbb{R},p}(M) = \mathbb{R}\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\}.$$

2. $T_{\mathbb{C},p}(M) = T_{\mathbb{R},p}(M) \otimes_{\mathbb{R}} \mathbb{C}$ is called the *complexified tangent space* to $M$ at $p$. It can be realized as the space of $\mathbb{C}$-linear derivations in the ring of complex-valued $C^\infty$ functions on $M$ around $p$. We can write

$$\begin{aligned} T_{\mathbb{C},p}(M) &= \mathbb{C}\left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i} \right\} \\ &= \mathbb{C}\left\{ \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i} \right\} \end{aligned}$$

where, as before,

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left( \frac{\partial}{\partial x_i} - \sqrt{-1}\,\frac{\partial}{\partial y_i} \right), \qquad \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left( \frac{\partial}{\partial x_i} + \sqrt{-1}\,\frac{\partial}{\partial y_i} \right).$$

3. $T_p'(M) = \mathbb{C}\{\partial/\partial z_i\} \subset T_{\mathbb{C},p}(M)$ is called the *holomorphic tangent space* to $M$ at $p$. It can be realized as the subspace of $T_{\mathbb{C},p}(M)$ consisting of derivations that vanish on antiholomorphic functions (i.e., $f$ such that $\bar{f}$ is holomorphic), and so is independent of the holomorphic coordinate system

$(z_1,\ldots,z_n)$ chosen. The subspace $T_p''(M)=\mathbb{C}\{\partial/\partial\bar{z}_i\}$ is called the *antiholomorphic tangent space* to $M$ at $p$; clearly

$$T_{\mathbb{C},p}(M) = T_p'(M)\oplus T_p''(M).$$

Observe that for $M,N$ complex manifolds any $C^\infty$ map $f:M\to N$ induces a linear map

$$f_*: T_{\mathbb{R},p}(M)\to T_{\mathbb{R},f(p)}(N)$$

and hence a map

$$f_*: T_{\mathbb{C},p}(M)\to T_{\mathbb{C},f(p)}(N),$$

but does not in general induce a map from $T_p'(M)$ to $T_{f(p)}'(N)$. In fact, *a map* f : M→N *is holomorphic if and only if*

$$f_*(T_p'(M))\subset T_{f(p)}'(N)$$

for all $p\in M$.

Note also that since $T_{\mathbb{C},p}(M)$ is given naturally as the real vector space $T_{\mathbb{R},p}(M)$ tensored with $\mathbb{C}$, the operation of conjugation sending $\partial/\partial z_i$ to $\partial/\partial\bar{z}_i$ is well-defined and

$$T_p''(M) = \overline{T_p'(M)}.$$

It follows that the projection

$$T_{\mathbb{R},p}(M)\to T_{\mathbb{C},p}(M)\to T_p'(M)$$

is an $\mathbb{R}$-linear isomorphism. This last feature allows us to "do geometry" purely in the holomorphic tangent space. For example, let $z(t)$ $(0<t<1)$ be a smooth arc in the complex $z$-plane. Then $z(t)=x(t)+\sqrt{-1}\,y(t)$, and the tangent to the arc may be taken either as

$$x'(t)\frac{\partial}{\partial x}+y'(t)\frac{\partial}{\partial y}\qquad \text{in } T_{\mathbb{R}}(\mathbb{C})$$

or

$$z'(t)\frac{\partial}{\partial z}\qquad \text{in } T'(\mathbb{C}),$$

and these two correspond under the projection $T_{\mathbb{R}}(\mathbb{C})\to T'(\mathbb{C})$.

Now let $M,N$ be complex manifolds, $z=(z_1,\ldots,z_n)$ be holomorphic coordinates centered at $p\in M$, $w=(w_1,\ldots,w_m)$ holomorphic coordinates centered at $q\in N$ and $f:M\to N$ a holomorphic map with $f(p)=q$. Corresponding to the various tangent spaces to $M$ and $N$ at $p$ and $q$, we have different notions of the *Jacobian* of $f$, as follows:

1. If we write $z_i=x_i+\sqrt{-1}\,y_i$ and $w_\alpha=u_\alpha+\sqrt{-1}\,v_\alpha$, then in terms of the bases $\{\partial/\partial x_i,\partial/\partial y_i\}$ and $\{\partial/\partial u_\alpha,\partial/\partial v_\alpha\}$ for $T_{\mathbb{R},p}(M)$ and $T_{\mathbb{R},q}(N)$, the

linear map $f_*$ is given by the $2m\times 2n$ matrix

$$\mathcal{J}_{\mathbf{R}}(f) = \left[\begin{array}{c|c} \dfrac{\partial u_\alpha}{\partial x_j} & \dfrac{\partial u_\alpha}{\partial y_j} \\ \hline \dfrac{\partial v_\alpha}{\partial x_j} & \dfrac{\partial v_\alpha}{\partial y_j} \end{array}\right].$$

In terms of the bases $\{\partial/\partial z_i, \partial/\partial \bar{z}_i\}$ and $\{\partial/\partial w_\alpha, \partial/\partial \bar{w}_\alpha\}$ for $T_{\mathbf{C},p}(M)$ and $T_{\mathbf{C},q}(N)$, $f_*$ is given by

$$\mathcal{J}_{\mathbf{C}}(f) = \begin{pmatrix} \mathcal{J}(f) & 0 \\ 0 & \overline{\mathcal{J}(f)} \end{pmatrix}$$

where

$$\mathcal{J}(f) = \left(\frac{\partial w_\alpha}{\partial z_j}\right).$$

Note in particular that rank $\mathcal{J}_{\mathbf{R}}(f) = 2\cdot\operatorname{rank}\mathcal{J}(f)$ and that if $m=n$, then

