1.2 Complex and Hermitian Structures

In this section, which is essentially a lesson in linear algebra, we shall study additional structures on a given real vector space, e.g. scalar products and (almost) complex structures. They induce linear operators on the exterior algebra (Hodge, Lefschetz, etc.), and we will be interested in the interaction between these operators.

In the following, V shall denote a finite-dimensional real vector space.

Definition 1.2.1 An endomorphism $I : V \to V$ with $I^2 = -\mathrm{id}$ is called an *almost complex structure* on V.

Clearly, if I is an almost complex structure then $I \in \mathrm{Gl}(V)$. If V is the real vector space underlying a complex vector space then $v \mapsto i \cdot v$ defines an almost complex structure I on V. The converse holds true as well:

Lemma 1.2.2 *If I is an almost complex structure on a real vector space V, then V admits in a natural way the structure of a complex vector space.*

Proof. The $\mathbb{C}$-module structure on V is defined by $(a + ib) \cdot v = a \cdot v + b \cdot I(v)$, where $a, b \in \mathbb{R}$. The $\mathbb{R}$-linearity of I and the assumption $I^2 = -\mathrm{id}$ yield $((a + ib)(c + id)) \cdot v = (a + ib)((c + id) \cdot v)$ and in particular $i(i \cdot v) = -v$. $\square$

Thus, almost complex structures and complex structures are equivalent notions for vector spaces. In particular, an almost complex structure can only exist on an even dimensional real vector space.

Corollary 1.2.3 *Any almost complex structure on V induces a natural orientation on V.*

Proof. Using the lemma, the assertion reduces to the statement that the real vector space $\mathbb{C}^n$ admits a natural orientation. We may assume $n = 1$ and use the orientation given by the basis $(1, i)$. The orientation is well-defined, as it does not change under $\mathbb{C}$-linear automorphisms. $\square$

For a real vector space V the complex vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ is denoted by $V_{\mathbb{C}}$. Thus, the real vector space V is naturally contained in the complex vector space $V_{\mathbb{C}}$ via the map $v \mapsto v \otimes 1$. Moreover, $V \subset V_{\mathbb{C}}$ is the part that is left invariant under complex conjugation on $V_{\mathbb{C}}$ which is defined by $\overline{(v \otimes \lambda)} := v \otimes \bar{\lambda}$ for $v \in V$ and $\lambda \in \mathbb{C}$.

Suppose that V is endowed with an almost complex structure I. Then we will also denote by I its $\mathbb{C}$-linear extension to an endomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$. Clearly, the only eigenvalues of I on $V_{\mathbb{C}}$ are $\pm i$.

Definition 1.2.4 Let I be an almost complex structure on a real vector space V and let $I : V_{\mathbb{C}} \to V_{\mathbb{C}}$ be its $\mathbb{C}$-linear extension. Then the $\pm i$ eigenspaces are denoted $V^{1,0}$ and $V^{0,1}$, respectively, i.e.

$$V^{1,0} = \{v \in V_{\mathbb{C}} \mid I(v) = i \cdot v\} \quad \text{and} \quad V^{0,1} = \{v \in V_{\mathbb{C}} \mid I(v) = -i \cdot v\}.$$

Lemma 1.2.5 *Let V be a real vector space endowed with an almost complex structure I. Then*

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}.$$

Complex conjugation on $V_{\mathbb{C}}$ induces an $\mathbb{R}$-linear isomorphism $V^{1,0} \cong V^{0,1}$.

Proof. Since $V^{1,0} \cap V^{0,1} = 0$, the canonical map

$$V^{1,0} \oplus V^{0,1} \longrightarrow V_{\mathbb{C}}$$

is injective. The first assertion follows from the existence of the inverse map

$$v \longmapsto \tfrac{1}{2}\left(v - iI(v)\right) \oplus \tfrac{1}{2}(v + iI(v))\,.$$

For the second assertion we write $v \in V_{\mathbb{C}}$ as $v = x + iy$ with $x, y \in V$. Then $\overline{(v - iI(v))} = (x - iy + iI(x) + I(y)) = (\overline{v} + iI(\overline{v}))$. Hence, complex conjugation interchanges the two factors. □

One should be aware of the existence of two almost complex structures on $V_{\mathbb{C}}$. One is given by I and the other one by i. They coincide on the subspace $V^{1,0}$ but differ by a sign on $V^{0,1}$. Obviously, $V^{1,0}$ and $V^{0,1}$ are complex subspaces of $V_{\mathbb{C}}$ with respect to both almost complex structures. In the sequel, we will always regard $V_{\mathbb{C}}$ as the complex vector space with respect to i. The $\mathbb{C}$-linear extension of I is the additional structure that gives rise to the above decomposition. If $V^{1,0}$ and $V^{0,1}$ are considered with the complex structure i, then the compositions $V \subset V_{\mathbb{C}} \to V^{1,0}$ and $V \subset V_{\mathbb{C}} \to V^{0,1}$ are complex linear respectively complex antilinear. Here, V is endowed with the almost complex structure I.

Lemma 1.2.6 *Let V be a real vector space endowed with an almost complex structure I. Then the dual space $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ has a natural almost complex structure given by $I(f)(v) = f(I(v))$. The induced decomposition on $(V^*)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) = (V_{\mathbb{C}})^*$ is given by*

$$\begin{aligned} (V^*)^{1,0} &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = if(v)\} &&= (V^{1,0})^* \\ (V^*)^{0,1} &= \{f \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid f(I(v)) = -if(v)\} &&= (V^{0,1})^*. \end{aligned}$$

Also note that $(V^*)^{1,0} = \mathrm{Hom}_{\mathbb{C}}((V, I), \mathbb{C})$. □

If V is a real vector space of dimension d, the natural decomposition of its exterior algebra is of the form

$$\bigwedge^* V = \bigoplus_{k=0}^{d} \bigwedge^k V.$$

Analogously, $\bigwedge^* V_\mathbb{C}$ denotes the exterior algebra of the complex vector space $V_\mathbb{C}$, which decomposes as

$$\bigwedge^* V_\mathbb{C} = \bigoplus_{k=0}^{d} \bigwedge^k V_\mathbb{C}. \tag{1.8}$$

Moreover, $\bigwedge^* V_\mathbb{C} = \bigwedge V \otimes_\mathbb{R} \mathbb{C}$ and $\bigwedge^* V$ is the real subspace of $\bigwedge^* V_\mathbb{C}$ that is left invariant under complex conjugation.

If V is endowed with an almost complex structure I, then its real dimension d is even, say $d = 2n$, and $V_\mathbb{C}$ decomposes as above $V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$ with $V^{1,0}$ and $V^{0,1}$ complex vector spaces of dimension n.

Definition 1.2.7 One defines

$$\bigwedge^{p,q} V := \bigwedge^p V^{1,0} \otimes_\mathbb{C} \bigwedge^q V^{0,1},$$

where the exterior products of $V^{1,0}$ and $V^{0,1}$ are taken as exterior products of complex vector spaces. An element $\alpha \in \bigwedge^{p,q} V$ is of bidegree (p, q).

