## Chapter 1

## Demailly's Holomorphic Morse Inequalities

The first aim of this chapter is to provide the background material on differential geometry for the whole book. Then, in the last two sections, we present a heat kernel proof of Demailly's holomorphic Morse inequalities, Theorem 1.7.1.

This chapter is organized as follows. In Section 1.1 we review the theory of connections on vector bundles. In Section 1.2, we explain different connections on the tangent bundle and their relations. In Section 1.3, we define the modified Dirac operator for an almost complex manifold and prove the related Lichnerowicz formula. We explain also the Atiyah-Singer index theorem for the modified Dirac operator. In Section 1.4, we show that the operator $\bar{\partial}^{E}+\bar{\partial}^{E, *}$ is a modified Dirac operator, and we establish the Lichnerowicz and Bochner-Kodaira-Nakano formulas for the Kodaira Laplacian. In Section 1.5, we deal with vanishing theorems for positive line bundles and the spectral gap property for the modified Dirac operator and the Kodaira Laplacian. In Section 1.6, we establish the asymptotic of the heat kernel which is the analytic core result of this chapter. Finally, in Section 1.7, we prove Demailly's holomorphic Morse inequalities.

### 1.1 Connections on vector bundles

In this section, we review the definition on connections and the associated curvatures. Section 1.1.1 reviews some general facts on connections on vector bundles, and we specify them to the holomorphic case in Section 1.1.2.

### 1.1.1 Hermitian connection

Let $E$ be a complex vector bundle over a smooth manifold $X$. Let $T X$ be the tangent bundle and $T^{*} X$ be the cotangent bundle. Let $\mathscr{C}^{\infty}(X, E)$ be the space of
smooth sections of $E$ on $X$. Let $\Omega^{r}(X, E)$ be the spaces of smooth $r$-forms on $X$ with values in $E$, and set $\mathscr{C}^{\infty}(X):=\mathscr{C}^{\infty}(X, \mathbb{C}), \Omega^{\bullet}(X):=\Omega^{\bullet}(X, \mathbb{C})$.
Let $d: \Omega^{\bullet}(X) \rightarrow \Omega^{\bullet+1}(X)$ be the exterior differential. It is characterized by
a) $d^{2}=0$;
b) for $\varphi \in \mathscr{C}^{\infty}(X), d \varphi$ is the one form such that $(d \varphi)(U)=U(\varphi)$ for a vector field $U$;
c) (Leibniz rule) for any $\alpha \in \Omega^{q}(X), \beta \in \Omega(X)$, then

$$
\begin{equation*}
d(\alpha \wedge \beta)=d \alpha \wedge \beta+(-1)^{q} \alpha \wedge d \beta \tag{1.1.1}
\end{equation*}
$$

Then we verify that for any 1-form $\alpha$, vector fields $U, V$ on $X$, we have

$$
\begin{equation*}
d \alpha(U, V)=U(\alpha(V))-V(\alpha(U))-\alpha([U, V]) \tag{1.1.2}
\end{equation*}
$$

here $[U, V]$ is the Lie bracket of $U$ and $V$.
A linear map $\nabla^{E}: \mathscr{C}^{\infty}(X, E) \rightarrow \mathscr{C}^{\infty}\left(X, T^{*} X \otimes E\right)$ is called a connection on $E$ if for any $\varphi \in \mathscr{C}^{\infty}(X), s \in \mathscr{C}^{\infty}(X, E)$ and $U \in T X$, we have

$$
\begin{equation*}
\nabla_{U}^{E}(\varphi s)=U(\varphi) s+\varphi \nabla_{U}^{E} s \tag{1.1.3}
\end{equation*}
$$

Connections on $E$ always exist. Indeed, let $\left\{V_{k}\right\}_{k}$ an open covering of $X$ such that $\left.E\right|_{V_{k}}$ is trivial. If $\left\{\eta_{k l}\right\}_{l}$ is a local frame of $\left.E\right|_{V_{k}}$, any section $s \in \mathscr{C}^{\infty}\left(V_{k}, E\right)$ has the form $s=\sum_{l} s_{l} \eta_{k l}$ with uniquely determined $s_{l} \in \mathscr{C}^{\infty}\left(V_{k}\right)$. We define a connection on $\left.E\right|_{V_{k}}$ by $\nabla_{k}^{E} s:=\sum_{l} d s_{l} \otimes \eta_{k l}$. Consider now a partition of unity $\left\{\psi_{k}\right\}_{k}$ subordinated to $\left\{V_{k}\right\}_{k}$. Then $\nabla^{E} s:=\sum_{k} \nabla_{k}^{E}\left(\psi_{k} s\right), s \in \mathscr{C}^{\infty}(X, E)$, defines a connection on $E$.

If $\nabla_{1}^{E}$ is another connection on $E$, then by (1.1.3), $\nabla_{1}^{E}-\nabla^{E} \in \Omega^{1}(X, \operatorname{End}(E))$. If $\nabla^{E}$ is a connection on $E$, then there exists a unique extension $\nabla^{E}$ : $\Omega^{\bullet}(X, E) \rightarrow \Omega^{\bullet+1}(X, E)$ verifying the Leibniz rule: for $\alpha \in \Omega^{q}(X), s \in \Omega^{r}(X, E)$, we have

$$
\begin{equation*}
\nabla^{E}(\alpha \wedge s)=d \alpha \wedge s+(-1)^{q} \alpha \wedge \nabla^{E} s \tag{1.1.4}
\end{equation*}
$$

From (1.1.2), for $s \in \mathscr{C}^{\infty}(X, E)$ and vector fields $U, V$ on $X$, we have

$$
\begin{equation*}
\left(\nabla^{E}\right)^{2}(U, V) s=\nabla_{U}^{E} \nabla_{V}^{E} s-\nabla_{V}^{E} \nabla_{U}^{E} s-\nabla_{[U, V]}^{E} s \tag{1.1.5}
\end{equation*}
$$

Then $\left(\nabla^{E}\right)^{2}(U, V)(\varphi s)=\left(\nabla^{E}\right)^{2}(U, \varphi V) s=\left(\nabla^{E}\right)^{2}(\varphi U, V) s=\varphi\left(\nabla^{E}\right)^{2}(U, V) s$ for any $\varphi \in \mathscr{C}^{\infty}(X)$. We deduce that:
Definition and Theorem 1.1.1. The operator $\left(\nabla^{E}\right)^{2}$ defines a bundle morphism $\left(\nabla^{E}\right)^{2}: E \rightarrow \Lambda^{2}\left(T^{*} X\right) \otimes E$, called the curvature operator. Therefore, there exists $R^{E} \in \Omega^{2}(X, \operatorname{End}(E))$, called the curvature of $\nabla^{E}$, such that $\left(\nabla^{E}\right)^{2}$ is given by multiplication with $R^{E}$, i.e., $\left(\nabla^{E}\right)^{2} s=R^{E} s \in \Omega^{2}(X, E)$ for $s \in \mathscr{C}{ }^{\infty}(X, E)$.

Let $h^{E}$ be a Hermitian metric on $E$, i.e., a smooth family $\left\{h_{x}^{E}\right\}_{x \in X}$ of sesquilinear maps $h_{x}^{E}: E_{x} \times E_{x} \rightarrow \mathbb{C}$ such that $h_{x}^{E}(\xi, \xi)>0$ for any $\xi \in E_{x} \backslash\{0\}$. We call $\left(E, h^{E}\right)$ a Hermitian vector bundle on $X$. There always exist Hermitian metrics on $E$ by using the partition of unity argument as above.

Definition 1.1.2. A connection $\nabla^{E}$ is said to be a Hermitian connection on $\left(E, h^{E}\right)$ if for any $s_{1}, s_{2} \in \mathscr{C}{ }^{\infty}(X, E)$,

$$
\begin{equation*}
d\left\langle s_{1}, s_{2}\right\rangle_{h^{E}}=\left\langle\nabla^{E} s_{1}, s_{2}\right\rangle_{h^{E}}+\left\langle s_{1}, \nabla^{E} s_{2}\right\rangle_{h^{E}} \tag{1.1.6}
\end{equation*}
$$

There always exist Hermitian connections. In fact, let $\nabla_{0}^{E}$ be a connection on $E$, then $\left\langle\nabla_{1}^{E} s_{1}, s_{2}\right\rangle_{h^{E}}=d\left\langle s_{1}, s_{2}\right\rangle_{h^{E}}-\left\langle s_{1}, \nabla_{0}^{E} s_{2}\right\rangle_{h^{E}}$ defines a connection $\nabla_{1}^{E}$ on $E$. Now $\nabla^{E}=\frac{1}{2}\left(\nabla_{0}^{E}+\nabla_{1}^{E}\right)$ is a Hermitian connection on $\left(E, h^{E}\right)$.

Let $\left\{\xi_{l}\right\}_{l=1}^{m}$ be a local frame of $E$. Denote by $h=\left(h_{l k}=\left\langle\xi_{k}, \xi_{l}\right\rangle_{h^{E}}\right)$ the matrix of $h^{E}$ with respect to $\left\{\xi_{l}\right\}_{l=1}^{m}$. The connection form $\theta=\left(\theta_{k}^{l}\right)$ of $\nabla^{E}$ with respect to $\left\{\xi_{l}\right\}_{l=1}^{m}$ is defined by, with local 1-forms $\theta_{k}^{l}$,

$$
\begin{equation*}
\nabla^{E} \xi_{k}=\theta_{k}^{l} \xi_{l} \tag{1.1.7}
\end{equation*}
$$

Remark 1.1.3. If $E$ is a real vector bundle on $X$, certainly, everything still holds, especially, a connection $\nabla^{E}$ is said to be an Euclidean connection on $\left(E, h^{E}\right)$ if it preserves the Euclidean metric $h^{E}$.

### 1.1.2 Chern connection

Let $E$ be a holomorphic vector bundle over a complex manifold $X$. Let $h^{E}$ be a Hermitian metric on $E$. We call $\left(E, h^{E}\right)$ a holomorphic Hermitian vector bundle.

The almost complex structure $J$ induces a splitting $T X \otimes_{\mathbb{R}} \mathbb{C}=T^{(1,0)} X \oplus$ $T^{(0,1)} X$, where $T^{(1,0)} X$ and $T^{(0,1)} X$ are the eigenbundles of $J$ corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Let $T^{*(1,0)} X$ and $T^{*(0,1)} X$ be the corresponding dual bundles. Let

$$
\Omega^{r, q}(X, E):=\mathscr{C}^{\infty}\left(X, \Lambda^{r}\left(T^{*(1,0)} X\right) \otimes \Lambda^{q}\left(T^{*(0,1)} X\right) \otimes E\right)
$$

be the spaces of smooth $(r, q)$-forms on $X$ with values in $E$.
The operator $\bar{\partial}^{E}: \mathscr{C}^{\infty}(X, E) \rightarrow \Omega^{0,1}(X, E)$ is well defined. Any section $s \in \mathscr{C}^{\infty}(X, E)$ has the local form $s=\sum_{l} \varphi_{l} \xi_{l}$ where $\left\{\xi_{l}\right\}_{l=1}^{m}$ is a local holomorphic frame of $E$ and $\varphi_{l}$ are smooth functions. We set $\bar{\partial}^{E} s=\sum_{l}\left(\bar{\partial} \varphi_{l}\right) \xi_{l}$, here $\bar{\partial} \varphi_{l}=$ $\sum_{j} d \bar{z}_{j} \frac{\partial}{\partial \bar{z}_{j}} \varphi_{l}$ in holomorphic coordinates $\left(z_{1}, \cdots, z_{n}\right)$.
Definition 1.1.4. A connection $\nabla^{E}$ on $E$ is said to be a holomorphic connection if $\nabla_{U}^{E} s=i_{U}\left(\bar{\partial}^{E} s\right)$ for any $U \in T^{(0,1)} X$ and $s \in \mathscr{C}^{\infty}(X, E)$.
Theorem 1.1.5. There exists a unique holomorphic Hermitian connection $\nabla^{E}$ on $\left(E, h^{E}\right)$, called the Chern connection. With respect to a local holomorphic frame, the connection matrix is given by $\theta=h^{-1} \cdot \partial h$.
Proof. By Definition 1.1.4, we have to define $\nabla_{U}^{E}$ just for $U \in T^{(1,0)} X$. Relation (1.1.6) implies for $U \in T^{(1,0)} X, s_{1}, s_{2} \in \mathscr{C}^{\infty}(X, E)$,

$$
\begin{equation*}
U\left\langle s_{1}, s_{2}\right\rangle_{h^{E}}=\left\langle\nabla_{U}^{E} s_{1}, s_{2}\right\rangle_{h^{E}}+\left\langle s_{1}, \nabla_{\bar{U}}^{E} s_{2}\right\rangle_{h^{E}} \tag{1.1.8}
\end{equation*}
$$

Since $\nabla \frac{E}{U} s_{2}=i_{\bar{U}}\left(\bar{\partial}^{E} s_{2}\right)$, the above equation defines $\nabla_{U}^{E}$ uniquely. Moreover, if $\left\{\xi_{l}\right\}_{l=1}^{m}$ is a local holomorphic frame, from (1.1.6) we deduce that $\theta=h^{-1} \cdot \partial h$.

Since $E$ is holomorphic, similar to (1.1.4), the operator $\bar{\partial}^{E}$ extends naturally to $\bar{\partial}^{E}: \Omega^{\bullet \bullet}(X, E) \longrightarrow \Omega^{\bullet \bullet+1}(X, E)$ and $\left(\bar{\partial}^{E}\right)^{2}=0$.

Let $\nabla^{E}$ be the holomorphic Hermitian connection on $\left(E, h^{E}\right)$. Then we have a decomposition of $\nabla^{E}$ after bidegree

$$
\left.\begin{array}{l}
\nabla^{E}=\left(\nabla^{E}\right)^{1,0}+\left(\nabla^{E}\right)^{0,1}, \quad\left(\nabla^{E}\right)^{0,1}=\bar{\partial}^{E}  \tag{1.1.9}\\
\left(\nabla^{E}\right)^{1,0}: \Omega^{\bullet} \cdot \bullet \\
(X, E) \longrightarrow \Omega^{\bullet+1} \bullet \\
\bullet
\end{array} X, E\right) .
$$

By (1.1.8), (1.1.9) and $\left(\bar{\partial}^{E}\right)^{2}=0$ we have

$$
\begin{equation*}
\left(\bar{\partial}^{E}\right)^{2}=\left(\left(\nabla^{E}\right)^{1,0}\right)^{2}=0, \quad\left(\nabla^{E}\right)^{2}=\bar{\partial}^{E}\left(\nabla^{E}\right)^{1,0}+\left(\nabla^{E}\right)^{1,0} \bar{\partial}^{E} \tag{1.1.10}
\end{equation*}
$$

Thus the curvature $R^{E} \in \Omega^{1,1}(X, \operatorname{End}(E))$. If $\operatorname{rk}(E)=1, \operatorname{End}(E)$ is trivial and $R^{E}$ is canonically identified to a $(1,1)$-form on $X$, such that $\sqrt{-1} R^{E}$ is real.