$$\begin{aligned} \det \mathcal{J}_{\mathbf{R}}(f) &= \det \mathcal{J}(f)\cdot \det \overline{\mathcal{J}(f)} \\ &= |\det \mathcal{J}(f)|^2 \geqslant 0, \end{aligned}$$

i.e., *holomorphic maps are orientation preserving*. We take the *natural orientation* on $\mathbf{C}^n$ to be given by the $2n$-form

$$\left(\frac{\sqrt{-1}}{2}\right)^n (dz_1\wedge d\bar{z}_1)\wedge(dz_2\wedge d\bar{z}_2)\wedge\cdots\wedge(dz_n\wedge d\bar{z}_n)$$
$$= dx_1\wedge dy_1\wedge\cdots\wedge dx_n\wedge dy_n;$$

it is clear that if $\varphi_\alpha : U_\alpha \to \mathbf{C}^n$, $\varphi_\beta : U_\beta \to \mathbf{C}^n$ are holomorphic coordinate maps on the complex manifold $M$, the pullbacks via $\varphi_\alpha$ and $\varphi_\beta$ of the natural orientation on $\mathbf{C}^n$ agree on $U_\alpha \cap U_\beta$. Thus *any complex manifold has a natural orientation* which is preserved under holomorphic maps.

## Submanifolds and Subvarieties

Now that we have established the relations among the various Jacobians of a holomorphic map, it is not hard to prove the

**Inverse Function Theorem.** *Let* U, V *be open sets in* $\mathbf{C}^n$ *with* $0 \in U$ *and* $f: U \to V$ *a holomorphic map with* $\mathcal{J}(f) = (\partial f_i/\partial z_j)$ *nonsingular at* 0. *Then* f *is one-to-one in a neighborhood of* 0, *and* $f^{-1}$ *is holomorphic at* f(0).

***Proof.*** First, since $\det|\mathcal{J}_{\mathbf{R}}(f)| = |\det(\mathcal{J}(f))|^2 \neq 0$ at 0, by the ordinary inverse function theorem $f$ has a $C^\infty$ inverse $f^{-1}$ near 0. Now we have

$$f^{-1}(f(z)) = z$$

so

$$\begin{aligned}0 &= \frac{\partial}{\partial \bar z_i}(f^{-1}(f(z)))\\ &= \sum_k \frac{\partial f_j^{-1}}{\partial z_k}\cdot\frac{\partial f_k}{\partial \bar z_i} + \sum_k \frac{\partial f_j^{-1}}{\partial \bar z_k}\cdot\frac{\partial \bar f_k}{\partial \bar z_i}\\ &= \sum_k \frac{\partial f_j^{-1}}{\partial \bar z_k}\cdot\left(\overline{\frac{\partial f_k}{\partial z_i}}\right) \qquad \text{for all } i,j.\end{aligned}$$

Since $(\partial f_k/\partial z_i)$ is nonsingular, this implies $\partial f_j^{-1}/\partial \bar z_k = 0$ for all $j,k$, so $f^{-1}$ is holomorphic. Q.E.D.

Similarly, we have the

**Implicit Function Theorem.** *Given* $\mathrm{f}_1,\ldots,\mathrm{f}_\mathrm{k} \in \mathcal{O}_\mathrm{n}$ *with*

$$\det\left(\frac{\partial f_i}{\partial z_j}(0)\right)_{1\leqslant i,j\leqslant k} \neq 0,$$

*there exist functions* $\mathrm{w}_1,\ldots,\mathrm{w}_\mathrm{k} \in \mathcal{O}_{\mathrm{n}-\mathrm{k}}$ *such that in a neighborhood of* $0$ *in* $\mathbb{C}^\mathrm{n}$,

$$f_1(z) = \cdots = f_k(z) = 0 \Leftrightarrow z_i = w_i(z_{k+1},\ldots,z_n), \qquad 1 \leqslant i \leqslant k.$$

***Proof.*** Again, by the $C^\infty$ implicit function theorem we can find $C^\infty$ functions $w_1,\ldots,w_k$ with the required property; to see that they are holomorphic we write, for $z=(z_{k+1},\ldots,z_n)$, $k+1\leqslant\alpha\leqslant n$,

$$\begin{aligned}0 &= \frac{\partial}{\partial \bar z_\alpha}(f_j(w(z),z))\\ &= \frac{\partial f_j}{\partial \bar z_\alpha}(w(z),z) + \sum \frac{\partial w_l}{\partial \bar z_\alpha}\cdot\frac{\partial f_j}{\partial w_l}(w(z),z) + \sum \frac{\partial \bar w_l}{\partial \bar z_\alpha}\cdot\frac{\partial f_j}{\partial \bar w_l}(w(z),z)\\ &= \sum \frac{\partial w_l}{\partial \bar z_\alpha}\cdot\frac{\partial f_j}{\partial w_l}(w(z),z)\\ &\Rightarrow \frac{\partial w_l}{\partial \bar z_\alpha} = 0 \qquad \text{for all } \alpha, l.\end{aligned}$$

Q.E.D.

One special feature of the holomorphic case is the following:

**Proposition.** *If* $\mathrm{f}: \mathrm{U} \to \mathrm{V}$ *is a one-to-one holomorphic map of open sets in* $\mathbb{C}^\mathrm{n}$ *then* $|\mathcal{J}(\mathrm{f})| \neq 0$, *i.e.,* $\mathrm{f}^{-1}$ *is holomorphic.*

***Proof.*** We prove this by induction on $n$; the case $n=1$ is clear. Let $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$ be coordinates on $U$ and $V$, respectively, and suppose $\mathcal{J}(f)$ has rank $k$ at $0\in U$; we may assume then that the matrix