Proposition 1.2.8 *For a real vector space V endowed with an almost complex structure I one has:*

i) $\bigwedge^{p,q} V$ *is in a canonical way a subspace of* $\bigwedge^{p+q} V_\mathbb{C}$.

ii) $\bigwedge^k V_\mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} V$.

iii) *Complex conjugation on $\bigwedge^* V_\mathbb{C}$ defines a ($\mathbb{C}$-antilinear) isomorphism $\bigwedge^{p,q} V \cong \bigwedge^{q,p} V$, i.e. $\overline{\bigwedge^{p,q} V} = \bigwedge^{q,p} V$.*

iv) *The exterior product is of bidegree $(0,0)$, i.e. $(\alpha, \beta) \mapsto \alpha \wedge \beta$ maps $\bigwedge^{p,q} V \times \bigwedge^{r,s} V$ to the subspace $\bigwedge^{p+r,q+s} V$.*

Proof. Let $v_1, \ldots, v_n \in \bigwedge^{1,0} V = V^{1,0}$ and $w_1, \ldots, w_n \in \bigwedge^{0,1} V = V^{0,1}$ be $\mathbb{C}$-basis. Then $v_{J_1} \otimes w_{J_2} \in \bigwedge^{p,q} V$ with $J_1 = \{i_1 < \ldots < i_p\}$ and $J_2 = \{j_1 < \ldots < j_q\}$ form a basis of $\bigwedge^{p,q} V$.

This shows i) and ii). Here, one could as well use the general fact that any direct sum decomposition $V_\mathbb{C} = W_1 \oplus W_2$ induces a direct sum decomposition $\bigwedge^k V_\mathbb{C} = \bigoplus_{p+q=k} \bigwedge^p W_1 \otimes \bigwedge^q W_2$.

Since complex conjugation is multiplicative, i.e. $\overline{w_1 \wedge w_2} = \overline{w_1} \wedge \overline{w_2}$, assertion iii) follows from $\overline{V^{1,0}} = V^{0,1}$. The last assertion holds again true for any decomposition $V_\mathbb{C} = W_1 \oplus W_2$. □

Any vector $v \in V_\mathbb{C}$ can be written as $v = x + iy$ with $x, y \in V$. Assume that $z_i = \frac{1}{2}(x_i - iy_i) \in V^{1,0}$ is a $\mathbb{C}$-basis of $V^{1,0}$ with $x_i, y_i \in V$. Since $I(z_i) = iz_i$, one finds $y_i = I(x_i)$ and $x_i = -I(y_i)$. Moreover, $x_i, y_i \in V$ form a real basis of V and, therefore, a basis of the complex vector space $V_\mathbb{C}$. A natural basis of the complex vector space $V^{0,1}$ is then provided by $\overline{z}_i = \frac{1}{2}(x_i + iy_i)$.

Conversely, if $v \in V$, then $\frac{1}{2}(v - iI(v)) \in V^{1,0}$. Therefore, if $\langle x_i, y_i := I(x_i)\rangle$ is a basis of the real vector space V, then $z_i = \frac{1}{2}(x_i - iy_i)$ is a basis of the complex vector space $V^{1,0}$. With these notations one has the following

Lemma 1.2.9 *For any $m \leq \dim_{\mathbb{C}} V^{1,0}$ one has*

$$(-2i)^m (z_1 \wedge \bar{z}_1) \wedge \ldots \wedge (z_m \wedge \bar{z}_m) = (x_1 \wedge y_1) \wedge \ldots \wedge (x_m \wedge y_m).$$

For $m = \dim_{\mathbb{C}} V^{1,0}$, this defines a positive oriented volume form for the natural orientation of V (cf. Corollary 1.2.3).

Proof. This is a straightforward calculation using induction on m. □

There is an analogous formula for the dual basis. Let $\langle x^i, y^i \rangle$ be the basis of V^* dual to $\langle x_i, y_i \rangle$. Then, $z^i = x^i + iy^i$ and $\bar{z}^i = x^i - iy^i$ are the basis of $V^{1,0^*}$ and $V^{0,1^*}$ dual to $\langle z_i \rangle$ respectively $\langle \bar{z}_i \rangle$. The above formula yields

$$\left(\frac{i}{2}\right)^m (z^1 \wedge \bar{z}^1) \wedge \ldots \wedge (z^m \wedge \bar{z}^m) = (x^1 \wedge y^1) \wedge \ldots \wedge (x^m \wedge y^m).$$

Note that $I(x^i) = -y^i$ and $I(y^i) = x^i$. We tacitly use the natural isomorphism $\bigwedge^k V^* \cong (\bigwedge^k V)^*$ given by $(\alpha_1 \wedge \ldots \wedge \alpha_k)(v_1 \wedge \ldots \wedge v_k) = \det\left(\alpha_j(v_j)\right)_{i,j}$.

Definition 1.2.10 With respect to the direct sum decompositions (1.8) and ii) of Proposition 1.2.8 one defines the natural projections

$$\Pi^k : \bigwedge^* V_{\mathbb{C}} \longrightarrow \bigwedge^k V_{\mathbb{C}} \quad \text{and} \quad \Pi^{p,q} : \bigwedge^* V_{\mathbb{C}} \longrightarrow \bigwedge^{p,q} V.$$

Furthermore, $\mathbf{I} : \bigwedge^* V_{\mathbb{C}} \to \bigwedge^* V_{\mathbb{C}}$ is the linear operator that acts on $\bigwedge^{p,q} V$ by multiplication with i^{p-q}, i.e.

$$\mathbf{I} = \sum_{p,q} i^{p-q} \cdot \Pi^{p,q}.$$

The operator Π^k does not depend on the almost complex structure I, but the operators $\mathbf{I}$ and $\Pi^{p,q}$ certainly do. Note that $\mathbf{I}$ is the multiplicative extension of the almost complex structure I on $V_{\mathbb{C}}$, but $\mathbf{I}$ is not an almost complex structure. Since I is defined on the real vector space V, also $\mathbf{I}$ is an endomorphism of the real exterior algebra $\bigwedge^* V$.

We denote the corresponding operators on the dual space $\bigwedge^* V_{\mathbb{C}}^*$ also by Π^k, $\Pi^{p,q}$, respectively $\mathbf{I}$. Note that $\mathbf{I}(\alpha)(v_1, \ldots, v_k) = \alpha(\mathbf{I}(v_1), \ldots, \mathbf{I}(v_k))$ for $\alpha \in \bigwedge^k V_{\mathbb{C}}^*$ and $v_i \in V_{\mathbb{C}}$.

Let $(V, \langle\ ,\ \rangle)$ be a finite-dimensional euclidian vector space, i.e. V is a real vector space and $\langle\ ,\ \rangle$ is a positive definite symmetric bilinear form.

Definition 1.2.11 An almost complex structure I on V is *compatible* with the scalar product $\langle\ ,\ \rangle$ if $\langle I(v), I(w) \rangle = \langle v, w \rangle$ for all $v, w \in V$, i.e. $I \in \mathrm{O}(V, \langle\ ,\ \rangle)$.

Before considering the general situation, let us study the two-dimensional case, where scalar products and almost complex structures are intimately related. It turns out that these two notions are almost equivalent. This definitely fails in higher dimensions.

Example 1.2.12 Let V be a real vector space of dimension two with a fixed orientation. If $\langle\ ,\ \rangle$ is a scalar product, then there exists a natural almost complex structure I on V associated to it which is defined as follows: For any $0 \neq v \in V$ the vector $I(v) \in V$ is uniquely determined by the following three conditions: $\langle v, I(v)\rangle = 0$, $\|I(v)\| = \|v\|$, and $\{v, I(v)\}$ is positively oriented. Equivalently, I is the rotation by $\pi/2$. Thus, $I^2 = -\mathrm{id}$, i.e. I is an almost complex structure. One also sees that $I \in \mathrm{SO(V)}$ and, thus, I is compatible with $\langle\ ,\ \rangle$.

Two scalar products $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ are called *conformal equivalent* if there exists a (positive) scalar λ with $\langle\ ,\ \rangle' = \lambda \cdot \langle\ ,\ \rangle$. Clearly, two conformally equivalent scalar products define the same almost complex structure. Conversely, for any given almost complex structure I there always exists a scalar product $\langle\ ,\ \rangle$ to which I is associated.