In general, let us introduce an auxiliary Riemannian $g^{T X}$ metric on $X$, compatible with the complex structure $J$ (i.e., $\left.g^{T X}(\cdot, \cdot)=g^{T X}(J \cdot, J \cdot)\right)$. Then $R^{E}$ induces a Hermitian matrix $\dot{R}^{E} \in \operatorname{End}\left(T^{(1,0)} X \otimes E\right)$ such that for $u, v \in T_{x}^{(1,0)} X$, $\xi, \eta \in E_{x}$, and $x \in X$,

$$
\begin{equation*}
\left\langle R^{E}(u, \bar{v}) \xi, \eta\right\rangle_{h^{E}}=\left\langle\dot{R}^{E}(u \otimes \xi), v \otimes \eta\right\rangle . \tag{1.1.11}
\end{equation*}
$$

Definition 1.1.6. We say that $\left(E, h^{E}\right)$ is Nakano positive (resp. semi-positive) if $\dot{R}^{E} \in \operatorname{End}\left(T^{(1,0)} X \otimes E\right)$ is positive definite (resp. semi-definite), and Griffiths positive (resp. semi-positive) if $\left\langle R^{E}(v, \bar{v}) \xi, \xi\right\rangle_{h^{E}}=\left\langle\dot{R}^{E}(v \otimes \xi), v \otimes \xi\right\rangle>0$ (resp. $\geqslant 0$ ) for all non-zero $v \in T_{x}^{(1,0)} X$ and all non-zero $\xi \in E_{x}$. Certainly, these definitions do not depend on the choice of $g^{T X}$.

### 1.2 Connections on the tangent bundle

On the tangent bundle of a complex manifold, we can define several connections: the Levi-Civita connection, the holomorphic Hermitian (i.e., Chern) connection and Bismut connection. In this section, we explain the relation between them. We shall see that these three connections coincide, if $X$ is a Kähler manifold.

We start by recalling in Section 1.2.1 some facts about the Levi-Civita connection. In Section 1.2.2, we study in detail the holomorphic Hermitian connection on the tangent bundle. In Section 1.2.3, we define the Bismut connection.

Let $(X, J)$ be a complex manifold with complex structure $J$ and $\operatorname{dim}_{\mathbb{C}} X=n$. Let $T_{h} X$ be the holomorphic tangent bundle on $X$, and let $T X$ be the corresponding real tangent bundle. Let $g^{T X}$ be any Riemannian metric on $T X$ compatible with $J$, i.e., $g^{T X}(J u, J v)=g^{T X}(u, v)$ for any $u, v \in T_{x} X, x \in X$. We will shortly express this relation by $g^{T X}(J \cdot, J \cdot)=g^{T X}(\cdot, \cdot)$.

### 1.2.1 Levi-Civita connection

The results of this section apply for any Riemannian manifold ( $X, g^{T X}$ ). We denote by $\langle\cdot, \cdot\rangle$ the $\mathbb{C}$-bilinear form on $T X \otimes_{\mathbb{R}} \mathbb{C}$ induced by the metric $g^{T X}$. Let $\nabla^{T X}$ be the Levi-Civita connection on $\left(T X, g^{T X}\right)$. By the explicit equation for $\left\langle\nabla^{T X} \cdot, \cdot\right\rangle$, for any $U, V, W, Y$ vector fields on $X$,

$$
\begin{align*}
2\left\langle\nabla_{U}^{T X} V, W\right\rangle=U\langle V, W\rangle+ & V\langle U, W\rangle-W\langle U, V\rangle \\
& -\langle U,[V, W]\rangle-\langle V,[U, W]\rangle+\langle W,[U, V]\rangle \tag{1.2.1}
\end{align*}
$$

$\nabla^{T X}$ is the unique connection on $T X$ which preserves the metric (satisfies (1.1.6)) and is torsion free, i.e.,

$$
\begin{equation*}
\nabla_{U}^{T X} V-\nabla_{V}^{T X} U=[U, V] \tag{1.2.2}
\end{equation*}
$$

The curvature $R^{T X} \in \Lambda^{2}\left(T^{*} X\right) \otimes \operatorname{End}(T X)$ of $\nabla^{T X}$ is defined by

$$
\begin{equation*}
R^{T X}(U, V)=\nabla_{U}^{T X} \nabla_{V}^{T X}-\nabla_{V}^{T X} \nabla_{U}^{T X}-\nabla_{[U, V]}^{T X} \tag{1.2.3}
\end{equation*}
$$

Then we have the following well-known facts

$$
\begin{align*}
& R^{T X}(U, V) W+R^{T X}(V, W) U+R^{T X}(W, U) V=0 \\
& \left\langle R^{T X}(U, V) W, Y\right\rangle=\left\langle R^{T X}(W, Y) U, V\right\rangle \tag{1.2.4}
\end{align*}
$$

Let $\left\{e_{i}\right\}_{i=1}^{2 n}$ be an orthonormal frame of $T X$ and $\left\{e^{i}\right\}_{i=1}^{2 n}$ its dual basis in $T^{*} X$. The Ricci curvature Ric and scalar curvature $r^{X}$ of $\left(T X, g^{T X}\right)$ are defined by

$$
\begin{equation*}
\operatorname{Ric}=-\sum_{j}\left\langle R^{T X}\left(\cdot, e_{j}\right) \cdot, e_{j}\right\rangle, \quad r^{X}=-\sum_{i j}\left\langle R^{T X}\left(e_{i}, e_{j}\right) e_{i}, e_{j}\right\rangle \tag{1.2.5}
\end{equation*}
$$

The Riemannian volume form $d v_{X}$ of $\left(T X, g^{T X}\right)$ has the form $d v_{X}=e^{1} \wedge$ $\cdots \wedge e^{2 n}$ if the orthonormal frame $\left\{e_{i}\right\}$ is oriented.

If $\alpha$ is a 1 -form on $X$, the function $\operatorname{Tr}(\nabla \alpha)$ is given by the formula

$$
\begin{equation*}
\operatorname{Tr}(\nabla \alpha)=\sum_{i} e_{i}\left(\alpha\left(e_{i}\right)\right)-\alpha\left(\nabla_{e_{i}}^{T X} e_{i}\right) \tag{1.2.6}
\end{equation*}
$$

The following formula is quite useful.
Proposition 1.2.1. For any $\mathscr{C}^{1} 1$-form $\alpha$ with compact support, we have

$$
\begin{equation*}
\int_{X} \operatorname{Tr}(\nabla \alpha) d v_{X}=0 \tag{1.2.7}
\end{equation*}
$$

Proof. Let $W$ be the vector field on $X$ corresponding to $\alpha$ under the Riemannian metric $g^{T X}$, so that $\langle W, Y\rangle=(\alpha, Y)$ for any $Y \in T X$.

We denote by $L_{W}$ the Lie derivative of the vector field $W$. Recall that for any vector field $Y$ on $X$,

$$
\begin{equation*}
L_{W} Y=[W, Y]=\nabla_{W}^{T X} Y-\nabla_{Y}^{T X} W \tag{1.2.8}
\end{equation*}
$$

Thus by (1.2.8) and $\left\langle\nabla_{W}^{T X} e_{j}, e_{j}\right\rangle=0$, we get

$$
\begin{align*}
& L_{W} d v_{X}=\left\langle L_{W} e^{j}, e_{j}\right\rangle d v_{X}=-\left\langle e_{j}, L_{W} e_{j}\right\rangle d v_{X} \\
&=\left\langle\nabla_{e_{j}}^{T X} W, e_{j}\right\rangle d v_{X}=\left(e_{j}\left\langle W, e_{j}\right\rangle-\left\langle W, \nabla_{e_{j}}^{T X} e_{j}\right\rangle\right) d v_{X} \\
&=\operatorname{Tr}(\nabla \alpha) d v_{X} \tag{1.2.9}
\end{align*}
$$

We will denote by $\wedge$ and $i$ the exterior and interior product respectively. E. Cartan's homotopy formula tells us that on the bundle of exterior differentials $\Lambda\left(T^{*} X\right)$,

$$
\begin{equation*}
L_{W}=d \cdot i_{W}+i_{W} \cdot d \tag{1.2.10}
\end{equation*}
$$

From (1.2.9) and (1.2.10), we get

$$
\begin{equation*}
0=\int_{X} L_{W} d v_{X}=\int_{X} \operatorname{Tr}(\nabla \alpha) d v_{X} \tag{1.2.11}
\end{equation*}
$$

The proof of Proposition 1.2.1 is complete.
For $x_{0} \in X, W \in T_{x_{0}} X$, let $\mathbb{R} \ni u \rightarrow x_{u}=\exp _{x_{0}}^{X}(u W)$ be the geodesic in $X$ such that $\left.x_{u}\right|_{u=0}=x_{0},\left.\frac{d x_{u}}{d u}\right|_{u=0}=W$. For $\varepsilon>0$, we denote by $B^{X}\left(x_{0}, \varepsilon\right)$ and $B^{T_{x_{0}} X}(0, \varepsilon)$ the open balls in $X$ and $T_{x_{0}} X$ with center $x_{0}$ and radius $\varepsilon$, respectively. Then the map $T_{x_{0}} X \ni Z \rightarrow \exp _{x_{0}}^{X}(Z) \in X$ is a diffeomorphism from $B^{T_{x_{0}} X}(0, \varepsilon)$ onto $B^{X}\left(x_{0}, \varepsilon\right)$ for $\varepsilon$ small enough; by identifying $Z=\sum Z_{i} e_{i} \in T_{x_{0}} X$ with $\left(Z_{1}, \ldots, Z_{2 n}\right) \in \mathbb{R}^{2 n}$, it yields a local chart for $X$ around $x_{0}$, called normal coordinate system at $x_{0}$. We will identify $B^{T_{x_{0}} X}(0, \varepsilon)$ with $B^{X}\left(x_{0}, \varepsilon\right)$ by this map.

Let $\left\{e_{i}\right\}_{i}$ be an oriented orthonormal basis of $T_{x_{0}} X$. We also denote by $\left\{e^{i}\right\}_{i}$ the dual basis of $\left\{e_{i}\right\}$. Let $\widetilde{e}_{i}(Z)$ be the parallel transport of $e_{i}$ with respect to $\nabla^{T X}$ along the curve $[0,1] \ni u \rightarrow u Z$. Then $e_{j}=\frac{\partial}{\partial Z_{j}}$.

The radial vector field $\mathcal{R}$ is the vector field defined by $\mathcal{R}=\sum_{i} Z_{i} e_{i}$ with $\left(Z_{1}, \ldots, Z_{2 n}\right)$ the coordinate functions.

Proposition 1.2.2. The following identities hold:

$$
\begin{align*}
& \mathcal{R}=\sum_{j} Z_{j} e_{j}=\sum_{j} Z_{j} \tilde{e}_{j}(Z)  \tag{1.2.12}\\
& \left\langle\mathcal{R}, e_{j}\right\rangle=Z_{j}
\end{align*}
$$

Proof. Note that $x_{u}:[0,1] \ni u \rightarrow u Z$ is a geodesic, and $\mathcal{R}\left(x_{u}\right)=u \frac{d x_{u}}{d u}$, thus by the geodesic equation $\nabla_{\frac{d x u}{d u}}^{T X} \frac{d x_{u}}{d u}=0$, we get

$$
\begin{equation*}
\nabla_{\mathcal{R}}^{T X} \mathcal{R}=u \nabla_{\frac{d x u}{d u}}^{T X}\left(u \frac{d x_{u}}{d u}\right)=u \frac{d x_{u}}{d u}=\mathcal{R} \tag{1.2.13}
\end{equation*}
$$

Thus we have

$$
\begin{equation*}
\mathcal{R}\left\langle\mathcal{R}, \widetilde{e}_{j}\right\rangle=\left\langle\nabla_{\mathcal{R}}^{T X} \mathcal{R}, \widetilde{e}_{j}\right\rangle+\left\langle\mathcal{R}, \nabla_{\mathcal{R}}^{T X} \widetilde{e}_{j}\right\rangle=\left\langle\mathcal{R}, \widetilde{e}_{j}\right\rangle \tag{1.2.14}
\end{equation*}
$$

This means that $\left\langle\mathcal{R}, \widetilde{e}_{j}\right\rangle$ is homogeneous of order 1. But

$$
\begin{equation*}
\left\langle\mathcal{R}, \widetilde{e}_{j}\right\rangle=\sum_{k} Z_{k}\left\langle e_{k}, \widetilde{e}_{j}\right\rangle=Z_{j}+\mathscr{O}\left(|Z|^{2}\right) \tag{1.2.15}
\end{equation*}
$$

Thus from (1.2.14) and (1.2.15), we infer the first equation of (1.2.12).
Since the Levi-Civita connection $\nabla^{T X}$ is torsion free and $\left[\mathcal{R}, e_{i}\right]=-e_{i}$, we have

$$
\begin{equation*}
\left\langle\mathcal{R}, \nabla_{\mathcal{R}}^{T X} e_{i}\right\rangle=\left\langle\mathcal{R}, \nabla_{e_{i}}^{T X} \mathcal{R}\right\rangle+\left\langle\mathcal{R},\left[\mathcal{R}, e_{i}\right]\right\rangle=\frac{1}{2} e_{i}\langle\mathcal{R}, \mathcal{R}\rangle-\left\langle\mathcal{R}, e_{i}\right\rangle \tag{1.2.16}
\end{equation*}
$$

From (1.2.13) and (1.2.16), we obtain

$$
\begin{equation*}
\mathcal{R}\left\langle\mathcal{R}, e_{i}\right\rangle=\left\langle\nabla_{\mathcal{R}}^{T X} \mathcal{R}, e_{i}\right\rangle+\left\langle\mathcal{R}, \nabla_{\mathcal{R}}^{T X} e_{i}\right\rangle=\frac{1}{2} e_{i}\langle\mathcal{R}, \mathcal{R}\rangle=Z_{i} . \tag{1.2.17}
\end{equation*}
$$

But $\left\langle\mathcal{R}, e_{i}\right\rangle=\sum_{j} Z_{j}\left\langle e_{j}, e_{i}\right\rangle=Z_{i}+\mathscr{O}\left(|Z|^{2}\right)$. Thus we get the second equation of (1.2.12).