$((\partial f_i/\partial z_j)(0))_{0<i,j<k}$ is nonsingular. Set

$$z_i' = f_i(z), \qquad 1 \leqslant i \leqslant k,$$
$$z_\alpha' = z_\alpha, \qquad k+1 \leqslant \alpha \leqslant n;$$

by the inverse function theorem, $z'=(z_1',\dots,z_n')$ is a holomorphic coordinate system for $U$ near 0. But now $f$ maps the locus $(z_1'=\cdots=z_k'=0)$ one-to-one onto the locus $(w_1=\cdots=w_k=0)$ and the Jacobian $(\partial f_\alpha/\partial z_\beta')$ of $f|_{(z_1'=\cdots=z_k'=0)}$ is singular at 0; by the induction hypothesis, either $k=0$ or $k=n$. We see then that the Jacobian matrix of $f$ vanishes identically wherever its determinant is zero, i.e., that $f$ maps every connected component of the locus $|\mathcal{J}(f)|=0$ to a single point in $V$. Since $f$ is one-to-one and the zero locus of the holomorphic function $|\mathcal{J}(f)|$ is positive-dimensional if nonempty, it follows that $|\mathcal{J}(f)|\neq 0$. Q.E.D.

Note that this proposition is in contrast to the real case, where the map $t\mapsto t^3$ on $\mathbb{R}$ is one-to-one but does not have a $C^\infty$ inverse.

Now we can make the

Definition. A *complex submanifold* $S$ of a complex manifold $M$ is a subset $S\subset M$ given locally either as the zeros of a collection $f_1,\dots,f_k$ of holomorphic functions with rank $\mathcal{J}(f)=k$, or as the image of an open set $U$ in $\mathbb{C}^{n-k}$ under a map $f\colon U\to M$ with rank $\mathcal{J}(f)=n-k$.

The implicit function theorem assures us that the two alternate conditions of the definition are in fact equivalent, and that the submanifold $S$ has naturally the structure of a complex manifold of dimension $n-k$.

Definition. An *analytic subvariety* $V$ of a complex manifold $M$ is a subset given locally as the zeros of a finite collection of holomorphic functions. A point $p\in V$ is called a *smooth point* of $V$ if $V$ is a submanifold of $M$ near $p$, that is, if $V$ is given in some neighborhood of $p$ by holomorphic functions $f_1,\dots,f_k$ with rank $\mathcal{J}(f)=k$; the locus of smooth points of $V$ is denoted $V^*$. A point $p\in V-V^*$ is called a *singular point* of $V$; the *singular locus* $V-V^*$ of $V$ is denoted $V_s$. $V$ is called *smooth* or *nonsingular* if $V=V^*$, i.e., if $V$ is a submanifold of $M$.

In particular, if $p$ is a point of an analytic hypersurface $V\subset M$ given in terms of local coordinates $z$ by the function $f$, we define the *multiplicity* $\operatorname{mult}_p(V)$ to be the order of vanishing of $f$ at $p$, that is, the greatest integer $m$ such that all partial derivatives

$$\frac{\partial^k f}{\partial z_{i_1}\cdots\partial z_{i_k}}(p)=0, \qquad k\leqslant m-1.$$

We should mention here a piece of terminology that is pervasive in algebraic geometry: the word *generic*. When we are dealing with a family of objects parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement that "a (or the) generic member of the family has a certain property" means exactly that "the set of objects in the family that do not have that property is contained in a subvariety of strictly smaller dimension".

In general, it will be clear how the objects in our family are to be parametrized. One exception will be a reference to "the generic $k$-plane in $\mathbb{P}^n$": until the section on Grassmannians, we have—at least officially—no way of parametrizing linear subspaces of projective space. The fastidious reader may substitute "the linear span of the generic $(k+1)$-tuple of points in $\mathbb{P}^n$."

A basic fact about analytic subvarieties is the

**Proposition.** *$V_s$ is contained in an analytic subvariety of $M$ not equal to $V$.*

***Proof.*** For $p \in V$ let $k$ be the largest integer such that there exist $k$ functions $f_1,\ldots,f_k$ in a neighborhood $U$ of $p$ vanishing on $V$ and such that $\mathcal{J}(f)$ has a $k \times k$ minor not everywhere singular on $V$; we may assume that $|(\partial f_i/\partial z_j)_{1 \leq i,j \leq k}| \not\equiv 0$ on $V$. Let $U' \subset U$ be the locus of $|(\partial f_i/\partial z_j)_{1 \leq i,j \leq k}| \neq 0$ and $V'$ the locus $f_1 = \cdots = f_k = 0$. Then $V' = V \cap U'$ is a complex submanifold of $U'$, and for any holomorphic function $f$ vanishing on $V$ the differential $df \equiv 0$ on $V'$, i.e., $f$ is constant on $V'$. It follows that for $q \in V'$ near $p$, $V = V'$ is a manifold in a neighborhood of $q$ and so $V_s \subset (|(\partial f_i/\partial z_j)_{1 \leq i,j \leq k}| = 0)$. Q.E.D.

It is in fact the case that $V_s$ is an analytic subvariety of $M$—if we choose local defining functions $f_1,\ldots,f_l$ for $V$ carefully, $V_s$ will be the common zero locus of the determinants of the $k \times k$ minors of $\mathcal{J}(f)$. For our purposes, however, we simply need to know that the singular locus of an analytic variety is comparatively small, and so we will not prove this stronger assertion.

We state one more result on analytic varieties:

**Proposition.** *An analytic variety $V$ is irreducible if and only if $V^*$ is connected.*

***Proof.*** One direction is clear: if $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneqq V$ analytic varieties, then $(V_1 \cap V_2) \subset V_s$, so $V^*$ is disconnected.

The converse is harder to prove in general; since we will use it only for analytic hypersurfaces, we will prove it in this case. Suppose $V^*$ is disconnected, and let $\{V_i\}$ denote the connected components of $V^*$; we want to show that $\overline{V_i}$ is an analytic variety. Let $p \in \overline{V_i}$ be any point, $f$ a defining function for $V$ near $p$, and $z = (z_1,\ldots,z_n)$ local coordinates around $p$; we may assume that $f$ is a Weierstrass polynomial of degree $k$ in $z_n$.