In this way one obtains a bijection between the set of conformal equivalence classes of scalar products on the two-dimensional oriented vector space V and the set of almost complex structures that induce the given orientation:

$$\{\langle\ ,\ \rangle\}_{/\sim_{\mathrm{conf}}} \longleftrightarrow \{I \in \mathrm{Gl}(V)_+ \mid I^2 = -\mathrm{id}\}.$$

Let us now come back to an euclidian vector space $(V, \langle\ ,\ \rangle)$ of arbitrary dimension endowed with a compatible almost complex structure I.

Definition 1.2.13 The *fundamental form* associated to $(V, \langle\ ,\ \rangle, I)$ is the form

$$\omega := -\langle(\ ), I(\ )\rangle = \langle I(\ ), (\ )\rangle.$$

Lemma 1.2.14 *Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space endowed with a compatible almost complex structure. Then, its fundamental form ω is real and of type $(1,1)$, i.e. $\omega \in \bigwedge^2 V^* \cap \bigwedge^{1,1} V^*$.*

Proof. Since

$$\langle v, I(w)\rangle) = \langle I(v), I(I(w))\rangle = -\langle I(v), w\rangle = -\langle w, I(v)\rangle$$

for all $v, w \in V$, the form ω is alternating, i.e. $\omega \in \bigwedge^2 V^*$.

Since

$$(\mathbf{I}\omega)(v,w) = \omega(\mathbf{I}(v), \mathbf{I}(w)) = \langle I(I(v)), I(w)\rangle = \omega(v,w),$$

one finds $\mathbf{I}(\omega) = \omega$, i.e. $\omega \in \bigwedge^{1,1} V_{\mathbb{C}}^*$. $\square$

Note that two of the three structures $\{\langle\ ,\ \rangle, I, \omega\}$ determine the remaining one.

Following a standard procedure, the scalar product and the fundamental form are encoded by a natural hermitian form.

Lemma 1.2.15 *Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space endowed with a compatible complex structure. The form $(\ ,\ ) := \langle\ ,\ \rangle - i \cdot \omega$ is a positive hermitian form on (V, I).*

Proof. The form $(\ ,\ )$ is clearly $\mathbb{R}$-linear and $(v, v) = \langle v, v\rangle > 0$ for $0 \neq v \in V$. Moreover, $(v, w) = \overline{(w, v)}$ and

$$\begin{aligned}(I(v), w) &= \langle I(v), w\rangle - i \cdot \omega(I(v), w)\\ &= \langle I(I(v)), I(w)\rangle + i \cdot \langle v, w\rangle\\ &= i \cdot (i \cdot \langle v, I(w)\rangle + \langle v, w\rangle) = i \cdot (v, w).\end{aligned}$$

□

One also considers the extension of the scalar product $\langle\ ,\ \rangle$ to a positive definite hermitian form $\langle\ ,\ \rangle_{\mathbb{C}}$ on $V_{\mathbb{C}}$. This is defined by

$$\langle v \otimes \lambda, w \otimes \mu\rangle_{\mathbb{C}} := (\lambda\overline{\mu}) \cdot \langle v, w\rangle$$

for $v, w \in V$ and $\lambda, \mu \in \mathbb{C}$.

Lemma 1.2.16 *If $(V, \langle\ ,\ \rangle)$ is an euclidian vector space with a compatible almost complex structure I. Then $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ is an orthogonal decomposition with respect to the hermitian product $\langle\ ,\ \rangle_{\mathbb{C}}$.*

Proof. Let $v - iI(v) \in V^{1,0}$ and $w + iI(w) \in V^{0,1}$ with $v, w \in V$. Then an easy calculation shows $\langle v - iI(v), w + iI(w)\rangle_{\mathbb{C}} = 0$. □

Let us now study the relation between $(\ ,\ )$ and $\langle\ ,\ \rangle_{\mathbb{C}}$.

Lemma 1.2.17 *Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space with a compatible almost complex structure I. Under the canonical isomorphism $(V, I) \cong (V^{1,0}, i)$ one has $\frac{1}{2}(\ ,\ ) = \langle\ ,\ \rangle_{\mathbb{C}}|_{V^{1,0}}$*

Proof. The natural isomorphism was given by $v \mapsto \frac{1}{2}(v - iI(v))$. Now use the definitions of $(\ ,\ )$ to conclude

$$\begin{aligned}&\langle (v - iI(v)), (v' - iI(v'))\rangle_{\mathbb{C}}\\ &= \langle v, v'\rangle + i\langle v, I(v')\rangle - i\langle I(v), v'\rangle + \langle I(v), I(v')\rangle\\ &= 2\langle v, v'\rangle + 2i\langle v, I(v')\rangle = 2(v, v')\end{aligned}$$

□

Often, it is useful to do calculations in coordinates. Let us see how the above products can be expressed explicitly once suitable basis have been chosen.

Let $z_1, \ldots, z_n$ be a $\mathbb{C}$-basis of $V^{1,0}$. Write $z_i = \frac{1}{2}(x_i - iI(x_i))$ with $x_i \in V$. Then $x_1, y_1 := I(x_1), \ldots, x_n, y_n := I(x_n)$ is a $\mathbb{R}$-basis of V and $x_1, \ldots, x_n$ is a

$\mathbb{C}$-basis of (V, I). The hermitian form $\langle\ ,\ \rangle_{\mathbb{C}}$ on $V^{1,0}$ with respect to the basis z_i is given by an hermitian matrix, say $\frac{1}{2}(h_{ij})$. Concretely,

$$\left\langle \sum_{i=1}^{n} a_i z_i, \sum_{j=1}^{n} b_j z_j \right\rangle_{\mathbb{C}} = \frac{1}{2} \sum_{i,j=1}^{n} h_{ij} a_i \bar{b}_j.$$

Using the lemma, we obtain $(x_i, x_j) = h_{ij}$. Since $(\ ,\ )$ is hermitian on (V, I), this yields $(x_i, y_j) = -ih_{ij}$ and $(y_i, y_j) = h_{ij}$.

By definition of $(\ ,\ )$, one has $\omega = -\mathrm{Im}(\ ,\ )$ and $\langle\ ,\ \rangle = \mathrm{Re}(\ ,\ )$. Hence, $\omega(x_i, x_j) = \omega(y_i, y_j) = -\mathrm{Im}(h_{ij})$, $\omega(x_i, y_j) = \mathrm{Re}(h_{ij})$, $\langle x_i, x_j\rangle = \langle y_i, y_j\rangle = \mathrm{Re}(h_{ij})$, and $\langle x_i, y_j\rangle = \mathrm{Im}(h_{ij})$. Thus,

$$\omega = -\sum_{i<j} \mathrm{Im}(h_{ij})(x^i \wedge x^j + y^i \wedge y^j) + \sum_{i,j=1}^{n} \mathrm{Re}(h_{ij}) x^i \wedge y^j.$$

Using $z^i \wedge \bar{z}^j = (x^i + iy^i) \wedge (x^j - iy^j) = x^i \wedge x^j - i(x^i \wedge y^j + x^j \wedge y^i) + y^i \wedge y^j$ this yields

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} z^i \wedge \bar{z}^j.$$

If $x_1, y_1, \ldots, x_n, y_n$ is an orthonormal basis of V with respect to $\langle\ ,\ \rangle$, i.e. $\langle\ ,\ \rangle = \sum_{i=1}^{n} x^i \otimes x^i + \sum_{i=1}^{n} y^i \otimes y^i$, then

$$\omega = \frac{i}{2} \sum_{i=1}^{n} z^i \wedge \bar{z}^i = \sum_{i=1}^{n} x^i \wedge y^i.$$

Note that there always exists an orthonormal basis as above. Indeed, pick $x_1 \neq 0$ arbitrary of norm one and define $y_1 = I(x_1)$, which is automatically orthogonal to x_1. Then continue with the orthogonal complement of $x_1\mathbb{R} \oplus y_1\mathbb{R}$.