For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{2 n}\right) \in \mathbb{N}^{2 n}$, set $Z^{\alpha}=Z_{1}^{\alpha_{1}} \ldots Z_{2 n}^{\alpha_{2 n}}$.
Lemma 1.2.3. If $\widetilde{e}_{i}(Z)$ is written in the basis $\left\{e_{i}\right\}$, its Taylor expansion up to order $r$ is determined by the Taylor expansion up to order $r-2$ of $R_{m q k l}=$ $\left\langle R^{T X}\left(e_{q}, e_{m}\right) e_{k}, e_{l}\right\rangle_{Z}$. Moreover we have

$$
\begin{equation*}
\widetilde{e}_{i}(Z)=e_{i}-\frac{1}{6} \sum_{j}\left\langle R_{x_{0}}^{T X}\left(\mathcal{R}, e_{i}\right) \mathcal{R}, e_{j}\right\rangle_{x_{0}} e_{j}+\sum_{|\alpha| \geqslant 3}\left(\frac{\partial^{\alpha}}{\partial Z^{\alpha}} \widetilde{e}_{i}\right)(0) \frac{Z^{\alpha}}{\alpha!} \tag{1.2.18}
\end{equation*}
$$

Thus the Taylor expansion up to order $r$ of $g_{i j}(Z)=g^{T X}\left(e_{i}, e_{j}\right)(Z)=\left\langle e_{i}, e_{j}\right\rangle_{Z}$ is a polynomial of the Taylor expansion up to order $r-2$ of $R_{m q k l}$; moreover

$$
\begin{equation*}
g_{i j}(Z)=\delta_{i j}+\frac{1}{3}\left\langle R_{x_{0}}^{T X}\left(\mathcal{R}, e_{i}\right) \mathcal{R}, e_{j}\right\rangle_{x_{0}}+\mathscr{O}\left(|Z|^{3}\right) \tag{1.2.19}
\end{equation*}
$$

Proof. Let $\Gamma^{T X}$ be the connection form of $\nabla^{T X}$ with respect to the frame $\left\{\widetilde{e}_{i}\right\}$ of $T X$. Then $\nabla^{T X}=d+\Gamma^{T X}$. Let $\partial_{i}=\nabla_{e_{i}}$ be the partial derivatives along $e_{i}$. By the definition of our fixed frame, we have $i_{\mathcal{R}} \Gamma^{T X}=0$. Thus

$$
\begin{equation*}
L_{\mathcal{R}} \Gamma^{T X}=\left[i_{\mathcal{R}}, d\right] \Gamma^{T X}=i_{\mathcal{R}}\left(d \Gamma^{T X}+\Gamma^{T X} \wedge \Gamma^{T X}\right)=i_{\mathcal{R}} R^{T X} \tag{1.2.20}
\end{equation*}
$$

Let $\widetilde{\theta}(Z)=\left(\theta_{j}^{i}(Z)\right)_{i, j=1}^{2 n}$ be the $2 n \times 2 n$-matrix such that

$$
\begin{equation*}
e_{i}=\sum_{j} \theta_{i}^{j}(Z) \widetilde{e}_{j}(Z), \quad \widetilde{e}_{j}(Z)=\left(\widetilde{\theta}(Z)^{-1}\right)_{j}^{k} e_{k} \tag{1.2.21}
\end{equation*}
$$

$\operatorname{Set} \theta^{j}(Z)=\sum_{i} \theta_{i}^{j}(Z) e^{i}$ and

$$
\begin{equation*}
\theta=\sum_{j} e^{j} \otimes e_{j}=\sum_{j} \theta^{j} \widetilde{e}_{j} \in T^{*} X \otimes T X \tag{1.2.22}
\end{equation*}
$$

As $\nabla^{T X}$ is torsion free, $\nabla^{T X} \theta=0$, thus the $\mathbb{R}^{2 n}$-valued 1-form $\theta=\left(\theta^{j}(Z)\right)$ satisfies the structure equation,

$$
\begin{equation*}
d \theta+\Gamma^{T X} \wedge \theta=0 \tag{1.2.23}
\end{equation*}
$$

Observe first that under our trivialization by $\left\{\widetilde{e}_{i}\right\}$, by (1.2.12), for the $\mathbb{R}^{2 n}$-valued function $i_{\mathcal{R}} \theta$,

$$
\begin{equation*}
i_{\mathcal{R}} \theta=\sum_{j} Z_{j} e_{j}=\left(Z_{1}, \ldots, Z_{2 n}\right)=: Z \tag{1.2.24}
\end{equation*}
$$

Substituting (1.2.12), (1.2.24) and $\left(L_{\mathcal{R}}-1\right) Z=0$, into the identity $i_{\mathcal{R}}(d \theta+$ $\left.\Gamma^{T X} \wedge \theta\right)=0$, from (1.2.20), we obtain

$$
\begin{equation*}
\left(L_{\mathcal{R}}-1\right) L_{\mathcal{R}} \theta=\left(L_{\mathcal{R}}-1\right)\left(d Z+\Gamma^{T X} Z\right)=\left(L_{\mathcal{R}} \Gamma^{T X}\right) Z=\left(i_{\mathcal{R}} R^{T X}\right) Z \tag{1.2.25}
\end{equation*}
$$

Where we consider the curvature $R^{T X}$ as a matrix of two-forms and $\theta$ is a $\mathbb{R}^{2 n}$ valued one-form. The $i$ th component of $R^{T X} Z, \theta$ is $\left\langle R^{T X} \mathcal{R}, \widetilde{e}_{i}\right\rangle, \theta^{i}$, from (1.2.25), we get

$$
\begin{equation*}
i_{e_{j}}\left(L_{\mathcal{R}}-1\right) L_{\mathcal{R}} \theta^{i}(Z)=\left\langle R^{T X}\left(\mathcal{R}, e_{j}\right) \mathcal{R}, \widetilde{e}_{i}\right\rangle(Z) \tag{1.2.26}
\end{equation*}
$$

By (1.2.12), $L_{\mathcal{R}} e^{j}=e^{j}$. Thus from the Taylor expansion of $\theta_{j}^{i}(Z)$, we get

$$
\begin{equation*}
\sum_{|\alpha| \geqslant 1}\left(|\alpha|^{2}+|\alpha|\right)\left(\partial^{\alpha} \theta_{j}^{i}\right)(0) \frac{Z^{\alpha}}{\alpha!}=\left\langle R^{T X}\left(\mathcal{R}, e_{j}\right) \mathcal{R}, \widetilde{e}_{i}\right\rangle(Z) \tag{1.2.27}
\end{equation*}
$$

Now by (1.2.21) and $\theta_{j}^{i}\left(x_{0}\right)=\delta_{i j},(1.2 .27)$ determines the Taylor expansion of $\theta_{j}^{i}(Z)$ up to order $m$ in terms of the Taylor expansion of the coefficients of $R^{T X}$ up to order $m-2$. And

$$
\begin{equation*}
\left(\widetilde{\theta}^{-1}\right)_{j}^{i}=\delta_{i j}-\frac{1}{6}\left\langle R_{x_{0}}^{T X}\left(\mathcal{R}, e_{i}\right) \mathcal{R}, e_{j}\right\rangle_{x_{0}}+\mathscr{O}\left(|Z|^{3}\right) \tag{1.2.28}
\end{equation*}
$$

By (1.2.21), (1.2.27), we infer (1.2.18).
From (1.2.21),

$$
\begin{equation*}
g_{i j}(Z)=\theta_{i}^{k}(Z) \theta_{j}^{k}(Z) \tag{1.2.29}
\end{equation*}
$$

Thus the rest of Lemma 1.2.3 follows from (1.2.28) and (1.2.29). The proof of Lemma 1.2.3 is complete.

Let $E$ be a complex vector bundle on $X$, and let $\nabla^{E}$ be a connection on $E$ with curvature $R^{E}:=\left(\nabla^{E}\right)^{2}$. Let $\left(\mathcal{U}, Z_{1}, \ldots, Z_{2 n}\right)$ be a local chart of $X$ such that $0 \in \mathcal{U}$ represents $x_{0} \in X$. Set $\mathcal{R}=\sum_{i} Z_{i} \frac{\partial}{\partial Z_{i}}$. Now we identify $E_{Z}$ to $E_{x_{0}}$ by parallel transport with respect to the connection $\nabla^{E}$ along the curve $[0,1] \ni u \rightarrow u Z$; this gives a trivialization of $E$ near 0 . We denote by $\Gamma^{E}$ the connection form with respect to this trivialization of $E$ near 0 . Then in the frame $e_{j}=\frac{\partial}{\partial Z_{j}}, \Gamma^{E}$ becomes a function with values in $\mathbb{R}^{2 n} \otimes \operatorname{End}\left(\mathbb{C}^{\mathrm{rk}(E)}\right)$ and $\nabla^{E}=d+\Gamma^{E}$.

Lemma 1.2.4. The Taylor coefficients of $\Gamma^{E}\left(e_{j}\right)(Z)$ at $x_{0}$ up to order $r$ are determined by Taylor coefficients of $R^{E}$ up to order $r-1$. More precisely,

$$
\begin{equation*}
\sum_{|\alpha|=r}\left(\partial^{\alpha} \Gamma^{E}\right)_{x_{0}}\left(e_{j}\right) \frac{Z^{\alpha}}{\alpha!}=\frac{1}{r+1} \sum_{|\alpha|=r-1}\left(\partial^{\alpha} R^{E}\right)_{x_{0}}\left(\mathcal{R}, e_{j}\right) \frac{Z^{\alpha}}{\alpha!} \tag{1.2.30}
\end{equation*}
$$

Especially,

$$
\begin{equation*}
\Gamma_{Z}^{E}\left(e_{j}\right)=\frac{1}{2} R_{x_{0}}^{E}\left(\mathcal{R}, e_{j}\right)+\mathscr{O}\left(|Z|^{2}\right) \tag{1.2.31}
\end{equation*}
$$

Proof. By the definition of our fixed frame, we have $R^{E}=d \Gamma^{E}+\Gamma^{E} \wedge \Gamma^{E}$ and

$$
\begin{equation*}
i_{\mathcal{R}} \Gamma^{E}=0, \quad L_{\mathcal{R}} \Gamma^{E}=\left[i_{\mathcal{R}}, d\right] \Gamma^{E}=i_{\mathcal{R}}\left(d \Gamma^{E}+\Gamma^{E} \wedge \Gamma^{E}\right)=i_{\mathcal{R}} R^{E} \tag{1.2.32}
\end{equation*}
$$

Using $L_{\mathcal{R}} d Z^{j}=d Z^{j}$ and expanding both sides of the second equation of (1.2.32) in Taylor's series of at $Z=0$, we obtain

$$
\begin{equation*}
\sum_{\alpha}(|\alpha|+1)\left(\partial^{\alpha} \Gamma^{E}\right)_{x_{0}}\left(e_{j}\right) \frac{Z^{\alpha}}{\alpha!}=\sum_{\alpha}\left(\partial^{\alpha} R^{E}\right)_{x_{0}}\left(\mathcal{R}, e_{j}\right) \frac{Z^{\alpha}}{\alpha!} \tag{1.2.33}
\end{equation*}
$$

By equating coefficients of $Z^{\alpha}$ of both sides, we get Lemma 1.2.4.

### 1.2.2 Chern connection

Recall that $T^{(1,0)} X$ is a holomorphic vector bundle with Hermitian metric $h^{T^{(1,0)} X}$ induced by $g^{T X}$. The map $T_{h} X \ni Y \rightarrow \frac{1}{2}(Y-\sqrt{-1} J Y) \in T^{(1,0)} X$ induces the natural identification of $T_{h} X$ and $T^{(1,0)} X$.

We will denote by $\langle\cdot, \cdot\rangle$ the $\mathbb{C}$-bilinear form on $T X \otimes_{\mathbb{R}} \mathbb{C}$ induced by $g^{T X}$. Note that $\langle\cdot, \cdot\rangle$ vanishes on $T^{(1,0)} X \times T^{(1,0)} X$ and on $T^{(0,1)} X \times T^{(0,1)} X$.

For $U \in T X \otimes_{\mathbb{R}} \mathbb{C}$, we will denote by $U^{(1,0)}, U^{(0,1)}$ its components in $T^{(1,0)} X$ and $T^{(0,1)} X$. Let $\left\{w_{j}\right\}_{j=1}^{n}$ be a local orthonormal frame of $T^{(1,0)} X$ with dual frame $\left\{w^{j}\right\}_{j=1}^{n}$. Then

$$
\begin{equation*}
e_{2 j-1}=\frac{1}{\sqrt{2}}\left(w_{j}+\bar{w}_{j}\right) \quad \text { and } \quad e_{2 j}=\frac{\sqrt{-1}}{\sqrt{2}}\left(w_{j}-\bar{w}_{j}\right), \quad j=1, \ldots, n \tag{1.2.34}
\end{equation*}
$$

form an orthonormal frame of $T X$. We fix this notation throughout the book and use it without further notice.

Let $\nabla^{T^{(1,0)} X}$ be the holomorphic Hermitian connection on $\left(T^{(1,0)} X, h^{T^{(1,0)} X}\right)$ with curvature $R^{T^{(1,0)} X}$. For $v \in \mathscr{C}^{\infty}\left(X, T^{(0,1)} X\right)$, we define

$$
\nabla^{T^{(0,1)} X} v:=\overline{\nabla^{T^{(1,0)} X} \bar{v}}
$$

Then $\nabla^{T^{(0,1)} X}$ defines a connection on $T^{(0,1)} X$. Set

$$
\begin{equation*}
\widetilde{\nabla}^{T X}=\nabla^{T^{(1,0)} X} \oplus \nabla^{T^{(0,1)} X} \tag{1.2.35}
\end{equation*}
$$

Then $\widetilde{\nabla}^{T X}$ is a connection on $T X \otimes_{\mathbb{R}} \mathbb{C}$ and it preserves $T X$; we still denote by $\widetilde{\nabla}^{T X}$ the induced connection on $T X$. Then $\widetilde{\nabla}^{T X}$ preserves the metric $g^{T X}$.