Write

$$g = \alpha \cdot f + \beta \cdot \frac{\partial f}{\partial z_n}, \qquad g \neq 0 \in \mathcal{O}_{n-1};$$

then for $\Delta$ some polydisc around $p$ and $\Delta'$ a polydisc in $\mathbb{C}^{n-1}$, the projection map $\pi$: $(z_1,\ldots,z_n) \mapsto (z_1,\ldots,z_{n-1})$ expresses $V_i \cap (\Delta - (g=0))$ as a covering space of $\Delta' - (g=0)$. Let $\{w_\nu(z')\}$ denote the $z_n$-coordinates of the points in $\pi^{-1}(z')$ for $z' = (z_1,\ldots,z_{n-1}) \in \Delta' - (g=0)$ and let $\sigma_1(z'),\ldots,\sigma_k(z')$ denote the elementary symmetric functions of the $w_\nu$. The functions $\sigma_i$ are well-defined and bounded on $\Delta' - (g=0)$, and so extend to $\Delta'$; the function

$$f_i(z) = z_n^k + \sigma_1(z')z_n^{k-1} + \cdots + \sigma_k(z')$$

is thus holomorphic in a neighborhood of $p$ and vanishes exactly on $\overline{V}_i$.
Q.E.D.

We take the *dimension* of an irreducible analytic variety $V$ to be the dimension of the complex manifold $V^*$; we say that a general analytic variety is of dimension $k$ if all of its irreducible components are.

A note: if $V \subset M$ is an analytic subvariety of a complex manifold $M$, then we may define the *tangent cone* $T_p(V) \subset T_p'(M)$ to $V$ at any point $p \in V$ as follows: if $V = (f=0)$ is an analytic hypersurface, and in terms of holomorphic coordinates $z_1,\ldots,z_n$ on $M$ centered around $p$ we write

$$f(z_1,\ldots,z_n) = f_m(z_1,\ldots,z_n) + f_{m+1}(z_1,\ldots,z_n) + \cdots$$

with $f_k(z_1,\ldots,z_n)$ a homogeneous polynomial of degree $k$ in $z_1,\ldots,z_n$, then the tangent cone to $V$ at $p$ is taken to be the subvariety of $T_p'(M) = \mathbb{C}\{\partial/\partial z_i\}$ defined by

$$\left\{ \sum \alpha_i \frac{\partial}{\partial z_i} : f_m(\alpha_1,\ldots,\alpha_n) = 0 \right\}.$$

In general, then, the tangent cone to an analytic variety $V \subset M$ at $p \in V$ is taken to be the intersection of the tangent cones at $p$ to all local analytic hypersurfaces in $M$ containing $V$. In case $V$ is smooth at $p$, of course, this is just the tangent space to $V$ at $p$.

More geometrically, the tangent cone $T_p(V) \subset T_p'(M)$ may be realized as the union of the tangent lines at $p$ to all analytic arcs $\gamma : \Delta \to V \subset M$.

The *multiplicity* of a subvariety $V$ of dimension $k$ in $M$ at a point $p$, denoted $\operatorname{mult}_p(V)$, is taken to be the number of sheets in the projection, in a small coordinate polydisc on $M$ around $p$, of $V$ onto a generic $k$-dimensional polydisc; note that $p$ is a smooth point of $V$ if and only if $\operatorname{mult}_p(V) = 1$. In general, if $W \subset M$ is an irreducible subvariety, we define the *multiplicity* $\operatorname{mult}_W(V)$ *of* $V$ *along* $W$ to be simply the multiplicity of $V$ at a generic point of $W$.

**De Rham and Dolbeault Cohomology**

Let $M$ be a differentiable manifold. Let $A^p(M,\mathbb{R})$ denote the space of differential forms of degree $p$ on $M$, and $Z^p(M,\mathbb{R})$ the subspace of closed $p$-forms. Since $d^2=0$, $d(A^{p-1}(M,\mathbb{R}))\subset Z^p(M,\mathbb{R})$; the quotient groups

$$H^p_{\mathrm{DR}}(M,\mathbb{R}) = \frac{Z^p(M,\mathbb{R})}{dA^{p-1}(M,\mathbb{R})}$$

of closed forms modulo exact forms are called the *de Rham cohomology groups of* $M$.

In the same way, we can let $A^p(M)$ and $Z^p(M)$ denote the spaces of complex-valued $p$-forms and closed complex-valued $p$-forms on $M$, respectively, and let

$$H^p_{\mathrm{DR}}(M) = \frac{Z^p(M)}{dA^{p-1}(M)}$$

be the corresponding quotient; clearly

$$H^p_{\mathrm{DR}}(M) = H^p_{\mathrm{DR}}(M,\mathbb{R})\otimes\mathbb{C}.$$

Now let $M$ be a complex manifold. By linear algebra, the decomposition

$$T^*_{\mathbb{C},z}(M) = T^{*\prime}_z(M)\oplus T^{*\prime\prime}_z(M)$$

of the cotangent space to $M$ at each point $z\in M$ gives a decomposition

$$\wedge^n T^*_{\mathbb{C},z}(M) = \bigoplus_{p+q=n}\left(\wedge^p T^{*\prime}_z(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\right).$$

Correspondingly, we can write

$$A^n(M) = \bigoplus_{p+q=n} A^{p,q}(M),$$

where

$$A^{p,q}(M) = \left\{\varphi\in A^n(M):\ \varphi(z)\in\wedge^p T^{*\prime}_z(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\text{ for all } z\in M\right\}.$$

A form $\varphi\in A^{p,q}(M)$ is said to be of *type* $(p,q)$. By way of notation, we denote by $\pi^{(p,q)}$ the projection maps

$$A^*(M)\to A^{p,q}(M),$$

so that for $\varphi\in A^*(M)$,

$$\varphi = \sum \pi^{(p,q)}\varphi;$$

we usually write $\varphi^{(p,q)}$ for $\pi^{(p,q)}\varphi$.