Definition 1.2.18 Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space and let I be a compatible almost complex structure. Furthermore, let ω be the associated fundamental form. Then the *Lefschetz operator* $L : \bigwedge^* V_{\mathbb{C}}^* \to \bigwedge^* V_{\mathbb{C}}^*$ is given by $\alpha \mapsto \omega \wedge \alpha$.

Remark 1.2.19 The following properties are easy to verify:
i) L is the $\mathbb{C}$-linear extension of the real operator $\bigwedge^* V^* \to \bigwedge^* V^*$, $\alpha \mapsto \omega \wedge \alpha$.
ii) The Lefschetz operator is of bidegree $(1, 1)$, i.e.

$$L\left(\bigwedge^{p,q} V^*\right) \subset \bigwedge^{p+1,q+1} V^*.$$

Furthermore the Lefschetz operator induces bijections

$$L^k : \bigwedge^k V^* \xrightarrow{\ \simeq\ } \bigwedge^{2n-k} V^*$$

for all $k \leq n$, where $\dim_{\mathbb{R}} V = 2n$. An elementary proof can be given by choosing a basis, but it is slightly cumbersome. A more elegant but less elementary argument, using $\mathfrak{sl}(2)$-representation theory, will be given in Proposition 1.2.30.

The Lefschetz operator comes along with its dual Λ. In order to define and to describe Λ we need to recall the Hodge $*$-operator on a real vector space.

Let $(V, \langle\ ,\ \rangle)$ be an oriented euclidian vector space of dimension d, then $\langle\ ,\ \rangle$ defines scalar products on all the exterior powers $\bigwedge^k V$. Explicitly, if $e_1, \ldots, e_d \in V$ is an orthonormal basis of V, then $e_I \in \bigwedge^k V$ with $I = \{i_1 < \ldots < i_k\}$ is an orthonormal basis of $\bigwedge^k V$. Let $\mathrm{vol} \in \bigwedge^d V$ be the orientation of V of norm 1 given by $\mathrm{vol} = e_1 \wedge \ldots \wedge e_d$.

Then the *Hodge $*$-operator* is defined by

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \cdot \mathrm{vol}$$

for $\alpha, \beta \in \bigwedge^* V$. This determines $*$, for the exterior product defines a non-degenerate pairing $\bigwedge^k V \times \bigwedge^{d-k} V \to \bigwedge^d V = \mathrm{vol} \cdot \mathbb{R}$. One easily sees that $* : \bigwedge^k V \to \bigwedge^{d-k} V$.

The most important properties of the Hodge $*$-operator are collected in the following proposition. Their proofs are all elementary.

Proposition 1.2.20 *Let $(V, \langle\ ,\ \rangle)$ be an oriented euclidian vector space of dimension d. Let $e_1, \ldots, e_d$ be an orthonormal basis of V and let $\mathrm{vol} \in \bigwedge^d V$ be the orientation of norm one given by $e_1 \wedge \ldots \wedge e_d$. The Hodge $*$-operator associated to $(V, \langle\ ,\ \rangle, \mathrm{vol})$ satisfies the following conditions:*

i) *If $\{i_1, \ldots, i_k, j_1, \ldots, j_{d-k}\} = \{1, \ldots d\}$ one has*

$$*(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \varepsilon \cdot e_{j_1} \wedge \ldots \wedge e_{j_{d-k}},$$

where $\varepsilon = \mathrm{sgn}(i_1, \ldots, i_k, j_1 \ldots j_{d-k})$. In particular, $*1 = \mathrm{vol}$.

ii) *The $*$-operator is self-adjoint up to sign: For $\alpha \in \bigwedge^k V$ one has*

$$\langle \alpha, *\beta \rangle = (-1)^{k(d-k)} \langle *\alpha, \beta \rangle.$$

iii) *The $*$-operator is involutive up to sign:*

$$(*|_{\bigwedge^k V})^2 = (-1)^{k(d-k)}.$$

iv) *The Hodge $*$-operator is an isometry on $(\bigwedge^* V, \langle\ ,\ \rangle)$.* □

In our situation we will usually have $d = 2n$ and $*$ and $\langle\ ,\ \rangle$ will be considered on the dual space $\bigwedge^* V^*$.

Let us now come back to the situation considered before. Associated to $(V, \langle\ ,\ \rangle, I)$ we had introduced the Lefschetz operator $L : \bigwedge^k V^* \to \bigwedge^{k+2} V^*$.

Definition 1.2.21 The *dual Lefschetz operator* Λ is the operator $\Lambda : \bigwedge^* V^* \to \bigwedge^* V^*$ that is adjoint to L with respect to $\langle\ ,\ \rangle$, i.e. $\Lambda\alpha$ is uniquely determined by the condition

$$\langle \Lambda\alpha, \beta\rangle = \langle \alpha, L\beta\rangle \quad \text{for all} \quad \beta \in \bigwedge^* V^*.$$

The $\mathbb{C}$-linear extension $\bigwedge^* V^*_{\mathbb{C}} \to \bigwedge^* V^*_{\mathbb{C}}$ of the dual Lefschetz operator will also be denoted by Λ.

Remark 1.2.22 Recall that I induces a natural orientation on V (Corollary 1.2.3). Thus, the Hodge $*$-operator is well-defined. Using an orthonormal basis $x_1, y_1 = I(x_1), \ldots, x_n, y_n = I(x_n)$ as above, a straightforward calculation yields

$$n! \cdot \omega^n = \mathrm{vol},$$

where ω is the associated fundamental form. See Exercise 1.2.9 for a far reaching generalization of this.

Lemma 1.2.23 *The dual Lefschetz operator Λ is of degree -2, i.e. $\Lambda(\bigwedge^k V^*) \subset \bigwedge^{k-2} V^*$. Moreover, one has $\Lambda = *^{-1} \circ L \circ *$.*

Proof. The first assertion follows from the fact that L is of degree two and that $\bigwedge^* V^* = \bigoplus \bigwedge^k V^*$ is orthogonal.

By definition of the Hodge $*$-operator one has $\langle \alpha, L\beta\rangle \cdot \mathrm{vol} = \langle L\beta, \alpha\rangle \cdot \mathrm{vol} = L\beta \wedge *\alpha = \omega \wedge \beta \wedge *\alpha = \beta \wedge (\omega \wedge *\alpha) = \langle \beta, *^{-1}(L(*\alpha))\rangle \cdot \mathrm{vol}$. □

Recall that $\langle\ ,\ \rangle_{\mathbb{C}}$ had been defined as the hermitian extension to $V^*_{\mathbb{C}}$ of the scalar product $\langle\ ,\ \rangle$ on V^*. It can further be extended to a positive definite hermitian form on $\bigwedge^* V^*_{\mathbb{C}}$. Equivalently, one could consider the extension of $\langle\ ,\ \rangle$ on $\bigwedge^* V^*$ to an hermitian form on $\bigwedge^* V^*_{\mathbb{C}}$. In any case, there is a natural positive hermitian product on $\bigwedge^* V^*_{\mathbb{C}}$ which will also be called $\langle\ ,\ \rangle_{\mathbb{C}}$.

The Hodge $*$-operator associated to $(V, \langle\ ,\ \rangle, \mathrm{vol})$ is extended $\mathbb{C}$-linearly to $* : \bigwedge^k V^*_{\mathbb{C}} \to \bigwedge^{2n-k} V^*_{\mathbb{C}}$. On $\bigwedge^* V^*_{\mathbb{C}}$ these two operators are now related by

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta\rangle_{\mathbb{C}} \cdot \mathrm{vol}.$$

Clearly, the Lefschetz operator L and its dual Λ on $\bigwedge^* V^*_{\mathbb{C}}$ are also formally adjoint to each other with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$. Moreover, $\Lambda = *^{-1} \circ L \circ *$ on $\bigwedge^* V^*_{\mathbb{C}}$.