Let $T$ be the torsion of the connection $\widetilde{\nabla}^{T X}$. Then $T \in \Lambda^{2}\left(T^{*} X\right) \otimes T X$ is defined by

$$
\begin{equation*}
T(U, V)=\widetilde{\nabla}_{U}^{T X} V-\widetilde{\nabla}_{V}^{T X} U-[U, V] \tag{1.2.36}
\end{equation*}
$$

for vector fields $U$ and $V$ on $X$. Hence

$$
T \operatorname{maps} T^{(1,0)} X \otimes T^{(1,0)} X\left(\text { resp. } T^{(0,1)} X \otimes T^{(0,1)} X\right) \text { into } T^{(1,0)} X
$$

$$
\begin{equation*}
\text { (resp. } \left.T^{(0,1)} X\right) \text { and vanishes on } T^{(1,0)} X \otimes T^{(0,1)} X \tag{1.2.37}
\end{equation*}
$$

Set

$$
\begin{equation*}
S=\widetilde{\nabla}^{T X}-\nabla^{T X}, \quad \mathcal{S}=\sum_{i} S\left(e_{i}\right) e_{i} \tag{1.2.38}
\end{equation*}
$$

Then $S$ is a real 1-form on $X$ taking values in the skew-adjoint endomorphisms of $T X$. Since $\nabla^{T X}$ is torsion free, we have for $U, V \in T X$,

$$
\begin{equation*}
T(U, V)=S(U) V-S(V) U \tag{1.2.39}
\end{equation*}
$$

Moreover, from (1.2.1), (1.2.36), (1.2.38) and since $\widetilde{\nabla}^{T X}$ preserves $g^{T X}$ we obtain directly

$$
\begin{equation*}
2\langle S(U) V, W\rangle-\langle T(U, V), W\rangle-\langle T(W, U), V\rangle+\langle T(V, W), U\rangle=0 \tag{1.2.40}
\end{equation*}
$$

By (1.2.37), (1.2.39) and (1.2.40), we get

$$
\begin{align*}
& \left\langle S\left(w_{i}\right) w_{k}, w_{j}\right\rangle=0  \tag{1.2.41}\\
& 2\left\langle S\left(w_{i} \bar{w}_{k}, w_{j}\right\rangle=2\left\langle S\left(\bar{w}_{k}\right) w_{i}, w_{j}\right\rangle=-\left\langle T\left(w_{i}, w_{j}\right), \bar{w}_{k}\right\rangle .\right.
\end{align*}
$$

Since $T\left(w_{i}, \bar{w}_{j}\right)=0, S\left(\bar{w}_{j}\right) w_{i}=S\left(w_{i}\right) \bar{w}_{j}$, and so

$$
\begin{align*}
\mathcal{S}=2 S\left(w_{j}\right) \bar{w}_{j} & =\left\langle T\left(w_{i}, w_{j}\right), \bar{w}_{j}\right\rangle \bar{w}_{i}+\left\langle T\left(\bar{w}_{i}, \bar{w}_{j}\right), w_{j}\right\rangle w_{i} \\
& =\left\langle T\left(e_{i}, e_{j}\right), e_{j}\right\rangle e_{i},  \tag{1.2.42}\\
2\left\langle S(\cdot) w_{j}, \bar{w}_{j}\right\rangle & =\left\langle T\left(w_{i}, w_{j}\right), \bar{w}_{j}\right\rangle w^{i}-\left\langle T\left(\bar{w}_{i}, \bar{w}_{j}\right), w_{j}\right\rangle \bar{w}^{i} .
\end{align*}
$$

The connection $\widetilde{\nabla}^{T X}$ on $T X$ induces naturally a covariant derivative on the exterior bundle $\Lambda\left(T^{*} X\right)$ and we still denote it by $\widetilde{\nabla}^{T X}$. For any differential forms $\alpha, \beta$ and vector field $Y$, it satisfies

$$
\begin{equation*}
\widetilde{\nabla}_{Y}^{T X}(\alpha \wedge \beta)=\left(\widetilde{\nabla}_{Y}^{T X} \alpha\right) \wedge \beta+\alpha \wedge \widetilde{\nabla}_{Y}^{T X} \beta \tag{1.2.43}
\end{equation*}
$$

For a 1-form $\alpha$ and vector fields $U, V$, we have $\left(\widetilde{\nabla}_{U}^{T X} \alpha, V\right)=U(\alpha, V)-\left(\alpha, \widetilde{\nabla}_{U}^{T X} V\right)$. Likewise, $\nabla^{T X}$ induces naturally a connection $\nabla^{T X}$ on $\Lambda\left(T^{*} X\right)$. We denote by $\varepsilon$ the exterior product $T^{*} X \otimes \Lambda^{\bullet}\left(T^{*} X\right) \rightarrow \Lambda^{\bullet+1}\left(T^{*} X\right)$.
Lemma 1.2.5. For the exterior differentiation operator $d$ acting on smooth sections of $\Lambda\left(T^{*} X\right)$, we have

$$
\begin{equation*}
d=\varepsilon \circ \widetilde{\nabla}^{T X}+i_{T}, \quad d=\varepsilon \circ \nabla^{T X} \tag{1.2.44}
\end{equation*}
$$

Proof. We write d:= $:=\widetilde{\nabla}^{T X}+i_{T}$. Then by using (1.2.43), we know that for any homogeneous differential forms $\alpha, \beta$, we have

$$
\begin{equation*}
\mathbf{d}(\alpha \wedge \beta)=(\mathbf{d} \alpha) \wedge \beta+(-1)^{\operatorname{deg} \alpha} \alpha \wedge \mathbf{d} \beta \tag{1.2.45}
\end{equation*}
$$

From Leibniz's rule (1.2.45), it suffices to show that $\mathbf{d}$ agrees with $d$ on functions (which is clear) and 1-forms. Now, for any smooth function $f$ on $X$, we have

$$
\begin{align*}
& \varepsilon \circ \widetilde{\nabla}^{T X} d f=e^{i} \wedge e^{j}\left\langle\widetilde{\nabla}_{e_{i}}^{T X} d f, e_{j}\right\rangle=e^{i} \wedge e^{j}\left(e_{i}\left(e_{j}(f)\right)-\left\langle d f, \widetilde{\nabla}_{e_{i}}^{T X} e_{j}\right\rangle\right) \\
& \quad=\frac{1}{2} e^{i} \wedge e^{j}\left(e_{i}\left(e_{j}(f)\right)-\left\langle d f, \widetilde{\nabla}_{e_{i}}^{T X} e_{j}\right\rangle-\left(e_{j}\left(e_{i}(f)\right)-\left\langle d f, \widetilde{\nabla}_{e_{j}}^{T X} e_{i}\right\rangle\right)\right)  \tag{1.2.46}\\
& \quad=-\frac{1}{2} e^{i} \wedge e^{j}\left\langle d f, T\left(e_{i}, e_{j}\right)\right\rangle=-i_{T} d f .
\end{align*}
$$

Thus $\mathbf{d}$ coincides also $d$ on 1 -forms. Thus we get the first equation of (1.2.44). As $\nabla^{T X}$ is torsion free, from the above argument, we obtain the second equation of (1.2.44).

If $B \in \Lambda^{2}\left(T^{*} X\right) \otimes T X$ we will denote by $B_{a s}$ the anti-symmetrization of the tensor $V, W, Y \rightarrow\langle B(V, W), Y\rangle$. Then

$$
\begin{equation*}
B_{a s}(V, W, Y)=\langle B(V, W), Y\rangle+\langle B(W, Y), V\rangle+\langle B(Y, V), W\rangle \tag{1.2.47}
\end{equation*}
$$

Especially from (1.2.37), we infer

$$
\begin{align*}
& T_{a s}=\frac{1}{2}\left\langle T\left(e_{i}, e_{j}\right), e_{k}\right\rangle e^{i} \wedge e^{j} \wedge e^{k} \\
& =\frac{1}{2}\left\langle T\left(w_{i}, w_{j}\right), \bar{w}_{k}\right\rangle w^{i} \wedge w^{j} \wedge \bar{w}^{k}+\frac{1}{2}\left\langle T\left(\bar{w}_{i}, \bar{w}_{j}\right), w_{k}\right\rangle \bar{w}^{i} \wedge \bar{w}^{j} \wedge w^{k}  \tag{1.2.48}\\
& =: T_{a s}^{(1,0)}+T_{a s}^{(0,1)}
\end{align*}
$$

Here $T_{a s}^{(1,0)}, T_{a s}^{(0,1)}$ are the anti-symmetrizations of the components $T^{(1,0)}, T^{(0,1)}$ of $T$ in $T^{(1,0)} X$ and $T^{(0,1)} X$.

Let $\Theta$ be the real $(1,1)$-form defined by

$$
\begin{equation*}
\Theta(X, Y)=g^{T X}(J X, Y) \tag{1.2.49}
\end{equation*}
$$

Note that the exterior differentiation operator $d$ acting on smooth sections of $\Lambda\left(T^{*} X\right)$ has the decomposition

$$
\begin{equation*}
d=\partial+\bar{\partial} \tag{1.2.50}
\end{equation*}
$$

Proposition 1.2.6. We have the identity of 3 -forms on $X$,

$$
\begin{equation*}
T_{a s}=-\sqrt{-1}(\partial-\bar{\partial}) \Theta \tag{1.2.51}
\end{equation*}
$$

Proof. By (1.2.34), we know that $\Theta=\sqrt{-1} \sum_{i} w^{i} \wedge \bar{w}^{i}$. Thus

$$
\begin{align*}
\widetilde{\nabla}^{T X} \Theta & =\sqrt{-1}\left(\left(\widetilde{\nabla}^{T X} w^{i}\right) \wedge \bar{w}^{i}+w^{i} \wedge \widetilde{\nabla}^{T X} \bar{w}^{i}\right) \\
& =\sqrt{-1}\left(-\left\langle\widetilde{\nabla}^{T X} w_{i}, \bar{w}_{j}\right\rangle-\left\langle w_{i}, \widetilde{\nabla}^{T X} \bar{w}_{j}\right\rangle\right) w^{i} \wedge \bar{w}^{j}=0 . \tag{1.2.52}
\end{align*}
$$

From (1.2.44), (1.2.48) and (1.2.52) we have

$$
\begin{equation*}
d \Theta=i_{T} \Theta=\sqrt{-1}\left(T_{a s}^{(1,0)}-T_{a s}^{(0,1)}\right) \tag{1.2.53}
\end{equation*}
$$

The relations (1.2.48) and (1.2.53) yield

$$
\begin{equation*}
\partial \Theta=\sqrt{-1} T_{a s}^{(1,0)}, \quad \bar{\partial} \Theta=-\sqrt{-1} T_{a s}^{(0,1)} \tag{1.2.54}
\end{equation*}
$$

(1.2.54) imply (1.2.51).

Definition 1.2.7. We call $\Theta$ as in (1.2.49) a Hermitian form on $X$ and $(X, J, \Theta)$ a complex Hermitian manifold. The metric $g^{T X}=\Theta(\cdot, J \cdot)$ on $T X$ is called a Kähler metric if $\Theta$ is a closed form, i.e., $d \Theta=0$. In this case, the form $\Theta$ is called a Kähler form on $X$, and the complex manifold $(X, J)$ is called a Kähler manifold .

Let $\nabla^{X} J \in T^{*} X \otimes \operatorname{End}(T X)$ be the covariant derivative of $J$ induced by the Levi-Civita connection $\nabla^{T X}$.
Theorem 1.2.8. $(X, J, \Theta)$ is Kähler if and only if the bundle $T^{(1,0)} X$ and $T^{(0,1)} X$ are preserved by the Levi-Civita connection $\nabla^{T X}$, or in other words, if and only if $\nabla^{X} J=0$. In this case,

$$
\begin{equation*}
\nabla^{T X}=\widetilde{\nabla}^{T X}, \quad S=0, \quad T=0 \tag{1.2.55}
\end{equation*}
$$

Proof. As $\Theta$ is a $(1,1)$-form, by $(1.2 .41),(1.2 .48)$ and $(1.2 .51), d \Theta=0$ is equivalent to $T_{a s}=0$ and equivalent to $S\left(\bar{w}_{k}\right) w_{i} \in T^{(1,0)} X$ for any $i, k$. But this means that the bundles $T^{(1,0)} X$ and $T^{(0,1)} X$ are preserved by $\nabla^{T X}$. Hence (1.2.55) is equivalent to $(X, \Theta)$ being Kähler. Moreover, as $J$ acts by multiplication with $\sqrt{-1}$ on $T^{(1,0)} X$, we get for $U \in T X$,

$$
\begin{align*}
\left\langle S(U) w_{i}, w_{j}\right\rangle & =-\left\langle\nabla_{U}^{T X} w_{i}, w_{j}\right\rangle=-\frac{1}{2}\left\langle\nabla_{U}^{T X}(1-\sqrt{-1} J) w_{i}, w_{j}\right\rangle \\
& =\frac{\sqrt{-1}}{2}\left\langle\left(\nabla_{U}^{X} J\right) w_{i}, w_{j}\right\rangle \tag{1.2.56}
\end{align*}
$$

by (1.2.38). Now, from $J^{2}=-1$ we deduce

$$
\begin{equation*}
J\left(\nabla^{X} J\right)+\left(\nabla^{X} J\right) J=0 \tag{1.2.57}
\end{equation*}
$$

This means that $\left(\nabla^{X} J\right)$ exchanges $T^{(1,0)} X$ and $T^{(0,1)} X$. By (1.2.44), and (1.2.56), $\nabla^{X} J=0$ is equivalent to $S\left(\bar{w}_{k}\right) w_{i} \in T^{(1,0)} X$ for any $i, k$. The proof of Theorem 1.2.8 is complete.

### 1.2.3 Bismut connection

Let $S^{B}$ denote the 1-form with values in the antisymmetric elements of $\operatorname{End}(T X)$ which satisfies for $U, V, W \in T X$,

$$
\begin{equation*}
\left\langle S^{B}(U) V, W\right\rangle=\frac{\sqrt{-1}}{2}((\partial-\bar{\partial}) \Theta)(U, V, W)=-\frac{1}{2} T_{a s}(U, V, W) \tag{1.2.58}
\end{equation*}
$$

By (1.2.40), (1.2.47), (1.2.58), we have for $U, V, W \in T X$,

$$
\begin{equation*}
\left\langle\left(S^{B}-S\right)(U) V, W\right\rangle=-\langle T(U, V), W\rangle+\langle T(U, W), V\rangle \tag{1.2.59}
\end{equation*}
$$

Relations (1.2.41), (1.2.48), and (1.2.58) yield

$$
\begin{align*}
\left\langle S^{B}\left(e_{j}\right) \omega_{l}, \bar{\omega}_{m}\right\rangle & =-\frac{1}{2}\left\langle T\left(e_{j}, \omega_{l}\right), \bar{\omega}_{m}\right\rangle+\frac{1}{2}\left\langle T\left(e_{j}, \bar{\omega}_{m}\right), \omega_{l}\right\rangle \\
& =-\left\langle S\left(e_{j}\right) \omega_{l}, \bar{\omega}_{m}\right\rangle  \tag{1.2.60}\\
\left\langle S^{B}\left(e_{j}\right) \omega_{l}, \omega_{m}\right\rangle & =-\frac{1}{2}\left\langle T\left(\omega_{l}, \omega_{m}\right), e_{j}\right\rangle=\left\langle S\left(e_{j}\right) \omega_{l}, \omega_{m}\right\rangle
\end{align*}
$$

Definition 1.2.9. The Bismut connection $\nabla^{B}$ on $T X$ is defined by

$$
\begin{equation*}
\nabla^{B}:=\nabla^{T X}+S^{B}=\widetilde{\nabla}^{T X}+S^{B}-S \tag{1.2.61}
\end{equation*}
$$

In view of (1.2.58), the torsion of $\nabla^{B}$ is $2 S^{B}$ which is a skew-symmetric tensor.

The connection $\nabla^{B}$ will be used in the Lichnerowicz formula (1.4.29).
Lemma 1.2.10. The connection $\nabla^{B}$ preserves the complex structure of $T X$.
Proof. Using (1.2.60), we find that for $V, W \in T^{(1,0)} X,\left\langle\left(S^{B}-S\right)(U) V, W\right\rangle=0$, for any $U \in T X$. Equivalently, $\left(S^{B}-S\right)(U)$ is a complex endomorphism of $T X$. Using (1.2.61), we find that $\nabla^{B}$ preserves the complex structure of $T X$.

## 1.3 $\operatorname{Spin}^{c}$ Dirac operator

This section is organized as follows. In Section 1.3.1, we define the Clifford connection. In Section 1.3.2, we define the spin ${ }^{c}$ Dirac operator on a complex manifold and prove the related Lichnerowicz formula. In Section 1.3.3, we obtain the Lichnerowicz formula for the modified Dirac operator. In Section 1.3.4, we explain also the Atiyah-Singer index theorem for the modified Dirac operator.