If $\varphi\in A^{p,q}(M)$, then for each $z\in M$,

$$d\varphi(z)\in\left(\wedge^p T^{*\prime}(M)\otimes\wedge^q T^{*\prime\prime}_z(M)\right)\wedge T^*_{\mathbb{C},z}(M),$$

i.e.,

$$d\varphi\in A^{p+1,q}(M)\oplus A^{p,q+1}(M).$$

We define the operators

$$\bar{\partial}: A^{p,q}(M) \to A^{p,q+1}(M)$$
$$\partial: A^{p,q}(M) \to A^{p+1,q}(M)$$

by

$$\bar{\partial} = \pi^{(p,q+1)} \circ d, \qquad \partial = \pi^{(p+1,q)} \circ d;$$

accordingly, we have

$$d = \partial + \bar{\partial}.$$

In terms of local coordinates $z = (z_1, \ldots, z_m)$, a form $\varphi \in A^n(M)$ is of type $(p,q)$ if we can write

$$\varphi(z) = \sum_{\substack{\# I = p \\ \# J = q}} \varphi_{IJ}(z)\, dz_I \wedge d\bar{z}_J,$$

where for each multiindex $I = \{i_1, \ldots, i_p\}$,

$$dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}.$$

The operators $\partial$ and $\bar{\partial}$ are then given by

$$\bar{\partial}\varphi(z) = \sum_{I,J,j} \frac{\partial}{\partial \bar{z}_j} \varphi_{IJ}(z)\, d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J,$$

$$\partial\varphi(z) = \sum_{I,J,i} \frac{\partial}{\partial z_i} \varphi_{IJ}(z)\, dz_i \wedge dz_I \wedge d\bar{z}_J.$$

In particular, we say that a form $\varphi$ of type $(q,0)$ is *holomorphic* if $\bar{\partial}\varphi = 0$; clearly this is the case if and only if

$$\varphi(z) = \sum_{\# I = q} \varphi_I(z)\, dz_I$$

with $\varphi_I(z)$ holomorphic.

Note that since the decomposition $T^*_{\mathbb{C},z} = T^{*\prime}_z \oplus T^{*\prime\prime}_z$ is preserved under holomorphic maps, so is the decomposition $A^* = \oplus A^{p,q}$. For $f\colon M \to N$ a holomorphic map of complex manifolds,

$$f^*(A^{p,q}(N)) \subset A^{p,q}(M)$$

and

$$\bar{\partial} \circ f^* = f^* \circ \bar{\partial} \qquad \text{on } A^{p,q}(N).$$

Let $Z^{p,q}_{\bar{\partial}}(M)$ denote the space of $\bar{\partial}$-closed forms of type $(p,q)$. Since $\partial^2/\partial\bar{z}_i\partial\bar{z}_j = \partial^2/\partial\bar{z}_j\partial\bar{z}_i$

$$\bar{\partial}^2 = 0$$

on $A^{p,q}(M)$, and we have

$$\bar{\partial}(A^{p,q}(M)) \subset Z^{p,q+1}_{\bar{\partial}}(M);$$

accordingly, we define the *Dolbeault cohomology groups* to be

$$H_{\bar\partial}^{p,q}(M) = \frac{Z_{\bar\partial}^{p,q}(M)}{\bar\partial(A^{p,q-1}(M))}.$$

Note in particular that if $f: M \to N$ is a holomorphic map of complex manifolds, $f$ induces a map

$$f^*: H_{\bar\partial}^{p,q}(N) \to H_{\bar\partial}^{p,q}(M).$$

The ordinary Poincaré lemma that every closed form on $\mathbb{R}^n$ is exact assures us that the de Rham groups are locally trivial. Analogously, a fundamental fact about the Dolbeault groups is the

**$\bar\partial$-Poincaré Lemma.** *For $\Delta = \Delta(r)$ a polycylinder in $\mathbb{C}^n$,*

$$H_{\bar\partial}^{p,q}(\Delta) = 0, \qquad q \geq 1.$$

***Proof.*** First note that if

$$\varphi = \sum_{\substack{{}^{\#}I=p \\ {}^{\#}J=q}} \varphi_{IJ} \cdot dz_I \wedge d\bar z_J$$

is a $\bar\partial$-closed form, then the forms

$$\varphi_I = \sum_{{}^{\#}J=q} \varphi_{IJ} \cdot d\bar z_J \in A^{0,q}(\Delta)$$

are again closed, and that if

$$\varphi_I = \bar\partial \eta_I$$

then

$$\varphi = \pm \bar\partial\Big(\sum_I dz_I \wedge \eta_I\Big);$$

thus it is sufficient to prove that the groups $H_{\bar\partial}^{0,q}(\Delta)$ vanish.

We first show that if $\varphi$ is a $\bar\partial$-closed $(0,q)$-form on $\Delta = \Delta(r)$, then for any $s < r$, we can find $\psi \in A^{0,q-1}(\Delta(s))$ with $\bar\partial\psi = \varphi$ in $\Delta(s)$. To see this, write

$$\varphi = \sum \varphi_I \, d\bar z_I;$$

we claim that if $\varphi \equiv 0$ modulo $(d\bar z_1, \ldots, d\bar z_k)$—that is, if $\varphi_I \equiv 0$ for $I \not\subset \{1, \ldots, k\}$—then we can find $\eta \in A^{0,q-1}(\Delta(s))$ such that

$$\varphi - \bar\partial\eta \equiv 0 \text{ modulo } (d\bar z_1, \ldots, d\bar z_{k-1});$$

this will clearly be sufficient. So assume $\varphi \equiv 0$ modulo $(d\bar z_1, \ldots, d\bar z_k)$ and set

$$\varphi_1 = \sum_{I:\, k \in I} \varphi_I \cdot d\bar z_{I-\{k\}},$$

$$\varphi_2 = \sum_{I:\, k \notin I} \varphi_I \cdot d\bar z_I,$$

so that $\varphi = \varphi_1' \wedge d\bar{z}_k + \varphi_2$, with $\varphi_2 \equiv 0$ modulo $(d\bar{z}_1, \ldots, d\bar{z}_{k-1})$. If $l > k$, $\bar{\partial}\varphi_2$ contains no terms with a factor $d\bar{z}_k \wedge d\bar{z}_l$; since $\bar{\partial}\varphi = \bar{\partial}\varphi_1 + \bar{\partial}\varphi_2 = 0$, it follows that

$$\frac{\partial}{\partial \bar{z}_l} \varphi_I = 0$$

for $l > k$ and $I$ such that $k \in I$.