Lemma 1.2.24 *Let $\langle\ ,\ \rangle_{\mathbb{C}}$, Λ, and $*$ be as above. Then*

i) *The decomposition $\bigwedge^k V^*_{\mathbb{C}} = \bigoplus \bigwedge^{p,q} V^*$ is orthogonal with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$.*

ii) *The Hodge $*$-operator maps $\bigwedge^{p,q} V^*$ to $\bigwedge^{n-q,n-p} V^*$, where $n = \dim_{\mathbb{C}}(V, I)$.*

iii) *The dual Lefschetz operator Λ is of bidegree $(-1,-1)$, i.e. $\Lambda(\bigwedge^{p,q} V^*) \subset \bigwedge^{p-1,q-1} V^*$.*

Proof. The first assertion follows directly from Lemma 1.2.16. The third assertion follows from the first and the fact that Λ is the formal adjoint of L with respect to $\langle\ ,\ \rangle_{\mathbb{C}}$. For the second assertion use $\alpha \wedge *\bar{\beta} = \langle \alpha, \beta\rangle_{\mathbb{C}} \cdot \mathrm{vol}$ and that $\gamma_1 \wedge \gamma_2 = 0$ for $\gamma_i \in \bigwedge^{p_i,q_i} V^*$ with $p_1 + p_2 + q_1 + q_2 = 2n$ but $(p_1 + p_2, q_1 + q_2) \neq (n, n)$. □

Definition 1.2.25 Let $H : \bigwedge^* V \to \bigwedge^* V$ be the *counting operator* defined by $H|_{\bigwedge^k V} = (k - n) \cdot \mathrm{id}$, where $\dim_{\mathbb{R}} V = 2n$. Equivalently,

$$H = \sum_{k=0}^{2n} (k - n) \cdot \Pi^k.$$

With H, L, Λ, Π, etc., we dispose of a large number of linear operators on $\bigwedge^* V^*$ and one might wonder whether they commute. In fact, they do not, but their commutators can be computed. This is done in the next proposition. We use the notation $[A, B] = A \circ B - B \circ A$.

Proposition 1.2.26 *Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space endowed with a compatible almost complex structure I. Consider the following linear operators on $\bigwedge^* V^*$: The associated Lefschetz operator L, its dual Λ, and the counting operator H. They satisfy:*

$$\text{i) } [H, L] = 2L, \quad \text{ii) } [H, \Lambda] = -2\Lambda, \quad \textit{and} \quad \text{iii) } [L, \Lambda] = H.$$

Proof. Let $\alpha \in \bigwedge^k V^*$. Then $[H, L](\alpha) = (k + 2 - n)(\omega \wedge \alpha) - \omega \wedge ((k - n)\alpha) = 2\omega \wedge \alpha$. Analogously, $[H, \Lambda](\alpha) = (k - 2 - n)(\Lambda\alpha) - \Lambda((k - n)\alpha) = -2\Lambda\alpha$.

The third assertion is the most difficult one. We will prove it by induction on the dimension of V. Assume we have a decomposition $V = W_1 \oplus W_2$ which is compatible with the scalar product and the almost complex structure, i.e. $(V, \langle\ ,\ \rangle, I) = (W_1, \langle\ ,\ \rangle_1, I_1) \oplus (W_2, \langle\ ,\ \rangle_2, I_2)$. Then $\bigwedge^* V^* = \bigwedge^* W_1^* \otimes \bigwedge^* W_2^*$ and in particular $\bigwedge^2 V^* = \bigwedge^2 W_1^* \oplus \bigwedge^2 W_2^* \oplus W_1^* \otimes W_2^*$. Since $V = W_1 \oplus W_2$ is orthogonal, the fundamental form ω on V decomposes as $\omega_1 \oplus \omega_2$, where ω_i is the fundamental form on W_i (no component in $W_1^* \otimes W_2^*$). Hence the Lefschetz operator L on $\bigwedge^* V^*$ is the direct sum of the Lefschetz operators L_1 and L_2 acting on $\bigwedge^* W_1^*$ and $\bigwedge^* W_2^*$, respectively, i.e. $L = L_1 + L_2$ with L_1 and L_2 acting as $L_1 \otimes 1$ respectively $1 \otimes L_2$ on $\bigwedge^* W_1^* \otimes \bigwedge^* W_2^*$.

Let $\alpha, \beta \in \bigwedge^* V^*$ and suppose that both are split, i.e. $\alpha = \alpha_1 \otimes \alpha_2$, $\beta = \beta_1 \otimes \beta_2$, with $\alpha_i, \beta_i \in \bigwedge^* W_i^*$. Then $\langle \alpha, \beta\rangle = \langle \alpha_1, \beta_1\rangle \cdot \langle \alpha_2, \beta_2\rangle$. Therefore,

$$\begin{aligned}
\langle \alpha, L\beta\rangle &= \langle \alpha, L_1(\beta_1) \otimes \beta_2\rangle + \langle \alpha, \beta_1 \otimes L_2(\beta_2)\rangle \\
&= \langle \alpha_1, L_1\beta_1\rangle\langle \alpha_2, \beta_2\rangle + \langle \alpha_1, \beta_1\rangle\langle \alpha_2, L_2\beta_2\rangle \\
&= \langle \Lambda_1\alpha_1, \beta_1\rangle\langle \alpha_2, \beta_2\rangle + \langle \alpha_1, \beta_1\rangle\langle \Lambda_2\alpha_2, \beta_2\rangle \\
&= \langle \Lambda_1(\alpha_1) \otimes \alpha_2, \beta_1 \otimes \beta_2\rangle + \langle \alpha_1 \otimes \Lambda_2(\alpha_2), \beta_2\rangle.
\end{aligned}$$

Hence, $\Lambda = \Lambda_1 + \Lambda_2$, where Λ_i is the dual Lefschetz operator on $\bigwedge^* W_i^*$. This yields

$$\begin{aligned}[L,\Lambda](\alpha_1\otimes\alpha_2) &= (L_1+L_2)(\Lambda_1(\alpha_1)\otimes\alpha_2+\alpha_1\otimes\Lambda_2(\alpha_2))\\ &\quad -(\Lambda_1+\Lambda_2)(L_1(\alpha_1)\otimes\alpha_2+\alpha_1\otimes L_2(\alpha_2))\\ &= [L_1,\Lambda_1](\alpha_1)\otimes\alpha_2+\alpha_1\otimes[L_2,\Lambda_2](\alpha_2).\end{aligned}$$

By induction hypothesis $[L_i,\Lambda_i]=H_i$ and, therefore,

$$\begin{aligned}[L,\alpha](\alpha_1\otimes\alpha_2) &= H_1(\alpha_1)\otimes\alpha_2+\alpha_1\otimes H_2(\alpha_2)\\ &= (k_1-n_1)(\alpha_1\otimes\alpha_2)+(k_2-n_2)(\alpha_1\otimes\alpha_2)\\ &= (k_1+k_2-n_1-n_2)(\alpha_1\otimes\alpha_2),\end{aligned}$$

for $\alpha_i\in\bigwedge^{k_i}W_i^*$ and $n_i=\dim_{\mathbb{C}}(W_i,I_i)$.