In this section, we work on a smooth manifold with an almost complex structure $J$.

### 1.3.1 Clifford connection

Let $(X, J)$ be a smooth manifold with $J$ an almost complex structure on $T X$. Let $g^{T X}$ be any Riemannian metric on $T X$ compatible with $J$. Let $h^{\Lambda^{0, \bullet}}$ be the Hermitian metric on $\Lambda\left(T^{*(0,1)} X\right)$ induced by $g^{T X}$.

The fundamental $\mathbb{Z}_{2}$ spinor bundle induced by $J$ is given by $\Lambda\left(T^{*(0,1)} X\right)$, whose $\mathbb{Z}_{2}$-grading is defined by $\Lambda\left(T^{*(0,1)} X\right)=\Lambda^{\text {even }}\left(T^{*(0,1)} X\right) \oplus \Lambda^{\text {odd }}\left(T^{*(0,1)} X\right)$. For any $v \in T X$ with decomposition $v=v^{(1,0)}+v^{(0,1)} \in T^{(1,0)} X \oplus T^{(0,1)} X$, let $\bar{v}^{(1,0), *} \in T^{*(0,1)} X$ be the metric dual of $v^{(1,0)}$. Then

$$
\begin{equation*}
c(v)=\sqrt{2}\left(\bar{v}^{(1,0), *} \wedge-i_{v^{(0,1)}}\right) \tag{1.3.1}
\end{equation*}
$$

defines the Clifford action of $v$ on $\Lambda\left(T^{*(0,1)} X\right)$, where $\wedge$ and $i$ denote the exterior and interior product, respectively. We verify easily that for $U, V \in T X$,

$$
\begin{equation*}
c(U) c(V)+c(V) c(U)=-2\langle U, V\rangle . \tag{1.3.2}
\end{equation*}
$$

For a skew-adjoint endomorphism $A$ of $T X$, from (1.3.1), using the notation of (1.2.34),

$$
\begin{align*}
& \frac{1}{4}\left\langle A e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right)=-\frac{1}{2}\left\langle A w_{j}, \bar{w}_{j}\right\rangle+\left\langle A w_{l}, \bar{w}_{m}\right\rangle \bar{w}^{m} \wedge i_{\bar{w}_{l}}  \tag{1.3.3}\\
& +\frac{1}{2}\left\langle A w_{l}, w_{m}\right\rangle i_{\bar{w}_{l}} i_{\bar{w}_{m}}+\frac{1}{2}\left\langle A \bar{w}_{l}, \bar{w}_{m}\right\rangle \bar{w}^{l} \wedge \bar{w}^{m} \wedge
\end{align*}
$$

Let $\nabla^{\text {det }}$ be a Hermitian connection on $\operatorname{det}\left(T^{(1,0)} X\right)$ endowed with metric induced by $g^{T X}$. Let $R^{\text {det }}$ be its curvature. Let $P^{T^{(1,0)} X}$ be the natural projection from $T X \otimes_{\mathbb{R}} \mathbb{C}$ onto $T^{(1,0)} X$. Then the connection $\nabla^{1,0}=P^{T^{(1,0)} X} \nabla^{T X} P^{T^{(1,0)} X}$ on $T^{(1,0)} X$ induces naturally a connection $\nabla^{\operatorname{det}_{1}}$ on $\operatorname{det}\left(T^{(1,0)} X\right)$.

Let $\Gamma^{T X} \in T^{*} X \otimes \operatorname{End}(T X), \Gamma^{\operatorname{det}}$ be the connection forms of $\nabla^{T X}, \nabla^{\operatorname{det}}$ associated to the frames $\left\{e_{j}\right\}, w_{1} \wedge \cdots \wedge w_{n}$, i.e.,

$$
\begin{align*}
& \nabla_{e_{i}}^{T X} e_{j}=\Gamma^{T X}\left(e_{i}\right) e_{j}, \quad \nabla^{\operatorname{det}}\left(w_{1} \wedge \cdots \wedge w_{n}\right)=\Gamma^{\operatorname{det}} w_{1} \wedge \cdots \wedge w_{n} \\
& \nabla^{\operatorname{det}_{1}}\left(w_{1} \wedge \cdots \wedge w_{n}\right)=\left(\sum_{j}\left\langle\Gamma^{T X} w_{j}, \bar{w}_{j}\right\rangle\right) w_{1} \wedge \cdots \wedge w_{n} \tag{1.3.4}
\end{align*}
$$

The Clifford connection $\nabla^{\mathrm{Cl}}$ on $\Lambda\left(T^{*(0,1)} X\right)$ is defined for the frame $\left\{\bar{w}^{j_{1}} \wedge\right.$ $\left.\cdots \wedge \bar{w}^{j_{k}}, 1 \leqslant j_{1}<\cdots<j_{k} \leqslant n\right\}$ by the local formula

$$
\begin{equation*}
\nabla^{\mathrm{Cl}}=d+\frac{1}{4}\left\langle\Gamma^{T X} e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right)+\frac{1}{2} \Gamma^{\mathrm{det}} . \tag{1.3.5}
\end{equation*}
$$

Proposition 1.3.1. $\nabla^{C l}$ defines a Hermitian connection on $\Lambda\left(T^{*(0,1)} X\right)$ and preserves its $\mathbb{Z}_{2}$-grading. For any $V$, $W$ vector fields of $T X$ on $X$, we have

$$
\begin{equation*}
\left[\nabla_{V}^{C l}, c(W)\right]=c\left(\nabla_{V}^{T X} W\right) \tag{1.3.6}
\end{equation*}
$$

Proof. At first, by (1.3.4) and (1.3.5), we have

$$
\begin{align*}
{\left[\nabla_{V}^{\mathrm{Cl}}, c\left(e_{k}\right)\right] } & =\frac{1}{4}\left[\left\langle\Gamma^{T X}(V) e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right), c\left(e_{k}\right)\right]  \tag{1.3.7}\\
& =\left\langle\Gamma^{T X}(V) e_{k}, e_{j}\right\rangle c\left(e_{j}\right)=c\left(\nabla_{V}^{T X} e_{k}\right)
\end{align*}
$$

Thus if $\nabla^{\mathrm{Cl}}$ is well defined, we get (1.3.6) from (1.3.7).
Now we observe that $c\left(w_{j_{1}}\right) \ldots c\left(w_{j_{k}}\right) 1,\left(1 \leqslant j_{1}<\cdots<j_{k} \leqslant n\right)$ generate a frame of $\Lambda\left(T^{*(0,1)} X\right)$. Taking into account (1.3.7), to verify that $\nabla^{\mathrm{Cl}}$ does not depend on the choice of our frame $\left\{w_{j}\right\}_{j=1}^{n}$, we only need to verify that $\nabla^{\mathrm{Cl}} 1$ is well defined.

Relations (1.2.38), (1.3.3), (1.3.4) and (1.3.5) entail

$$
\begin{align*}
\nabla^{\mathrm{Cl}}=d+ & \frac{1}{2}\left(\nabla^{\mathrm{det}}-\nabla^{\operatorname{det}_{1}}\right)+\left\langle\Gamma^{T X} w_{l}, \bar{w}_{m}\right\rangle \bar{w}^{m} \wedge i_{\bar{w}_{l}} \\
& -\frac{1}{2}\left\langle S w_{l}, w_{m}\right\rangle i_{\bar{w}_{l}} i_{\bar{w}_{m}}-\frac{1}{2}\left\langle S \bar{w}_{l}, \bar{w}_{m}\right\rangle \bar{w}^{l} \wedge \bar{w}^{m} \wedge \tag{1.3.8}
\end{align*}
$$

From (1.3.8), we know

$$
\begin{equation*}
\nabla^{\mathrm{Cl}} 1=\frac{1}{2}\left(\nabla^{\mathrm{det}}-\nabla^{\mathrm{det}_{1}}\right)-\frac{1}{2} \sum_{l m}\left\langle S \bar{w}_{l}, \bar{w}_{m}\right\rangle \bar{w}^{l} \wedge \bar{w}^{m} \tag{1.3.9}
\end{equation*}
$$

Clearly, $\nabla^{\text {det }}-\nabla^{\text {det }_{1}}$ is a 1 -form on $X$, and the right-hand side of (1.3.9) does not depend on the choice of the frame $w_{j}$. Thus $\nabla^{\mathrm{Cl}} 1$ is well defined.

Let $c\left(e_{i}\right)^{*}$ be the adjoint of $c\left(e_{i}\right)$ with respect to the Hermitian product on $\Lambda\left(T^{*(0,1)} X\right)$. By (1.3.1), we have

$$
\begin{equation*}
c\left(e_{i}\right)^{*}=-c\left(e_{i}\right) \tag{1.3.10}
\end{equation*}
$$

Using (1.3.5), (1.3.10) and the anti-symmetry of $\left\langle\Gamma^{T X} e_{i}, e_{j}\right\rangle$ in $i, j$, we see that $\nabla^{\mathrm{Cl}}$ preserves the Hermitian metric on $\Lambda\left(T^{*(0,1)} X\right)$.

Finally, from (1.3.5), $\nabla^{\mathrm{Cl}}$ preserves the $\mathbb{Z}_{2}$-grading on $\Lambda\left(T^{*(0,1)} X\right)$. The proof of Proposition 1.3.1 is complete.

Let $R^{\mathrm{Cl}}$ be the curvature of $\nabla^{\mathrm{Cl}}$.
Proposition 1.3.2. We have the following identity:

$$
\begin{equation*}
R^{C l}=\frac{1}{4}\left\langle R^{T X} e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right)+\frac{1}{2} R^{\mathrm{det}} \tag{1.3.11}
\end{equation*}
$$

Proof. At first, observe that if $i, j, k, l$ are different, then $\left[c\left(e_{i}\right) c\left(e_{j}\right), c\left(e_{k}\right) c\left(e_{l}\right)\right]=0$. Thus from (1.3.2),

$$
\begin{align*}
& {\left[\left\langle\Gamma^{T X}(W) e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right),\left\langle\Gamma^{T X}(V) e_{k}, e_{l}\right\rangle c\left(e_{k}\right) c\left(e_{l}\right)\right]} \\
& \quad=4 \sum_{i \neq j \neq k}\left\langle\Gamma^{T X}(W) e_{i}, e_{j}\right\rangle\left\langle\Gamma^{T X}(V) e_{k}, e_{j}\right\rangle\left[c\left(e_{i}\right) c\left(e_{j}\right), c\left(e_{k}\right) c\left(e_{j}\right)\right]  \tag{1.3.12}\\
& \quad=4\left\langle\Gamma^{T X}(W) e_{i}, \Gamma^{T X}(V) e_{k}\right\rangle\left(c\left(e_{i}\right) c\left(e_{k}\right)-c\left(e_{k}\right) c\left(e_{i}\right)\right) \\
& \quad=4\left\langle\left(\Gamma^{T X} \wedge \Gamma^{T X}\right)(W, V) e_{i}, e_{k}\right\rangle c\left(e_{i}\right) c\left(e_{k}\right)
\end{align*}
$$

Moreover, we have

$$
\begin{align*}
& R^{T X}=d \Gamma^{T X}+\Gamma^{T X} \wedge \Gamma^{T X}  \tag{1.3.13}\\
& R^{\mathrm{Cl}}\left(e_{l}, e_{m}\right)=\nabla_{e_{l}}^{\mathrm{Cl}} \nabla_{e_{m}}^{\mathrm{Cl}}-\nabla_{e_{m}}^{\mathrm{Cl}} \nabla_{e_{l}}^{\mathrm{Cl}}-\nabla_{\left[e_{l}, e_{m}\right]}^{\mathrm{Cl}}
\end{align*}
$$

Finally, (1.3.5), (1.3.12) and (1.3.13) yield (1.3.11).

### 1.3.2 Dirac operator and Lichnerowicz formula

Let $\left(E, h^{E}\right)$ be a Hermitian vector bundle on $X$. Let $\nabla^{E}$ be a Hermitian connection on $\left(E, h^{E}\right)$ with curvature $R^{E}$.

Set $\mathbf{E}^{q}=\Lambda^{q}\left(T^{*(0,1)} X\right) \otimes E, \mathbf{E}=\oplus_{q=0}^{n} E^{q}$. We still denote by $\nabla^{\mathrm{Cl}}$ the connection on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ induced by $\nabla^{\mathrm{Cl}}$ and $\nabla^{E}$. Let $\Omega^{0, q}(X, E):=\mathscr{C}^{\infty}\left(X, \mathbf{E}^{q}\right)$ be the set of smooth sections of $\mathbf{E}^{q}$ on $X$.

Along the fibers of $\Lambda\left(T^{*(0,1)} X\right) \otimes E$, we consider the pointwise Hermitian product $\langle\cdot, \cdot\rangle_{\Lambda^{0}, \bullet E}$ induced by $g^{T X}$ and $h^{E}$. The $L^{2}$-scalar product on $\Omega^{0, \bullet}(X, E)$ is given by

$$
\begin{equation*}
\left\langle s_{1}, s_{2}\right\rangle=\int_{X}\left\langle s_{1}(x), s_{2}(x)\right\rangle_{\Lambda^{0} \cdot \bullet E} d v_{X}(x) \tag{1.3.14}
\end{equation*}
$$

We denote the corresponding norm with $\|\cdot\|_{L^{2}}$, and by $L^{2}\left(X, \Lambda\left(T^{*(0,1)} X\right) \otimes E\right)$ or $L_{0, \bullet}^{2}(X, E)$, the $L^{2}$ completion of $\Omega_{0}^{0, \bullet}(X, E)$, which is the subspace of $\Omega^{0} \bullet \bullet(X, E)$ consisting of elements with compact support.

Definition 1.3.3. The spin ${ }^{c}$ Dirac operator $D^{c}$ is defined by

$$
\begin{equation*}
D^{c}=\sum_{j=1}^{2 n} c\left(e_{j}\right) \nabla_{e_{j}}^{\mathrm{Cl}}: \Omega^{0, \bullet}(X, E) \longrightarrow \Omega^{0, \bullet}(X, E) \tag{1.3.15}
\end{equation*}
$$

By Proposition 1.3.1 and equation (1.3.1), $D^{c}$ interchanges $\Omega^{0, \text { even }}(X, E)$ and $\Omega^{0, \text { odd }}(X, E)$. We write

$$
\begin{equation*}
D_{+}^{c}=\left.D^{c}\right|_{\Omega^{0, \mathrm{even}}(X, E)}, \quad D_{-}^{c}=\left.D^{c}\right|_{\Omega^{0, \text { odd }(X, E)}} \tag{1.3.16}
\end{equation*}
$$

Lemma 1.3.4. $D^{c}$ is a formally self-adjoint, first order elliptic differential operator on $\Omega^{0, \bullet}(X, E)$.