Now set

$$\eta = \sum_{I:\, k \in I} \eta_I \, d\bar{z}_{I - \{k\}}$$

where

$$\eta_I(z) = \frac{1}{2\pi\sqrt{-1}} \int_{|w_k| < s_k} \varphi_I(z_1, \ldots, w_k, \ldots, z_n) \frac{dw_k \wedge d\bar{w}_k}{w_k - z_k}.$$

By the proposition on p. 5, we have

$$\frac{\partial}{\partial \bar{z}_k} \eta_I(z) = \varphi_I(z),$$

and for $l > k$,

$$\begin{aligned} \frac{\partial}{\partial \bar{z}_l} \eta_I(z) &= \frac{1}{2\pi\sqrt{-1}} \int_{|w_k| < s_k} \frac{\partial}{\partial \bar{z}_l} \varphi_I(z_1, \ldots, w_k, \ldots, z_n) \frac{dw_k \wedge d\bar{w}_k}{w_k - z_k} \\ &= 0 \end{aligned}$$

Thus

$$\varphi - \bar{\partial}\eta \equiv 0 \text{ modulo } (d\bar{z}_1, \ldots, d\bar{z}_{k-1})$$

in $\Delta(s)$ as was desired.

To prove the full $\bar{\partial}$-Poincaré lemma let $\{r_i\}$ be a monotone increasing sequence tending to $r$. By the first step, we can find $\psi_k \in A^{0,q-1}(\Delta)$ such that $\bar{\partial}\psi_k = \varphi$ in $\Delta(r_k)$—take $\psi_k' \in A^{0,q-1}(\Delta(r_{k+1}))$ with $\bar{\partial}\psi_k' = \varphi$, $\rho_k$ a $C^\infty$ bump function $\equiv 1$ on $\Delta(r_k)$ and having compact support in $\Delta(r_{k+1})$, and set $\psi_k = \rho_k \cdot \psi_k'$—the problem is to show that we can choose $\{\psi_k\}$ so that they converge suitably on compact sets. We do this by induction on $q$. Suppose we have $\psi_k$ as above. Take $\alpha \in A^{0,q-1}(\Delta)$ with $\bar{\partial}\alpha = \varphi$ in $\Delta(r_{k+1})$; then

$$\bar{\partial}(\psi_k - \alpha) = 0 \qquad \text{in } \Delta(r_k),$$

and, if $q \geq 2$, then by the induction hypothesis we can find $\beta \in A^{0,q-2}(\Delta)$ with

$$\bar{\partial}\beta = \psi_k - \alpha \qquad \text{in } \Delta(r_{k-1}).$$

Set

$$\psi_{k+1} = \alpha + \bar{\partial}\beta;$$

then $\bar\partial\psi_{k+1}=\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$ and

$$\psi_{k+1}=\psi_k \qquad \text{in } \Delta(r_{k-1}).$$

Thus the sequence $\{\psi_k\}$ chosen in this way converges uniformly on compact sets.

It remains to consider the case $q=1$. Again, say $\psi_k\in C^\infty(\Delta)$ with $\bar\partial\psi_k=\varphi$ in $\Delta(r_k)$, $\alpha\in C^\infty(\Delta)$ with $\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$; then $\psi_k-\alpha$ is a holomorphic function in $\Delta(r_k)$ and hence has a power series expansion around the origin in $\mathbb{C}^n$. Truncate this series expansion to obtain a polynomial $\beta$ with

$$\sup_{\Delta(r_{k-1})}|(\psi_k-\alpha)-\beta|<\frac{1}{2^k},$$

and set

$$\psi_{k+1}=\alpha+\beta.$$

Then $\bar\partial\psi_{k+1}=\bar\partial\alpha=\varphi$ in $\Delta(r_{k+1})$, $\psi_{k+1}-\psi_k$ is holomorphic in $\Delta(r_k)$, and

$$\sup_{\Delta(r_{k-1})}|\psi_{k+1}-\psi_k|<\frac{1}{2^k},$$

so $\psi=\lim\psi_k$ exists, and $\bar\partial\psi=\varphi$. Q.E.D.

Note that the proof works for $r=\infty$.

We leave it as an exercise for the reader to prove, using a similar argument with annuli and Laurent expansions, that

$$H_{\bar\partial}^{p,q}(\Delta^{*k}\times\Delta^l)=0 \qquad \text{for } q\geqslant 1,$$

where $\Delta^*$ is the punctured disc $\Delta-\{0\}$.