It remains to prove the case $\dim_{\mathbb{C}}(V,I)=1$. With respect to a basis x_1,y_1 of V one has

$$\begin{aligned}\textstyle\bigwedge^*V^* &= \textstyle\bigwedge^0V^* \oplus \bigwedge^1V^* \oplus \bigwedge^2V^*\\ &= \mathbb{R} \oplus (x^1\mathbb{R}\oplus y^1\mathbb{R}) \oplus \omega\mathbb{R}\end{aligned}$$

Moreover, $L:\bigwedge^0V^*\to\bigwedge^2V^*$ and $\Lambda:\bigwedge^2V^*\to\bigwedge^0V^*$ are given by $1\mapsto\omega$ and $\omega\mapsto 1$, respectively. Hence, $[L,\Lambda]|_{\bigwedge^0V^*}=-\Lambda L|_{\bigwedge^0V^*}=-1$, $[L,\Lambda]|_{\bigwedge^1V^*}=0$, and $[L,\Lambda]|_{\bigwedge^2V^*}=1$. □

Corollary 1.2.27 *Let $(V,\langle\ ,\ \rangle,I)$ be an euclidian vector space with a compatible almost complex structure. The action of L, Λ, and H defines a natural $\mathfrak{sl}(2)$-representation on $\bigwedge^* V^*$.*

Proof. Recall, that $\mathfrak{sl}(2)$ is the three-dimensional (over $\mathbb{C}$ or over $\mathbb{R}$) Lie algebra of all 2×2-matrices of trace zero. A basis is given by $X=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, $Y=\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)$, and $B=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. A quick calculation shows that they satisfy $[B,X]=2X$, $[B,Y]=-2Y$, and $[X,Y]=B$. Thus mapping $X\mapsto L$, $Y\mapsto\Lambda$, and $B\mapsto H$ defines a Lie algebra homomorphism $\mathfrak{sl}(2)\to\mathrm{End}(\bigwedge^* V^*)$. The $\mathfrak{sl}(2,\mathbb{C})$-representation is obtained by tensorizing with $\mathbb{C}$. □

Assertion iii) of Proposition 1.2.26 can be generalized to

Corollary 1.2.28 $[L^i,\Lambda](\alpha)=i(k-n+i-1)L^{i-1}(\alpha)$ *for all* $\alpha\in\bigwedge^k V^*$.

Proof. This is easily seen by induction on i as follows:

$$\begin{aligned}[L^i,\Lambda](\alpha) &= L^i\Lambda\alpha-\Lambda L^i\alpha\\ &= L(L^{i-1}\Lambda\alpha-\Lambda L^{i-1}\alpha)+L\Lambda L^{i-1}\alpha-\Lambda LL^{i-1}\alpha\\ &= L[L^{i-1},\Lambda](\alpha)+[L,\Lambda](L^{i-1}\alpha)\\ &= (i-1)(k-n+(i-1)-1)L^{i-1}(\alpha)+(2i-2+k-n)L^{i-1}(\alpha)\\ &= i(k-n+i-1)L^{i-1}(\alpha).\end{aligned}$$

□

Definition 1.2.29 Let $(V, \langle\ ,\ \rangle, I)$ and the induced operators L, Λ, and H be as before. An element $\alpha \in \bigwedge^k V^*$ is called *primitive* if $\Lambda\alpha = 0$. The linear subspace of all primitive elements $\alpha \in \bigwedge^k V^*$ is denoted by $P^k \subset \bigwedge^k V^*$.

Accordingly, an element $\alpha \in \bigwedge^k V^*_{\mathbb{C}}$ is called primitive if $\Lambda\alpha = 0$. Clearly, the subspace of those is just the complexification of P^k.

Proposition 1.2.30 *Let $(V, \langle\ ,\ \rangle, I)$ be an euclidian vector space of dimension $2n$ with a compatible almost complex structure and let L and Λ be the associated Lefschetz operators.*

i) *There exists a direct sum decomposition of the form:*

$$\bigwedge^k V^* = \bigoplus_{i \geq 0} L^i(P^{k-2i}). \tag{1.9}$$

This is the Lefschetz decomposition. *Moreover, (1.9) is orthogonal with respect to $\langle\ ,\ \rangle$.*

ii) *If $k > n$, then $P^k = 0$.*

iii) *The map $L^{n-k} : P^k \to \bigwedge^{2n-k} V^*$ is injective for $k \leq n$.*

iv) *The map $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ is bijective for $k \leq n$.*

v) *If $k \leq n$, then $P^k = \{\alpha \in \bigwedge^k V^* \mid L^{n-k+1}\alpha = 0\}$.*

The following two diagrams might to help memorize the above facts:

$$\begin{tikzcd} \bigwedge^{k-2} V^* \arrow[r, bend left, "L"] & \bigwedge^{k} V^* \arrow[l, bend left, "\Lambda"] \arrow[r, bend left, "L"] & \bigwedge^{k+2} V^* \arrow[l, bend left, "\Lambda"] \arrow[r, bend left, "L"] & \bigwedge^{k+4} V^* \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

$$\begin{tikzcd} \bigwedge^{n-2} V^* \arrow[rrrr, bend left, "L^2", "\cong"'] & \bigwedge^{n-1} V^* \arrow[rr, bend right, "\cong", "L"'] & \bigwedge^{n} V^* & \bigwedge^{n+1} V^* & \bigwedge^{n+2} V^* \end{tikzcd}$$

Proof. i) The easiest way to prove i) is to apply some small amount of representation theory. Since $\bigwedge^* V^*_{\mathbb{C}}$ is a finite-dimensional $\mathfrak{sl}(2)$-representation, it is a direct sum of irreducible ones. Any finite-dimensional $\mathfrak{sl}(2)$-representation admits a primitive vector v, i.e. $\Lambda v = 0$. Indeed, for any vector v the sequence $\Lambda^i v$ for $i = 0, 1, \ldots$ has to terminate by dimension reasons. (Use $H\Lambda^i v = (\deg(v) - 2i - n)\Lambda^i v$.) Using Corollary 1.2.28 one finds that for any primitive v the subspace $v, Lv, L^2v, \ldots$ defines a subrepresentation. Thus, the irreducible $\mathfrak{sl}(2)$-representations are of this form. Altogether this proves the existence of the direct sum decomposition (1.9). The orthogonality with respect to $\langle\ ,\ \rangle$ follows from Corollary 1.2.28.

ii) If $\alpha \in P^k$, $k > n$, and $0 < i$ minimal with $L^i\alpha = 0$, then by Corollary 1.2.28 one has $0 = [L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}\alpha$. This yields $i = 0$, i.e. $\alpha = 0$.

iii) Let $0 \neq \alpha \in P^k$, $k \leq n$ and $0 < i$ minimal with $L^i\alpha = 0$. Then again by Corollary 1.2.28 one finds $0 = [L^i, \Lambda](\alpha) = i(k-n+i-1)L^{i-1}\alpha$ and, therefore, $k-n+i-1 = 0$. In particular, $L^{n-k}(\alpha) \neq 0$. Moreover, $L^{n-k+1}\alpha = 0$, which will be used in the proof of v).

Assertion iv) follows from i), ii), and iii).

v) We have seen already that $P^k \subset \mathrm{Ker}(L^{n-k+1})$. Conversely, let $\alpha \in \bigwedge^k V^*$ with $L^{n-k+1}\alpha = 0$. Then $L^{n-k+2}\Lambda\alpha = L^{n-k+2}\Lambda\alpha - \Lambda L^{n-k+2}\alpha = (n-k+2)L^{n-k+1}\alpha = 0$. But by iv) the map L^{n-k+2} is injective on $\bigwedge^{k-2} V^*$. Hence, $\Lambda\alpha = 0$. $\square$

Let us consider a few special cases. Obviously, $\bigwedge^0 V^* = P^0 = \mathbb{R}$ and $\bigwedge^1 V^* = P^1$. In degree two and four one has $\bigwedge^2 V^* = \omega\mathbb{R} \oplus P^2$ and $\bigwedge^4 V^* = \omega^2\mathbb{R} \oplus L(P^2) \oplus P^4$.