Proof. Let $s_{1}, s_{2} \in \Omega^{0, \bullet}(X, E)$ with compact support and let $\alpha$ be the 1-form on $X$ given by $\alpha(Y)=\left\langle c(Y) s_{1}, s_{2}\right\rangle_{\Lambda^{0} \bullet \bullet E}$, for any vector field $Y$ on $X$. Proposition 1.3.1 and (1.3.10) imply that for $x \in X$,

$$
\begin{equation*}
\left\langle s_{1}, D^{c} s_{2}\right\rangle_{\Lambda^{0}, \bullet \otimes E, x}=\left\langle D^{c} s_{1}, s_{2}\right\rangle_{\Lambda^{0}, \bullet \otimes E, x}-\operatorname{Tr}(\nabla \alpha)_{x} \tag{1.3.17}
\end{equation*}
$$

The integral over $X$ of the last term vanishes by Proposition 1.2.1. Thus $D^{c}$ is formally self-adjoint.

For $\zeta \in T^{*} X$, let $\zeta^{*} \in T X$ be the metric dual of $\zeta$. The principal symbol $\sigma\left(D^{c}\right)$ of $D^{c}$ is

$$
\begin{equation*}
\sigma\left(D^{c}\right)(\zeta)=\sqrt{-1} c\left(\zeta^{*}\right) \tag{1.3.18}
\end{equation*}
$$

By (1.3.2), $\left(\sigma\left(D^{c}\right)(\zeta)\right)^{2}=|\zeta|^{2}$, which means, that $\sigma\left(D^{c}\right)(\zeta)$ is invertible for any $\zeta \neq 0$. Thus $D^{c}$ is a first order elliptic differential operator.

Let $\left(F, h^{F}\right)$ be a Hermitian vector bundle on $X$ and let $\nabla^{F}$ be a Hermitian connection on $F$. Then the usual Bochner Laplacians $\Delta^{F}, \Delta$ are defined by

$$
\begin{equation*}
\Delta^{F}:=-\sum_{i=1}^{2 n}\left(\left(\nabla_{e_{i}}^{F}\right)^{2}-\nabla_{\nabla_{e_{i}}^{T X}}^{F}\right), \quad \Delta=\Delta^{\mathbb{C}} \tag{1.3.19}
\end{equation*}
$$

Let $s_{1}, s_{2} \in \mathscr{C}^{\infty}(X, F)$, with compact support and let $\alpha$ be the 1-form on $X$ given by $\alpha(Y)(x)=\left\langle\nabla_{Y}^{F} s_{1}, s_{2}\right\rangle(x)$, for any $Y \in T_{x} X$. Then by (1.2.6), (1.2.7), we get the following useful equation:

$$
\begin{align*}
\int_{X}\left\langle\Delta^{F} s_{1}, s_{2}\right\rangle d v_{X} & =\int_{X}\left\langle\nabla^{F} s_{1}, \nabla^{F} s_{2}\right\rangle d v_{X}-\int_{X} \operatorname{Tr}(\nabla \alpha) d v_{X}  \tag{1.3.20}\\
& =\int_{X}\left\langle\nabla^{F} s_{1}, \nabla^{F} s_{2}\right\rangle d v_{X}
\end{align*}
$$

We denote by $\Delta^{\mathrm{Cl}}$ the Bochner Laplacian on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ associated to $\nabla^{\mathrm{Cl}}$ as in (1.3.19). Now we prove the Lichnerowicz formula for $D^{c}$.

Theorem 1.3.5.

$$
\begin{equation*}
\left(D^{c}\right)^{2}=\Delta^{C l}+\frac{r^{X}}{4}+\frac{1}{2}\left(R^{E}+\frac{1}{2} R^{\mathrm{det}}\right)\left(e_{i}, e_{j}\right) c\left(e_{i}\right) c\left(e_{j}\right) . \tag{1.3.21}
\end{equation*}
$$

Proof. By (1.3.2), (1.3.6) and (1.3.15),

$$
\begin{align*}
\left(D^{c}\right)^{2}= & \frac{1}{2} \sum_{i j}\left\{c\left(e_{i}\right) \nabla_{e_{i}}^{\mathrm{Cl}} c\left(e_{j}\right) \nabla_{e_{j}}^{\mathrm{Cl}}+c\left(e_{j}\right) \nabla_{e_{j}}^{\mathrm{Cl}} c\left(e_{i}\right) \nabla_{e_{i}}^{\mathrm{Cl}}\right\} \\
= & \frac{1}{2} \sum_{i j}\left\{\left(c\left(e_{i}\right) c\left(e_{j}\right)+c\left(e_{j}\right) c\left(e_{i}\right)\right) \nabla_{e_{i}}^{\mathrm{Cl}} \nabla_{e_{j}}^{\mathrm{Cl}}+c\left(e_{i}\right)\left[\nabla_{e_{i}}^{\mathrm{Cl}}, c\left(e_{j}\right)\right] \nabla_{e_{j}}^{\mathrm{Cl}}\right. \\
& \left.+c\left(e_{j}\right)\left[\nabla_{e_{j}}^{\mathrm{Cl}}, c\left(e_{i}\right)\right] \nabla_{e_{i}}^{\mathrm{Cl}}+c\left(e_{j}\right) c\left(e_{i}\right)\left[\nabla_{e_{j}}^{\mathrm{Cl}}, \nabla_{e_{i}}^{\mathrm{Cl}}\right]\right\}  \tag{1.3.22}\\
= & -\sum_{i}\left(\nabla_{e_{i}}^{\mathrm{Cl}}\right)^{2}+\sum_{i j k}\left\langle\nabla_{e_{i}}^{T X} e_{j}, e_{k}\right\rangle c\left(e_{i}\right) c\left(e_{k}\right) \nabla_{e_{j}}^{\mathrm{Cl}} \\
& +\frac{1}{2} \sum_{i j} c\left(e_{j}\right) c\left(e_{i}\right)\left[\nabla_{e_{j}}^{\mathrm{Cl}}, \nabla_{e_{i}}^{\mathrm{Cl}}\right] .
\end{align*}
$$

But we have

$$
\begin{equation*}
\left\langle\nabla_{e_{i}}^{T X} e_{j}, e_{k}\right\rangle=-\left\langle e_{j}, \nabla_{e_{i}}^{T X} e_{k}\right\rangle \tag{1.3.23}
\end{equation*}
$$

In view of (1.3.2), (1.3.23), we obtain

$$
\begin{align*}
& \left\langle\nabla_{e_{i}}^{T X} e_{j}, e_{k}\right\rangle c\left(e_{i}\right) c\left(e_{k}\right) \nabla_{e_{j}}^{\mathrm{Cl}}=-c\left(e_{i}\right) c\left(e_{k}\right) \nabla_{\nabla_{e_{i}}^{T X}}^{\mathrm{Cl}} e_{k} \\
& =\nabla_{\nabla_{e_{i}}^{T X} e_{i}}^{\mathrm{Cl}}-\frac{1}{2} \sum_{i \neq k} c\left(e_{i}\right) c\left(e_{k}\right)\left(\nabla_{\nabla_{e_{i}}^{T X}}^{\mathrm{Cl}} e_{k}-\nabla_{\nabla_{e_{k}}^{T X} e_{i}}^{\mathrm{Cl}}\right)  \tag{1.3.24}\\
& \quad=\nabla_{\nabla_{e_{i}}^{T X} e_{i}}^{\mathrm{Cl}}-\frac{1}{2} c\left(e_{i}\right) c\left(e_{k}\right) \nabla_{\left[e_{i}, e_{k}\right]}^{\mathrm{Cl}} .
\end{align*}
$$

Comparing to (1.3.13), we have here

$$
\begin{equation*}
\left(R^{\mathrm{Cl}}+R^{E}\right)\left(e_{l}, e_{m}\right)=\nabla_{e_{l}}^{\mathrm{Cl}} \nabla_{e_{m}}^{\mathrm{Cl}}-\nabla_{e_{m}}^{\mathrm{Cl}} \nabla_{e_{l}}^{\mathrm{Cl}}-\nabla_{\left[e_{l}, e_{m}\right]}^{\mathrm{Cl}} . \tag{1.3.25}
\end{equation*}
$$

(1.3.22)-(1.3.25) yield

$$
\begin{equation*}
\left(D^{c}\right)^{2}=-\sum_{i}\left(\left(\nabla_{e_{i}}^{\mathrm{Cl}}\right)^{2}-\nabla_{\nabla_{e_{i}}^{T X} e_{i}}^{\mathrm{Cl}}\right)+\frac{1}{2} c\left(e_{j}\right) c\left(e_{i}\right)\left(R^{\mathrm{Cl}}+R^{E}\right)\left(e_{j}, e_{i}\right) . \tag{1.3.26}
\end{equation*}
$$

To simplify the notation, set

$$
\begin{equation*}
R_{i j k l}:=\left\langle R^{T X}\left(e_{j}, e_{i}\right) e_{k}, e_{l}\right\rangle \tag{1.3.27}
\end{equation*}
$$

By Proposition 1.3.2, we get

$$
\begin{align*}
c\left(e_{j}\right) c\left(e_{i}\right) R^{\mathrm{Cl}}\left(e_{j}, e_{i}\right)= & -\frac{1}{4} R_{i j k l} c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right) c\left(e_{l}\right) \\
& +\frac{1}{2} c\left(e_{i}\right) c\left(e_{j}\right) R^{\mathrm{det}}\left(e_{i}, e_{j}\right) . \tag{1.3.28}
\end{align*}
$$

By the second equation of (1.2.4) and (1.3.2),

$$
\sum_{i \neq k \neq j} R_{i j k l} c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right)=2 \sum_{i<j<k}\left(R_{i j k l}+R_{j k i l}+R_{k i j l}\right) c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right)=0
$$

Thus

$$
\begin{align*}
R_{i j k l} c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right) c\left(e_{l}\right) & =-R_{i j j l} c\left(e_{i}\right) c\left(e_{l}\right)+R_{i j i l} c\left(e_{j}\right) c\left(e_{l}\right)  \tag{1.3.29}\\
& =2 c\left(e_{j}\right) c\left(e_{l}\right) R_{i j i l}=-2 R_{i j i j}
\end{align*}
$$

In the last equation of (1.3.29), we use that $R_{i j i l}$ is symmetric in $j, l$ (which follows by the first equation of (1.2.4)). By (1.2.5) and (1.3.27), we get the right-hand side of (1.3.29) equals $-2 r^{X}$. Hence (1.3.26)-(1.3.29) imply (1.3.21).

### 1.3.3 Modified Dirac operator

For any $\mathbb{Z}_{2}$-graded vector space $V=V^{+} \oplus V^{-}$, the natural $\mathbb{Z}_{2}$-grading on $\operatorname{End}(V)$ is defined by
$\operatorname{End}(V)^{+}=\operatorname{End}\left(V^{+}\right) \oplus \operatorname{End}\left(V^{-}\right), \quad \operatorname{End}(V)^{-}=\operatorname{Hom}\left(V^{+}, V^{-}\right) \oplus \operatorname{Hom}\left(V^{-}, V^{+}\right)$,
and we define $\operatorname{deg} B=0$ for $B \in \operatorname{End}(V)^{+}$, and $\operatorname{deg} B=1$ for $B \in \operatorname{End}(V)^{-}$. For $B, C \in \operatorname{End}(V)$, we define their supercommutator (or graded Lie bracket) by

$$
\begin{equation*}
[B, C]=B C-(-1)^{\operatorname{deg} B \cdot \operatorname{deg} C} C B \tag{1.3.30}
\end{equation*}
$$

For $B, B^{\prime}, C \in \operatorname{End}(V)$, the Jacobi identity holds:

$$
\begin{align*}
&(-1)^{\operatorname{deg} C \cdot \operatorname{deg} B^{\prime}}\left[B^{\prime},[B, C]\right]+(-1)^{\operatorname{deg} B^{\prime} \cdot \operatorname{deg} B}\left[B,\left[C, B^{\prime}\right]\right] \\
&+(-1)^{\operatorname{deg} B \cdot \operatorname{deg} C}\left[C,\left[B^{\prime}, B\right]\right]=0 \tag{1.3.31}
\end{align*}
$$

We will apply the above notation for spaces $\Lambda\left(T^{*(0,1)} X\right)$ and $\Omega^{0, \bullet}(X, E)$ with natural $\mathbb{Z}_{2}$-grading induced by the parity of the degree.
For $i_{1}<\cdots<i_{j}$, we define

$$
\begin{equation*}
{ }^{c}\left(e^{i_{1}} \wedge \cdots \wedge e^{i_{j}}\right)=c\left(e_{i_{1}}\right) \ldots c\left(e_{i_{j}}\right) . \tag{1.3.32}
\end{equation*}
$$

Then by extending $\mathbb{C}$-linearly, ${ }^{c} B$ is defined for any $B \in \Lambda\left(T^{*} X \otimes_{\mathbb{R}} \mathbb{C}\right)$.
For $A \in \Lambda^{3}\left(T^{*} X\right)$, set $|A|^{2}=\sum_{i<j<k}\left|A\left(e_{i}, e_{j}, e_{k}\right)\right|^{2}$. Now let $A$ be a smooth section of $\Lambda^{3}\left(T^{*} X\right)$. Let

$$
\begin{equation*}
\nabla_{U}^{A}=\nabla_{U}^{\mathrm{Cl}}+{ }^{c}\left(i_{U} A\right) \quad \text { for } U \in T X \tag{1.3.33}
\end{equation*}
$$

be the Hermitian connection on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ induced by $\nabla^{\mathrm{Cl}}$ and $A$. Let $\Delta^{A}$ be the Bochner Laplacian defined by $\nabla^{A}$ as in (1.3.19).
Definition 1.3.6. The modified Dirac operators $D^{c, A}, D_{ \pm}^{c, A}$ are defined by

$$
\begin{equation*}
D^{c, A}:=D^{c}+{ }^{c} A, \quad D_{ \pm}^{c, A}:=D_{ \pm}^{c}+{ }^{c} A \tag{1.3.34}
\end{equation*}
$$

Theorem 1.3.7. The modified Dirac operator $D^{c, A}$ is formally self-adjoint and

$$
\begin{equation*}
\left(D^{c, A}\right)^{2}=\Delta^{A}+\frac{r^{X}}{4}+{ }^{c}\left(R^{E}+\frac{1}{2} R^{\mathrm{det}}\right)+{ }^{c}(d A)-2|A|^{2} \tag{1.3.35}
\end{equation*}
$$

Proof. By Lemma 1.3.4 and (1.3.10), the operator $D^{c}+{ }^{c} A$ is formally self-adjoint.
By (1.3.6), $\nabla_{e_{i}}^{\mathrm{Clc}} A={ }^{c}\left(\nabla_{e_{i}}^{T X} A\right)$. From (1.2.44) and (1.3.2) and since $A$ is odd degree, we have

$$
\begin{align*}
& {\left[c\left(e_{i}\right),{ }^{c} A\right]=-2^{c}\left(i_{e_{i}} A\right)}  \tag{1.3.36}\\
& c\left(e_{i}\right)\left(\nabla_{e_{i}}^{\mathrm{Cl}} A\right)-\left(\nabla_{e_{i}}^{\mathrm{Cl}} A\right) c\left(e_{i}\right)=2^{c}\left(e^{i} \wedge \nabla_{e_{i}}^{T X} A\right)=2^{c}(d A) .
\end{align*}
$$