## Calculus on Complex Manifolds

Let $M$ be a complex manifold of dimension $n$. A *hermitian metric* on $M$ is given by a positive definite hermitian inner product

$$(\ ,\ )_z:\ T_z'(M)\otimes\overline{T_z'(M)}\to\mathbb{C}$$

on the holomorphic tangent space at $z$ for each $z\in M$, depending smoothly on $z$—that is, such that for local coordinates $z$ on $M$ the functions

$$h_{ij}(z)=\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right)_z$$

are $C^\infty$. Writing $(\ ,\ )_z$ in terms of the basis $\{dz_i\otimes d\bar z_j\}$ for

$$\left(T_z'(M)\otimes\overline{T_z'(M)}\right)^*=T_z^{*\prime}(M)\otimes T_z^{*\prime\prime}(M),$$

the hermitian metric is given by

$$ds^2=\sum_{i,j}h_{ij}(z)\,dz_i\otimes d\bar z_j.$$

A *coframe* for the hermitian metric is an $n$-tuple of forms $(\varphi_1,\dots,\varphi_n)$ of type $(1,0)$ such that

$$ds^2 = \sum_i \varphi_i \otimes \bar{\varphi}_i,$$

i.e., such that, in terms of the inner product induced on $T_z^{*\prime}(M)$ by $(\ ,\ )_z$ on $T_z'(M)$, $(\varphi_1(z),\dots,\varphi_n(z))$ is an orthonormal basis for $T_z^{*\prime}(M)$. From this description it is clear that coframes always exist locally: we can construct one by applying the Gram-Schmidt process to the basis $(dz_1,\dots,dz_n)$ for $T_z^{*\prime}(M)$ at each $z$.

The real and imaginary parts of a hermitian inner product on a complex vector space give an ordinary inner product and an alternating quadratic form, respectively, on the underlying real vector space. Since we have a natural $\mathbb{R}$-linear isomorphism

$$T_{\mathbb{R},z}(M) \longrightarrow T_z'(M),$$

we see that for a hermitian metric $ds^2$ on $M$,

$$\operatorname{Re} ds^2\colon\ T_{\mathbb{R},z}(M) \otimes T_{\mathbb{R},z}(M) \to \mathbb{R}$$

is a *Riemannian metric* on $M$, called the induced Riemannian metric of the hermitian metric. When we speak of distance, area, or volume on a complex manifold with hermitian metric, we always refer to the induced Riemannian metric.

We also see that since the quadratic form

$$\operatorname{Im} ds^2\colon\ T_{\mathbb{R},p}(M) \otimes T_{\mathbb{R},p}(M) \to \mathbb{R}$$

is alternating, it represents a real differential form of degree 2; $\omega = -\frac{1}{2}\operatorname{Im} ds^2$ is called the *associated (1,1)-form* of the metric.

Explicitly, if $(\varphi_1,\dots,\varphi_n)$ is a coframe for $ds^2$, we write

$$\varphi_i = \alpha_i + \sqrt{-1}\,\beta_i,$$

where $\alpha_i, \beta_i$ are real differential forms; then

$$\begin{aligned} ds^2 &= \left(\sum (\alpha_i + \sqrt{-1}\,\beta_i)\right) \otimes \left(\sum (\alpha_i - \sqrt{-1}\,\beta_i)\right) \\ &= \sum_i (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i) + \sqrt{-1}\,\sum_i (-\alpha_i \otimes \beta_i + \beta_i \otimes \alpha_i). \end{aligned}$$

The induced Riemannian metric is given by

$$\operatorname{Re} ds^2 = \sum (\alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i),$$

and the associated (1,1)-form of the metric is given by

$$\begin{aligned} \omega &= -\tfrac{1}{2}\operatorname{Im} ds^2 \\ &= \sum \alpha_i \wedge \beta_i \\ &= \frac{\sqrt{-1}}{2} \sum \varphi_i \wedge \bar{\varphi}_i. \end{aligned}$$

It follows from this last representation that the metric $ds^2=\sum\varphi_i\otimes\bar{\varphi}_i$ may be directly recovered from its associated (1,1)-form $\omega=\frac{1}{2}\sqrt{-1}\sum\varphi_i\wedge\bar{\varphi}_i$. Indeed, any real differential form $\omega$ of type (1,1) on $M$ gives a hermitian form $H(\ ,\ )$ on each tangent space $T'_z(M)$. The form $H$ will be positive definite—i.e., will induce a hermitian metric on $M$—if and only if for every $z\in M$ and holomorphic tangent vector $v\in T'_z(M)$,

$$-\sqrt{-1}\cdot\langle\omega(z),v\wedge\bar{v}\rangle>0.$$

Such a differential form $\omega$ is called a *positive (1,1)-form*; in terms of local holomorphic coordinates $z=(z_1,\dots,z_n)$ on $M$, a form $\omega$ is positive if

$$\omega(z)=\frac{\sqrt{-1}}{2}\sum_{i,j}h_{ij}(z)\,dz_i\wedge d\bar{z}_j$$

with $H(z)=(h_{ij}(z))$ a positive definite hermitian matrix for each $z$.

If $S\subset M$ is a complex submanifold, then for $z\in S$ we have a natural inclusion

$$T'_z(S)\subset T'_z(M);$$

consequently a hermitian metric on $M$ induces the same on $S$ by restriction. More generally, if $f:N\to M$ is any holomorphic map such that

$$f_*:\ T'_z(N)\to T'_{f(z)}(M)$$

is injective for all $z\in N$, a metric on $M$ induces a metric on $N$ by setting

$$\left(\frac{\partial}{\partial w_\alpha},\frac{\partial}{\partial w_\beta}\right)_z=\left(f_*\frac{\partial}{\partial w_\alpha},f_*\frac{\partial}{\partial w_\beta}\right)_{f(z)}.$$

Note that in this case we can always find, for $U\subset N$ small, a coframe $(\varphi_1,\dots,\varphi_n)$ on $f(U)\subset M$ with $\varphi_{k+1},\dots,\varphi_n\in\operatorname{Ker}f^*:T^{*\prime}_{f(z)}(M)\to T^{*\prime}_z(N)$; then $f^*\varphi_1,\dots,f^*\varphi_k$ form a coframe on $U$ for the induced metric on $N$. The associated (1,1)-form $\omega_N$ on $N$ is thus given by

$$\begin{aligned}\omega_N&=\frac{\sqrt{-1}}{2}\sum_{i=1}^{k}f^*\varphi_i\wedge f^*\bar{\varphi}_i\\&=f^*\left(\frac{\sqrt{-1}}{2}\sum_{i=1}^{k}\varphi_i\wedge\bar{\varphi}_i\right)\\&=f^*\left(\frac{\sqrt{-1}}{2}\sum_{i=1}^{n}\varphi_i\wedge\bar{\varphi}_i\right)\\&=f^*\omega_M,\end{aligned}$$

i.e., *the associated (1,1)-form of the induced metric on* N *is the pullback of the associated (1,1)-form of the metric on* M.