Roughly, the Lefschetz operators and its dual Λ induce a reflection of $\bigwedge^* V^*$ in the middle exterior product $\bigwedge^n V^*$. But there is another operator with this property, namely the Hodge $*$-operator. The interplay between these two is described in the following mysterious but extremely useful proposition.

Proposition 1.2.31 *For all $\alpha \in P^k$ one has*

$$*L^j\alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} \cdot L^{n-k-j}\mathbf{I}(\alpha).$$

Proof. The proof will be given by induction. Suppose that $\dim_{\mathbb{C}}(V) = 1$. Choose an orthonormal basis $V = x_1\mathbb{R} \oplus y_1\mathbb{R}$ such that $I(x_1) = y_1$. Thus, $\omega = x^1 \wedge y^1$. Moreover, $\bigwedge^* V^* = \bigwedge^0 V^* \oplus \bigwedge^1 V^* \oplus \bigwedge^2 V^*$ and the primitive part of $\bigwedge^* V^*$ is $\bigwedge^0 V^* \oplus \bigwedge^1 V^*$. Thus, in order to prove the assertion in the one-dimensional case one has to compare $*1 = \omega$, $*\omega = 1$, $*x^1 = y^1$, and $*y^1 = -x^1$ with the corresponding expressions on the right hand side. Using $\mathbf{I}(x^1) = -y^1$ this is easily verified.

Next, let V be of arbitrary dimension and let $(V, \langle\ ,\ \rangle, I) = (W_1, \langle\ ,\ \rangle_1, I_1) \oplus (W_2, \langle\ ,\ \rangle_2, I_2)$ be a direct sum decomposition. As has been used already in the proof of Proposition 1.2.26, one has $L = L_1 \otimes 1 + 1 \otimes L_2$ and $\Lambda = \Lambda_1 \otimes 1 + 1 \otimes \Lambda_2$ on $\bigwedge^* V^* = \bigwedge^* W_1^* \otimes \bigwedge^* W_2^*$. Moreover, for $\delta_i \in \bigwedge^{k_i} W_i^*$, $i = 1, 2$, the Hodge $*$-operator of $\delta_1 \otimes \delta_2$ is given by $*(\delta_1 \otimes \delta_2) = (-1)^{k_1k_2}(*_1\delta_1) \otimes (*_2\delta_2)$.

Assuming the assertion for W_1 and W_2 one could in principle deduce the assertion for V. However, as the Lefschetz decomposition of $\bigwedge^* V^*$ is not the product of the Lefschetz decompositions of $\bigwedge^* W_1^*$ and $\bigwedge^* W_2^*$, the calculation is slightly cumbersome. It is actually more convenient to assume in addition that W_2 is complex one-dimensional. Of course, the induction argument is still valid. So, we let $W_2 = x_1\mathbb{R} \oplus y_1\mathbb{R}$ as in the one-dimensional case.

Any $\alpha \in \bigwedge^k V^*$ can thus be written as

$$\alpha = \beta_k + \beta'_{k-1} \otimes x^1 + \beta''_{k-1} \otimes y^1 + \beta_{k-2} \otimes \omega,$$

where $\beta_k \in \bigwedge^k W_1^*$, $\beta'_{k-1}, \beta''_{k-1} \in \bigwedge^{k-1} W_1^*$, and $\beta_{k-2} \in \bigwedge^{k-2} W_1^*$. Hence, $\Lambda\alpha = \Lambda_1\beta_k + (\Lambda_1\beta'_{k-1}) \otimes x^1 + (\Lambda_1\beta''_{k-1}) \otimes y^1 + (\Lambda_1\beta_{k-2}) \otimes \omega + \beta_{k-2}$. Thus, α is primitive if and only if $\beta'_{k-1}, \beta''_{k-1}, \beta_{k-2} \in \bigwedge^* W_1^*$ are primitive and $\Lambda_1\beta_k + \beta_{k-2} = 0$. The latter condition holds true if and only if the Lefschetz decomposition of β_k is of the form $\beta_k = \gamma_k + L_1\gamma_{k-2}$ and $\beta_{k-2} = (k-n-1)\gamma_{k-2}$.

Next one computes $L^j\alpha$. Since W_2 is one-dimensional, one has $L^j = L_1^j \otimes 1 + jL_1^{j-1} \otimes L_2$ and, therefore,

$$\begin{aligned} L^j\alpha = {} & L_1^j\gamma_k + L_1^{j+1}\gamma_{k-2} + j(L_1^{j-1}\gamma_k) \otimes \omega + j(L_1^j\gamma_{k-2}) \otimes \omega \\ & +(L_1^j\beta'_{k-1}) \otimes x^1 + (L_1^j\beta''_{k-1}) \otimes y^1 + (k-n-1)(L_1^j\gamma_{k-2}) \otimes \omega. \end{aligned}$$

In order to compute $*L^j\alpha$, one uses this equation and the induction hypothesis:

$$\begin{aligned} *_1 L_1^\ell\gamma_k &= (-1)^{\frac{k(k+1)}{2}} \frac{\ell!}{(n-1-k-\ell)!} L_1^{n-1-k-\ell}\mathbf{I}_1(\gamma_k), \quad \ell = j-1, j \\ *_1 L_1^\ell\gamma_{k-2} &= (-1)^{\frac{(k-2)(k-1)}{2}} \frac{\ell!}{(n+1-k-\ell)!} L_1^{n+1-k-\ell}\mathbf{I}_1(\gamma_{k-2}), \quad \ell = j, j+1 \\ *_1 L_1^j\beta'^{(\prime)}_{k-1} &= (-1)^{\frac{k(k-1)}{2}} \frac{j!}{(n-k-j)!} L_1^{n-k-j}\mathbf{I}_1(\beta'^{(\prime)}_{k-1}). \end{aligned}$$

This yields

$$\begin{aligned} & (-1)^{\frac{k(k+1)}{2}} \frac{(n-k-j)!}{j!} * L^j\alpha \\ & = (n-k-j)(L_1^{n-k-j)}\mathbf{I}_1(\gamma_k)) \otimes \omega - (j+1)(L_1^{n-k-j}\mathbf{I}_1(\gamma_{k-2})) \otimes \omega \\ & \quad -(L_1^{n-k-j}\mathbf{I}_1(\beta'_{k-1})) \otimes *_2(x^1) - (L_1^{n-k-j}\mathbf{I}_1(\beta''_{k-1})) \otimes *_2(y^1) \\ & \quad +L_1^{n-k-j}\mathbf{I}_1(\gamma_k) - L_1^{n+1-k-j}\mathbf{I}_1(\gamma_{k-2}) \end{aligned}$$

On the other hand,

$$\begin{aligned} L^{n-k-j}\mathbf{I}(\alpha) = {} & L_1^{n-k-j}\mathbf{I}_1(\gamma_k) + (n-k-j)(L_1^{n-k-j-1}\mathbf{I}_1(\gamma_k)) \otimes \omega \\ & +(L_1^{n-k-j+1}\mathbf{I}_1(\gamma_{k-2}) + (n-k-j)(L_1^{n-k-j}\mathbf{I}_1(\gamma_{k-2})) \otimes \omega \\ & +(L_1^{n-k-j}\mathbf{I}_1(\beta'_{k-1})) \otimes (-y^1) + (L_1^{n-k-j}\mathbf{I}_1(\beta''_{k-1})) \otimes x^1 \\ & +(k-n-1)(L_1^{n-k-j}\mathbf{I}_1(\gamma_{k-2})) \otimes \omega \end{aligned}$$

Comparing both expressions yields the result. $\square$

Observe that the above proposition shows once again that L^{n-k} is bijective on $\bigwedge^k V^*$ for $k \leq n$ (cf. iv), Proposition 1.2.30).