By (1.3.19), (1.3.33) and the first equation of (1.3.36),

$$
\begin{align*}
\Delta^{A}= & \Delta^{\mathrm{Cl}}+\frac{1}{2}\left(\nabla_{e_{i}}^{\mathrm{Cl}}\left[c\left(e_{i}\right),{ }^{c} A\right]+\left[c\left(e_{i}\right),{ }^{c} A\right] \nabla_{e_{i}}^{\mathrm{Cl}}\right) \\
& -\frac{1}{2}\left[c\left(\nabla_{e_{i}}^{T X} e_{i}\right),{ }^{c} A\right]-\frac{1}{4} \sum_{i}\left[c\left(e_{i}\right),{ }^{c} A\right]^{2}  \tag{1.3.37}\\
= & \Delta^{\mathrm{Cl}}-2^{c}\left(i_{e_{i}} A\right) \nabla_{e_{i}}^{\mathrm{Cl}}+\frac{1}{2}\left[c\left(e_{i}\right), \nabla_{e_{i}}^{\mathrm{Cl}}{ }^{c} A\right]-\sum_{i}{ }^{c}\left(i_{e_{i}} A\right)^{2} .
\end{align*}
$$

Then Theorem 1.3.5, (1.3.33), (1.3.36) and (1.3.37) imply

$$
\begin{align*}
\left(D^{c}+{ }^{c} A\right)^{2}= & \left(D^{c}\right)^{2}+\left[c\left(e_{i}\right),{ }^{c} A\right] \nabla_{e_{i}}^{\mathrm{Cl}}+c\left(e_{i}\right)\left(\nabla_{e_{i}}^{\mathrm{Cl} c} A\right)+\left({ }^{c} A\right)^{2} \\
= & \Delta^{A}+\left({ }^{c} A\right)^{2}+\sum_{i}{ }^{c}\left(i_{e_{i}} A\right)^{2}+c\left(e_{i}\right)\left(\nabla_{e_{i}}^{\mathrm{Cl} c} A\right)  \tag{1.3.38}\\
& -\frac{1}{2}\left[c\left(e_{i}\right),\left(\nabla_{e_{i}}^{\mathrm{Cl}} A\right)\right]+\frac{r^{X}}{4}+{ }^{c}\left(R^{E}+\frac{1}{2} R^{\mathrm{det}}\right)
\end{align*}
$$

Relations (1.3.36) and (1.3.38) yield

$$
\begin{equation*}
\left(D^{c}+{ }^{c} A\right)^{2}=\Delta^{A}+\left({ }^{c} A\right)^{2}+\sum_{i}{ }^{c}\left(i_{e_{i}} A\right)^{2}+{ }^{c}(d A)+\frac{r^{X}}{4}+{ }^{c}\left(R^{E}+\frac{1}{2} R^{\mathrm{det}}\right) \tag{1.3.39}
\end{equation*}
$$

Let $I=\left\{i_{1}, \ldots, i_{m}\right\}$ be an ordered subset of $\{1, \ldots, 2 n\}$, and assume that all $i_{j} \in I$ are distinct. Let $|I|$ be the cardinal of $I$. Set ${ }^{c} e_{I}=c\left(e_{i_{1}}\right) \ldots c\left(e_{i_{m}}\right)$. Take $k \leqslant 2 n$, and let $I, J$ be two ordered subsets of $\{k+1, \ldots, 2 n\}$ such that $I \cap J=\emptyset$. Then

$$
\begin{equation*}
{ }^{c} e_{1 \ldots k}{ }^{c} e_{I}{ }^{c} e_{1 \ldots k}{ }^{c} e_{J}=(-1)^{k|I|}\left({ }^{c} e_{1 \ldots k}\right)^{2} e^{c} e^{c} e_{J}=(-1)^{k|I|+\frac{k(k+1)}{2} c} e_{I}^{c} e_{J} \tag{1.3.40}
\end{equation*}
$$

Since $A$ is odd degree, (1.3.40) imply

$$
\begin{align*}
& { }^{c}\left(i_{e_{i}} A\right)^{2}=\sum_{k=0}^{2} \sum_{i_{1}<\cdots<i_{k}}(-1)^{\frac{k(k-1)}{2} c}\left(\left(i_{e_{i_{1}}} \ldots i_{e_{i_{k}}} i_{e_{i}} A\right)^{2}\right), \\
& { }^{c}(A)^{2}=\sum_{k=0}^{3} \sum_{i_{1}<\cdots<i_{k}}(-1)^{\frac{k(k+1)}{2} c}\left(\left(i_{e_{i_{1}}} \ldots i_{e_{i_{k}}} A\right)^{2}\right) . \tag{1.3.41}
\end{align*}
$$

Observe that since $A \in \Lambda^{3}\left(T^{*} X\right), A^{2}=0$ and $\left(i_{e_{i_{1}}} i_{e_{i_{2}}} A\right)^{2}=0$. Thus

$$
\begin{equation*}
\left({ }^{c} A\right)^{2}+\sum_{i}{ }^{c}\left(i_{e_{i}} A\right)^{2}=-2 \sum_{i_{1}<i_{2}<i_{3}}\left(i_{e_{i_{1}}} i_{e_{i_{2}}} i_{e_{i_{3}}} A\right)^{2}=-2|A|^{2} . \tag{1.3.42}
\end{equation*}
$$

From (1.3.39) and (1.3.42), we infer (1.3.35).

### 1.3.4 Atiyah-Singer index theorem

Theorem 1.3.8. If $X$ is compact, the modified Dirac operator $D^{c, A}$ is an essentially self-adjoint Fredholm operator, thus its kernel $\operatorname{Ker}\left(D^{c, A}\right)$ is a finite-dimensional complex vector space.

Proof. At first, if $s_{k} \in L_{0, \bullet}^{2}(X, E), D^{c, A} s_{k}=0$ and $\lim _{k \rightarrow \infty} s_{k}=s \in L_{0, \bullet}^{2}(X, E)$, then $D^{c, A} s=0$ in the sense of distributions. By Theorem A.3.4, $s \in \Omega^{0, \bullet}(X, E)$ and
$s \in \operatorname{Ker}\left(D^{c, A}\right)$. Thus the space $\operatorname{Ker}\left(D^{c, A}\right)$ is closed, so a Hilbert space. Since $X$ is compact, Theorems A.3.1, A.3.2 and Lemma 1.3.4 imply that $D^{c, A}$ is essentially self-adjoint and the unit ball

$$
\begin{equation*}
B=\left\{s \in L_{0, \bullet}^{2}(X, E):\|s\|_{L^{2}} \leqslant 1, D^{c, A} s=0\right\} \subset \operatorname{Ker}\left(D^{c, A}\right) \tag{1.3.43}
\end{equation*}
$$

is compact. Thus $\operatorname{Ker}\left(D^{c, A}\right)$ is finite-dimensional and $D^{c, A}$ is Fredhlom.
When $X$ is compact, we define the index $\operatorname{Ind}\left(D_{+}^{c, A}\right)$ of $D_{+}^{c, A}$ as

$$
\begin{align*}
\operatorname{Ind}\left(D_{+}^{c, A}\right) & :=\operatorname{dim} \operatorname{Ker}\left(D_{+}^{c, A}\right)-\operatorname{dim} \operatorname{Coker}\left(D_{+}^{c, A}\right) \\
& =\operatorname{dim} \operatorname{Ker}\left(D_{+}^{c, A}\right)-\operatorname{dim} \operatorname{Ker}\left(D_{-}^{c, A}\right) \tag{1.3.44}
\end{align*}
$$

For any Hermitian (complex) vector bundle $\left(F, h^{F}\right)$ with Hermitian connection $\nabla^{F}$ and curvature $R^{F}$ on $X$, set

$$
\begin{align*}
\operatorname{ch}\left(F, \nabla^{F}\right) & :=\operatorname{Tr}\left[\exp \left(\frac{-R^{F}}{2 \pi \sqrt{-1}}\right)\right] \\
c_{1}\left(F, \nabla^{F}\right) & :=\operatorname{Tr}\left[\frac{-R^{F}}{2 \pi \sqrt{-1}}\right]  \tag{1.3.45}\\
\operatorname{Td}\left(F, \nabla^{F}\right) & :=\operatorname{det}\left(\frac{R^{F} /(2 \pi \sqrt{-1})}{\exp \left(R^{F} /(2 \pi \sqrt{-1})\right)-1}\right) .
\end{align*}
$$

By Appendix B. 5 these are closed real differential forms on $X$ and their cohomology classes do not depend on the choice of the metric $h^{F}$ and connection $\nabla^{F}$. The corresponding cohomology classes are called the Chern class of $F$, the first Chern class of $F$, the Todd class of $F$, respectively, and we denote them by $\operatorname{ch}(F), c_{1}(F)$, $\operatorname{Td}(F) \in H^{*}(X, \mathbb{R})$ (see Example B.5.5) .

Theorem 1.3.9 (Atiyah-Singer index theorem). If $X$ is compact, $\operatorname{Ind}\left(D_{+}^{c, A}\right)$ is a topological invariant given by

$$
\begin{equation*}
\operatorname{Ind}\left(D_{+}^{c, A}\right)=\int_{X} \operatorname{Td}\left(T^{(1,0)} X\right) \operatorname{ch}(E) \tag{1.3.46}
\end{equation*}
$$

### 1.4 Lichnerowicz formula for $\square^{E}$

This section is organized as follows. In Section 1.4.1, we exhibit the relation between the operator $\bar{\partial}^{E}+\bar{\partial}^{E, *}$ and the Dirac operator $D^{c}$. In Section 1.4.2, we prove Bismut's Lichnerowicz formula for the Kodaira Laplacian $\square^{E}$. In Section 1.4.3, we establish the Bochner-Kodaira-Nakano formula for $\square^{E}$. In Section 1.4.4, we prove the Bochner-Kodaira-Nakano formula with boundary term.

We will use the notation from Sections 1.2, 1.3.

### 1.4.1 The operator $\bar{\partial}^{E}+\bar{\partial}^{E, *}$

Let $(X, J)$ be a complex manifold with complex structure $J$ and $\operatorname{dim}_{\mathbb{C}} X=n$, and let $g^{T X}$ be any Riemannian metric on $T X$ compatible with $J$. We consider a holomorphic Hermitian vector bundle $\left(E, h^{E}\right)$ on $X$. Let $\nabla^{E}$ be the holomorphic Hermitian (i.e., Chern) connection on $\left(E, h^{E}\right)$ whose curvature is $R^{E}$. Let $\bar{\partial}^{E}$ be the Dolbeault operator acting on $\Omega^{0, \bullet}(X, E):=\oplus_{q} \Omega^{0, q}(X, E)$. Then

$$
\begin{equation*}
\left(\bar{\partial}^{E}\right)^{2}=0 \tag{1.4.1}
\end{equation*}
$$

The complex $\left(\Omega^{0, \bullet}(X, E), \bar{\partial}^{E}\right)$ is called the Dolbeault complex and its cohomology, called Dolbeault cohomology of $X$ with values in $E$, is denoted by $H^{0, \bullet}(X, E)$.

By the Dolbeault isomorphism (Theorem B.4.4), $H^{0, \bullet}(X, E)$ is canonically isomorphic to the $q$ th cohomology group $H^{q}\left(X, \mathscr{O}_{X}(E)\right)$ of the sheaf $\mathscr{O}_{X}(E)$ of holomorphic sections of $E$ over $X$. We shortly denote $H^{q}(X, E):=H^{q}\left(X, \mathscr{O}_{X}(E)\right)$. Especially for $q=0$,

$$
\begin{equation*}
H^{0,0}(X, E)=H^{0}\left(X, \mathscr{O}_{X}(E)\right)=H^{0}(X, E) \tag{1.4.2}
\end{equation*}
$$

Let $\bar{\partial}^{E, *}$ be the formal adjoint of $\bar{\partial}^{E}$ on the Dolbeault complex $\Omega^{0, \bullet}(X, E)$ with respect to the scalar product $\langle\cdot, \cdot\rangle$ in (1.3.14). Set

$$
\begin{align*}
& D=\sqrt{2}\left(\bar{\partial}^{E}+\bar{\partial}^{E, *}\right)  \tag{1.4.3}\\
& \square^{E}=\bar{\partial}^{E} \bar{\partial}^{E, *}+\bar{\partial}^{E, *} \bar{\partial}^{E}
\end{align*}
$$

Then $\square^{E}$ is called the Kodaira Laplacian and

$$
\begin{equation*}
D^{2}=2 \square^{E} \tag{1.4.4}
\end{equation*}
$$

Thus $D^{2}$ preserves the $\mathbb{Z}$-grading of $\Omega^{0, \bullet}(X, E)$. It is a fundamental result, that the elements of $\operatorname{Ker}\left(\square^{E}\right)$, called harmonic forms, represent the Dolbeault cohomology. The following theorem follows from the more general Theorem 3.1.8 on noncompact manifolds (cf. Remark 3.1.10).
Theorem 1.4.1 (Hodge theory). If $X$ is a compact complex manifold, then for any $q \in \mathbb{N}$, we have the following direct sum decomposition:

$$
\begin{align*}
\Omega^{0, q}(X, E) & =\operatorname{Ker}\left(\left.D\right|_{\Omega^{0, q}}\right) \oplus \operatorname{Im}\left(\left.\square^{E}\right|_{\Omega^{0, q}}\right) \\
& =\operatorname{Ker}\left(\left.D\right|_{\Omega^{0, q}}\right) \oplus \operatorname{Im}\left(\left.\bar{\partial}^{E}\right|_{\Omega^{0, q-1}}\right) \oplus \operatorname{Im}\left(\left.\bar{\partial}^{E, *}\right|_{\Omega^{0, q+1}}\right) \tag{1.4.5}
\end{align*}
$$

Thus for any $q \in \mathbb{N}$, we have the canonical isomorphism,

$$
\begin{equation*}
\operatorname{Ker}\left(\left.D\right|_{\Omega^{0, q}}\right)=\operatorname{Ker}\left(\left.D^{2}\right|_{\Omega^{0, q}}\right) \simeq H^{0, q}(X, E) \tag{1.4.6}
\end{equation*}
$$

Especially, $H^{q}(X, E) \simeq H^{0, q}(X, E)$ is finite-dimensional.

Definition 1.4.2. The Bergman kernel of $E$ is $P\left(x, x^{\prime}\right),\left(x, x^{\prime} \in X\right)$, the Schwartz kernel of $P$, the orthogonal projection from $\left(L^{2}\left(X, \Lambda\left(T^{*(0,1)} X\right) \otimes E\right),\langle \rangle\right)$ onto $\operatorname{Ker}(D)$, the kernel of $D$ acting on $\Omega^{0, \bullet}(X, E) \cap L^{2}\left(X, \Lambda\left(T^{*(0,1)} X\right) \otimes E\right)$, with respect to the Riemannian volume form $d v_{X}\left(x^{\prime}\right)$. Especially,

$$
P\left(x, x^{\prime}\right) \in\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x} \otimes\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x^{\prime}}^{*}
$$

Remark 1.4.3. From Theorem 1.4.1, the Bergman kernel $P\left(x, x^{\prime}\right)$ is smooth on $x, x^{\prime} \in X$ when $X$ is compact. In general, by the ellipticity of $D$ and Schwartz kernel theorem, we know $P\left(x, x^{\prime}\right)$ is $\mathscr{C}^{\infty}$ (cf. Problem 1.5).