*Examples*

1. The hermitian metric on $\mathbb{C}^n$ given by

$$ds^2 = \sum_{i=1}^{n} dz_i \otimes d\bar{z}_i$$

is called the *Euclidean* or *standard* metric; the induced Riemannian metric is, of course, the standard metric on $\mathbb{C}^n = \mathbb{R}^{2n}$.

2. If $\Lambda \subset \mathbb{C}^n$ is a full lattice, then the metric given on the complex torus $\mathbb{C}^n/\Lambda$ by

$$ds^2 = \sum dz_i \otimes d\bar{z}_i$$

is again called the *Euclidean metric* on $\mathbb{C}^n/\Lambda$.

3. Let $Z_0, \ldots, Z_n$ be coordinates on $\mathbb{C}^{n+1}$ and denote by $\pi: \mathbb{C}^{n+1} - \{0\} \to \mathbb{P}^n$ the standard projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z: U \to \mathbb{C}^{n+1} - \{0\}$ a lifting of $U$, i.e., a holomorphic map with $\pi \circ Z = id$; consider the differential form

$$\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z\|^2.$$

If $Z': U \to \mathbb{C}^{n+1} - \{0\}$ is another lifting, then

$$Z' = f \cdot Z$$

with $f$ a nonzero holomorphic function, so that

$$\begin{aligned} \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|Z'\|^2 &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\left(\log \|Z\|^2 + \log f + \log \bar{f}\right) \\ &= \omega + \frac{\sqrt{-1}}{2\pi}\left(\partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f}\right) \\ &= \omega. \end{aligned}$$

Therefore $\omega$ is independent of the lifting chosen; since liftings always exist locally, $\omega$ is a globally defined differential form in $\mathbb{P}^n$. Clearly $\omega$ is of type $(1,1)$. To see that $\omega$ is positive, first note that the unitary group $U(n+1)$ acts transitively on $\mathbb{P}^n$ and leaves the form $\omega$ invariant, so that $\omega$ is positive everywhere if it is positive at one point. Now let $\{w_i = Z_i/Z_0\}$ be coordinates on the open set $U_0 = (Z_0 \neq 0)$ in $\mathbb{P}^n$ and use the lifting $Z = (1, w_1, \ldots, w_n)$ on $U_0$; we have

$$\begin{aligned} \omega &= \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log\left(1 + \sum w_i \bar{w}_i\right) \\ &= \frac{\sqrt{-1}}{2\pi} \partial\left(\frac{\sum w_i\, d\bar{w}_i}{1 + \sum w_i \bar{w}_i}\right) \\ &= \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dw_i \wedge d\bar{w}_i}{1 + \sum w_i \bar{w}_i} - \frac{\left(\sum \bar{w}_i\, dw_i\right) \wedge \left(\sum w_i\, d\bar{w}_i\right)}{\left(1 + \sum w_i \bar{w}_i\right)^2}\right]. \end{aligned}$$

At the point $[1,0,\ldots,0]$,

$$\omega = \frac{\sqrt{-1}}{2\pi}\sum dw_i \wedge d\bar{w}_i > 0.$$

Thus $\omega$ defines a hermitian metric on $\mathbb{P}^n$, called the *Fubini-Study metric.*

*The Wirtinger Theorem.* The interplay between the real and imaginary parts of a hermitian metric now gives us the Wirtinger theorem, which expresses another fundamental difference between Riemannian and hermitian differential geometry. Let $M$ be a complex manifold, $z=(z_1,\ldots,z_n)$ local coordinates on $M$, and

$$ds^2 = \sum \varphi_i \otimes \bar{\varphi}_i$$

a hermitian metric on $M$ with associated $(1,1)$-form $\omega$. Write $\varphi_i = \alpha_i + \sqrt{-1}\,\beta_i$; then the associated Riemannian metric on $M$ is

$$\operatorname{Re}(ds^2) = \sum_{i,j} \alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i,$$

and the volume element associated to $\operatorname{Re}(ds^2)$ is given by

$$d\mu = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n.$$

On the other hand, we have

$$\omega = \sum \alpha_i \wedge \beta_i,$$

so that the $n^{\text{th}}$ exterior power

$$\begin{aligned}\omega^n &= n!\cdot\alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_n \wedge \beta_n\\ &= n!\cdot d\mu.\end{aligned}$$

Now let $S \subset M$ be a complex submanifold of dimension $d$. As we have observed, the $(1,1)$-form associated to the metric induced on $S$ by $ds^2$ is just $\omega|_S$, and applying the above to the induced metric on $S$, we have the

**Wirtinger Theorem**

$$\operatorname{vol}(S) = \frac{1}{d!}\int_S \omega^d.$$

The fact that the volume of a complex submanifold $S$ of the complex manifold $M$ is expressed as the integral over $S$ of a globally defined differential form on $M$ is quite different from the real case. For a $C^\infty$ arc

$$t \mapsto (x(t), y(t))$$

in $\mathbb{R}^2$, for example, the element of arc length is given by

$$\left(x'(t)^2 + y'(t)^2\right)^{1/2} dt,$$

which is not, in general, the pullback of any differential form in $\mathbb{R}^2$.

To close this section, we discuss integration over analytic subvarieties of a complex manifold $M$. To begin with, we define the integral of a