Example 1.2.32 Here are a few instructive special cases. Let $j = k = 0$ and $\alpha = 1$, then we obtain $*1 = \frac{1}{n!}L^n 1 = \frac{\omega^n}{n!}$. Thus, $\mathrm{vol} = \frac{\omega^n}{n!}$ as was claimed before (Remark 1.2.22).

For $k = 0$, $\alpha = 1$, and $j = 1$, the proposition yields $*\omega = \frac{1}{(n-1)!}\omega^{n-1}$.

If α is a primitive $(1,1)$-form, i.e. $\alpha \in P^2 \cap \bigwedge^{1,1} V^*$, then $*\alpha = \frac{-1}{(n-2)!}\omega^{n-2} \wedge \alpha$.

Remark 1.2.33 Since L, Λ, and H are of pure type $(1,1)$, $(-1,-1)$ and $(0,0)$, respectively, the Lefschetz decomposition is compatible with the bidegree decomposition. Thus, $P^k_{\mathbb{C}} = \bigoplus_{p+q=k} P^{p,q}$, where $P^{p,q} = P^k_{\mathbb{C}} \cap \bigwedge^{p,q} V^*$. Since Λ and L are real, one also has $\overline{P^{p,q}} = P^{q,p}$.

Example 1.2.34 In particular, $\bigwedge^0 V^*_{\mathbb{C}} = P^{0,0} = P^0_{\mathbb{C}} = \mathbb{C}$, $\bigwedge^1 V^*_{\mathbb{C}} = P^{1,0} \oplus P^{0,1}$, and

$$\begin{aligned} \bigwedge^2 V^*_{\mathbb{C}} &= \bigwedge^{2,0} V^* \oplus \bigwedge^{1,1} V^* \oplus \bigwedge^{0,2} V^* \\ &= P^{2,0} \oplus (P^{1,1} \oplus \omega\mathbb{C}) \oplus P^{0,2}. \end{aligned}$$

Definition 1.2.35 Let $(V, \langle\ ,\ \rangle, I)$ be as before and let ω be the associated fundamental form. The *Hodge–Riemann pairing* is the bilinear form

$$Q : \bigwedge^k V^* \times \bigwedge^k V^* \longrightarrow \mathbb{R}, \quad (\alpha, \beta) \longmapsto (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \beta \wedge \omega^{n-k},$$

where $\bigwedge^{2n} V^*$ is identified with $\mathbb{R}$ via the volume form vol.

By definition $Q = 0$ on $\bigwedge^k V^*$ for $k > n$. We will also denote by Q the $\mathbb{C}$-linear extension of the Hodge–Riemann pairing to $\bigwedge^* V^*_{\mathbb{C}}$.

Corollary 1.2.36 (Hodge–Riemann bilinear relation) *Let $(V, \langle\ ,\ \rangle, I)$ be an euclidian vector space endowed with a compatible almost complex structure. Then the associated Hodge–Riemann pairing Q satisfies:*

$$Q(\bigwedge^{p,q} V^*, \bigwedge^{p',q'} V^*) = 0$$

for $(p,q) \neq (q',p')$ and

$$i^{p-q} Q(\alpha, \bar{\alpha}) = (n - (p+q))! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$$

for $0 \neq \alpha \in P^{p,q}$ with $p + q \leq n$.

Proof. Only the second assertion needs a proof. By definition

$$\begin{aligned} Q(\alpha, \bar{\alpha}) \cdot \mathrm{vol} &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge \bar{\alpha} \wedge \omega^{n-k} \\ &= (-1)^{\frac{k(k-1)}{2}} \alpha \wedge L^{n-k} \bar{\alpha} \\ &= (-1)^{\frac{k(k-1)}{2}} \langle \alpha, \beta \rangle_{\mathbb{C}} \cdot \mathrm{vol}, \end{aligned}$$

where $k = p+q$ and $\beta \in \bigwedge^k V^*$ such that $*\bar{\beta} = L^{n-k}\bar{\alpha}$. Hence, $*^2\bar{\beta} = (-1)^k\bar{\beta}$ and, on the other hand,

$$*^2\bar{\beta} = *L^{n-k}\bar{\alpha} = (-1)^{\frac{k(k+1)}{2}}(n-k)!i^{q-p}\bar{\alpha}$$

by Proposition 1.2.31. Thus, $\beta = (-1)^{k+\frac{k(k+1)}{2}}(n-k)! \cdot i^{p-q}\alpha$ and, therefore,

$$Q(\alpha, \bar{\alpha}) = (-1)^{k+\frac{k(k+1)}{2}+\frac{k(k-1)}{2}}(n-k)! \cdot i^{q-p} \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}}.$$

This yields $i^{p-q}Q(\alpha, \bar{\alpha}) = (n-k)! \cdot \langle \alpha, \alpha \rangle_{\mathbb{C}} > 0$ for $0 \neq \alpha \in P^{p,q}$. □

Example 1.2.37 Suppose $n \geq 2$ and consider the decomposition $(\bigwedge^{1,1} V^*)_{\mathbb{R}} = \omega\mathbb{R} \oplus P^{1,1}_{\mathbb{R}}$, where $(\ )_{\mathbb{R}}$ denotes the intersection with $\bigwedge^2 V^*$. Then, the decomposition is Q-orthogonal, because $(\alpha \wedge \omega) \wedge \omega^{n-2} = \alpha \wedge \omega^{n-1} = 0$ for $\alpha \in P^2$. Moreover, Q is a positive definite symmetric bilinear form on $\omega\mathbb{R}$ and a negative definite symmetric bilinear from on $P^{1,1}_{\mathbb{R}}$. This is what will lead to the Hodge index theorem in Section 3.3.

Exercises

1.2.1 Let $(V, \langle\ ,\ \rangle)$ be a four-dimensional euclidian vector space. Show that the set of all compatible almost complex structures consist of two copies of S^2.

1.2.2 Show that the two decompositions $\bigwedge^k V^* = \bigoplus_{0 \leq i} L^i P^{k-2i}$ and $L^i P^{k-2i} = \bigoplus_{p+q=k-2i} L^i P^{p,q}$ are orthogonal with respect to the Hodge–Riemann pairing.

1.2.3 Prove the following identities: $*\Pi^{p,q} = \Pi^{n-q,n-p}*$ and $[L, \mathbf{I}] = [\Lambda, \mathbf{I}] = 0$.

1.2.4 Is the product of two primitive forms again primitive?

1.2.5 Let $(V, \langle\ ,\ \rangle)$ be an euclidian vector space and let I, J, and K be compatible almost complex structures where $K = I \circ J = -J \circ I$. Show that V becomes in a natural way a vector space over the quaternions. The associated fundamental forms are denoted by ω_I, ω_J, and ω_K. Show that $\omega_J + i\omega_K$ with respect to I is a form of type $(2, 0)$. How many natural almost complex structures do you see in this context?

1.2.6 Let $\omega \in \bigwedge^2 V^*$ be non-degenerate, i.e. the induced homomorphism $\tilde{\omega} : V \to V^*$ is bijective. Study the relation between the two isomorphisms $L^{n-k} : \bigwedge^k V^* \to \bigwedge^{2n-k} V^*$ and $\bigwedge^k V^* \cong \bigwedge^{2n-k} V \cong \bigwedge^{2n-k} V^*$, where the latter is given by $\bigwedge^{2n-k} \tilde{\omega}$. Here, $2n = \dim_{\mathbb{R}}(V)$.

1.2.7 Let V be a vector space endowed with a scalar product and a compatible almost complex structure. What is the signature of the pairing $(\alpha, \beta) \mapsto \frac{\alpha \wedge \beta \wedge \omega^{n-2}}{\mathrm{vol}}$ on $\bigwedge^2 V^*$?