Recall that the tensors $S, T, \mathcal{S}, T_{a s}$ were defined in (1.2.38) and (1.2.48).
Lemma 1.4.4. For the operators $\bar{\partial}^{E},\left(\nabla^{E}\right)^{1,0}$ acting on $\Omega^{\bullet \bullet}(X, E)$ in (1.1.9), we have

$$
\begin{align*}
\bar{\partial}^{E} & =\bar{w}^{j} \wedge \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+i_{T^{(0,1)}} \\
& =\bar{w}^{j} \wedge \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+\frac{1}{2}\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{m}\right\rangle \bar{w}^{j} \wedge \bar{w}^{k} \wedge i_{\bar{w}_{m}}  \tag{1.4.7}\\
\left(\nabla^{E}\right)^{1,0} & =w^{j} \wedge \widetilde{\nabla}_{w_{j}}^{T X}+i_{T^{(1,0)}} \\
& =w^{j} \wedge \widetilde{\nabla}_{w_{j}}^{T X}+\frac{1}{2}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle w^{j} \wedge w^{k} \wedge i_{w_{m}} \tag{1.4.8}
\end{align*}
$$

For the formal adjoints $\bar{\partial}^{E, *}$ and $\left(\nabla^{E}\right)^{1,0 *}$ of $\bar{\partial}^{E}$ and $\left(\nabla^{E}\right)^{1,0}$ with respect to (1.3.14), we have

$$
\begin{align*}
\bar{\partial}^{E, *}= & -i \bar{w}_{j} \widetilde{\nabla}_{w_{j}}^{T X}-\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{k}\right\rangle i_{\bar{w}_{j}} \\
& +\frac{1}{2}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle \bar{w}^{m} \wedge i i_{\bar{w}_{k}} \wedge i_{\bar{w}_{j}}  \tag{1.4.9}\\
\left(\nabla^{E}\right)^{1,0 *}= & -i_{w_{j}} \widetilde{\nabla}_{\bar{w}_{j}}^{T X}-\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{k}\right\rangle i_{w_{j}} \\
& +\frac{1}{2}\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{m}\right\rangle w^{m} \wedge i_{w_{k}} i_{w_{j}} . \tag{1.4.10}
\end{align*}
$$

Proof. The operator $\bar{\partial}^{E}$ on $E$ is given by

$$
\begin{equation*}
\bar{\partial}^{E}=\sum_{i=1}^{n} \bar{w}^{i} \wedge \nabla \frac{E}{\bar{w}_{i}} \tag{1.4.11}
\end{equation*}
$$

We still denote by $\widetilde{\nabla}^{T X}$ the connection $\widetilde{\nabla}^{T X} \otimes 1+1 \otimes \nabla^{E}$ and by $i_{T}$ the operator $i_{T} \otimes 1$ on $\Lambda^{\bullet \bullet}\left(T^{*} X\right) \otimes E$. From (1.2.44), we deduce

$$
\begin{equation*}
\nabla^{E}=\varepsilon \circ \widetilde{\nabla}^{T X}+i_{T} \tag{1.4.12}
\end{equation*}
$$

Relations (1.2.37) and (1.4.12) imply (1.4.7) and (1.4.8), by decomposition after bidegree and the definition of $T$. Observe that from $(1.2 .38)$, the $(0,1)$ and $(1,0)$ components of $\mathcal{S}$ are

$$
\begin{align*}
& \mathcal{S}^{(0,1)}=\left(\left\langle\widetilde{\nabla}_{w_{i}}^{T X} \bar{w}_{i}, w_{j}\right\rangle-\left\langle\nabla_{e_{k}}^{T X} e_{k}, w_{j}\right\rangle\right) \bar{w}_{j} \\
& \mathcal{S}^{(1,0)}=\left(\left\langle\widetilde{\nabla}_{\bar{w}_{i}}^{T X} w_{i}, \bar{w}_{j}\right\rangle-\left\langle\nabla_{e_{k}}^{T X} e_{k}, \bar{w}_{j}\right\rangle\right) w_{j} \tag{1.4.13}
\end{align*}
$$

Let $s_{1}, s_{2} \in \Omega_{0}^{\bullet \bullet \bullet}(X, E)$ and let $\alpha$ be the $(0,1)$-form on $X$ given for any vector field $U=U^{(1,0)} \oplus U^{(0,1)} \in T^{(1,0)} X \oplus T^{(0,1)} X$ on $X$, by $\alpha(U)=-\left\langle i_{U^{(0,1)}} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet E}$. Note that from (1.2.6),

$$
\begin{equation*}
\operatorname{Tr}(\nabla \alpha)=w_{j} \alpha\left(\bar{w}_{j}\right)+\bar{w}_{j} \alpha\left(w_{j}\right)-\alpha\left(\nabla_{e_{k}}^{T X} e_{k}\right) \tag{1.4.14}
\end{equation*}
$$

Proceeding as in the proof of (1.3.17), (1.4.13) and (1.4.14) entail the following relation between pointwise scalar products:

$$
\begin{align*}
\left\langle s_{1}, \bar{w}^{i} \widetilde{\nabla}_{\bar{w}_{i}}^{T X} s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet} \bullet E, x & =-\left\langle i_{\bar{w}_{i}} \widetilde{\nabla}_{w_{i}}^{T X} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet} \otimes E, x \\
& \quad \operatorname{Tr}(\nabla \alpha)_{x}+i_{\mathcal{S}^{(0,1)}} \alpha . \tag{1.4.15}
\end{align*}
$$

The integral of the last term vanishes by Proposition 1.2.1, so integrating (1.4.15) and (1.2.42) over $X$, we infer (1.4.9).

Let $\beta$ be the $(1,0)$ form on $X$ given by $\beta(U)=-\left\langle i_{U^{(1,0)}} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \otimes E}$. Then as in (1.4.15),

$$
\begin{align*}
&\left.\left\langle s_{1}, w^{j} \widetilde{\nabla}_{w_{j}}^{T X} s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet}\right)^{\prime}, x=-\left\langle i_{w_{j}} \widetilde{\nabla}_{\bar{w}_{j}}^{T X} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet} \otimes E, x \\
&-\operatorname{Tr}(\nabla \beta)_{x}+i_{\mathcal{S}^{(1,0)}} \beta . \tag{1.4.16}
\end{align*}
$$

Integration of (1.4.16) and (1.2.42) gives (1.4.10).
In this section, in the definition (1.3.15) of the $\operatorname{spin}^{c}$ Dirac operator $D^{c}$, we choose $\nabla^{\text {det }}$ to be the holomorphic Hermitian connection on $\operatorname{det}\left(T^{(1,0)} X\right)$. Consequently $D$ is a modified Dirac operator.
Theorem 1.4.5. We have the following identity:

$$
\begin{equation*}
D=D^{c}-\frac{1}{4}^{c}\left(T_{a s}\right) . \tag{1.4.17}
\end{equation*}
$$

Proof. In view of (1.3.1), (1.4.7) and (1.4.9), we have

$$
\begin{align*}
\sqrt{2} \bar{\partial}^{E} & =c\left(w_{i}\right) \widetilde{\nabla}_{\bar{w}_{i}}^{T X}-\frac{1}{4} c\left(w_{i}\right) c\left(w_{j}\right) c\left(T\left(\bar{w}_{i}, \bar{w}_{j}\right)\right), \\
\sqrt{2} \bar{\partial}^{E, *} & =c\left(\bar{w}_{i}\right) \widetilde{\nabla}_{w_{i}}^{T X}+\frac{\sqrt{2}}{2}\left\langle T\left(w_{i}, w_{j}\right), \bar{w}_{k}\right\rangle i_{\bar{w}_{j}} i_{\bar{w}_{i}} \wedge \bar{w}^{k}  \tag{1.4.18}\\
& =c\left(\bar{w}_{i}\right) \widetilde{\nabla}_{w_{i}}^{T X}+\frac{1}{4} c\left(\bar{w}_{j}\right) c\left(\bar{w}_{i}\right) c\left(T\left(w_{i}, w_{j}\right)\right) .
\end{align*}
$$

Taking into account (1.4.3) and (1.4.18), we get

$$
\begin{align*}
D= & c\left(w_{i}\right) \widetilde{\nabla} \bar{w}_{i}^{T X}+c\left(\bar{w}_{i}\right) \widetilde{\nabla}_{w_{i}}^{T X} \\
& -\frac{1}{4} c\left(w_{i}\right) c\left(w_{j}\right) c\left(T\left(\bar{w}_{i}, \bar{w}_{j}\right)\right)-\frac{1}{4} c\left(\bar{w}_{i}\right) c\left(\bar{w}_{j}\right) c\left(T\left(w_{i}, w_{j}\right)\right) . \tag{1.4.19}
\end{align*}
$$

Let $\Gamma^{T^{(1,0)} X} \in T^{*} X \otimes \operatorname{End}\left(T^{(1,0)} X\right)$ be the connection form of $\nabla^{T^{(1,0)} X}$ associated to the frames $\left\{w_{j}\right\}$. Note that for the frame $\left\{\bar{w}^{j_{1}} \wedge \cdots \wedge \bar{w}^{j_{k}}, 1 \leqslant j_{1}<\cdots<j_{k} \leqslant n\right\}$,

$$
\begin{align*}
& \widetilde{\nabla}^{T X}=d+\left\langle\Gamma^{T^{(1,0)} X} w_{l}, \bar{w}_{m}\right\rangle \bar{w}^{m} \wedge i_{\bar{w}_{l}} \\
& \Gamma^{\operatorname{det}}=\operatorname{Tr}\left[\Gamma^{T^{(1,0)} X}\right] \tag{1.4.20}
\end{align*}
$$

Comparing with (1.2.38), (1.3.3), (1.3.5), we obtain

$$
\begin{equation*}
\widetilde{\nabla}^{T X}=\nabla^{\mathrm{Cl}}+\frac{1}{4} \sum_{i j}\left\langle S(\cdot) e_{i}, e_{j}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right) \tag{1.4.21}
\end{equation*}
$$

Clearly, by (1.2.38),

$$
\begin{equation*}
\frac{1}{4}\left(\left\langle S\left(e_{i}\right) e_{i}, e_{j}\right\rangle\left(c\left(e_{i}\right)\right)^{2} c\left(e_{j}\right)+\left\langle S\left(e_{i}\right) e_{j}, e_{i}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{i}\right)\right)=-\frac{1}{2} c(\mathcal{S}) \tag{1.4.22}
\end{equation*}
$$

Thus (1.2.39), (1.4.21), (1.4.22) imply

$$
\begin{align*}
& c\left(w_{i}\right) \widetilde{\nabla} \frac{\bar{\nabla}_{w_{i}} X}{}+c\left(\bar{w}_{i}\right) \widetilde{\nabla}_{w_{i}}^{T X} \\
& \quad=D^{c}-\frac{1}{2} c(\mathcal{S})+\frac{1}{4} \sum_{j \neq i \neq k}\left\langle S\left(e_{i}\right) e_{j}, e_{k}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right)  \tag{1.4.23}\\
& \quad=D^{c}-\frac{1}{2} c(\mathcal{S})+\frac{1}{4}{ }^{c}\left(T_{a s}\right) .
\end{align*}
$$

Using (1.2.42), we get

$$
\begin{align*}
& \frac{1}{4} c\left(\bar{w}_{i}\right) c\left(\bar{w}_{j}\right) c\left(T\left(w_{i}, w_{j}\right)\right)+\frac{1}{4} c\left(w_{i}\right) c\left(w_{j}\right) c\left(T\left(\bar{w}_{i}, \bar{w}_{j}\right)\right)  \tag{1.4.24}\\
& \quad=\frac{1}{4}\left\langle T\left(e_{i}, e_{j}\right), e_{k}\right\rangle c\left(e_{i}\right) c\left(e_{j}\right) c\left(e_{k}\right)=\frac{1}{2}{ }^{c}\left(T_{a s}\right)-\frac{1}{2} c(\mathcal{S}) .
\end{align*}
$$

Finally (1.4.19), (1.4.23) and (1.4.24) imply (1.4.17).
When $X$ is compact, the Euler number $\chi(X, E)$ of the holomorphic vector bundle $E$ is defined by

$$
\begin{equation*}
\chi(X, E)=\sum_{q=0}^{n}(-1)^{q} \operatorname{dim} H^{q}(X, E) \tag{1.4.25}
\end{equation*}
$$

From Theorems 1.3.9, 1.4.1, 1.4.5, we obtain:
Theorem 1.4.6 (Riemann-Roch-Hirzebruch theorem). If $X$ is compact, then

$$
\begin{equation*}
\chi(X, E)=\int_{X} \operatorname{Td}\left(T_{h} X\right) \operatorname{ch}(E) \tag{1.4.26}
\end{equation*}
$$

### 1.4.2 Bismut's Lichnerowicz formula for $\square^{E}$

Recall that the Bismut connection $\nabla^{B}$ preserves the complex structure on $T X$ by Lemma 1.2.10, thus, as in (1.2.43), it induces a natural connection $\nabla^{B}$ on $\Lambda\left(T^{*(0,1)} X\right)$ which preserves its $\mathbb{Z}$-grading. Let $\nabla^{B, \Lambda^{0, \bullet}}, \nabla^{B, \Lambda^{0, \bullet} \otimes E}$ be the connections on $\Lambda\left(T^{*(0,1)} X\right), \Lambda\left(T^{*(0,1)} X\right) \otimes E$ defined by

$$
\begin{align*}
& \nabla^{B, \Lambda^{0} \bullet \bullet}=\nabla^{B}+\left\langle S(\cdot) w_{j}, \bar{w}_{j}\right\rangle \\
& \nabla^{B, \Lambda^{0} \cdot \bullet} \otimes E=\nabla^{B, \Lambda^{0} \cdot \bullet} \otimes 1+1 \otimes \nabla^{E} \tag{1.4.27}
\end{align*}
$$

By (1.2.42), $\left\langle S(\cdot) w_{j}, \bar{w}_{j}\right\rangle$ is a purely imaginary form, thus $\nabla^{B, \Lambda^{0, \bullet} \otimes E}$ is a Hermitian connection on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ which preserves its $\mathbb{Z}$-grading. We denote by $R^{B, \Lambda^{0, \bullet}}$ the curvature of $\nabla^{B, \Lambda^{0, \bullet}}$.
By (1.2.60), (1.3.3) and (1.3.8), as in (1.4.21), we get for $U \in T X$,

$$
\begin{equation*}
\nabla_{U}^{B, \Lambda^{0, \bullet} \otimes E}=\nabla_{U}^{\mathrm{Cl}}+\frac{1}{2} c\left(S^{B}(U)\right)=\nabla_{U}^{\mathrm{Cl}}-\frac{1}{4} c\left(i_{U} T_{a s}\right) \tag{1.4.28}
\end{equation*}
$$

As in (1.3.19), we denote by $\Delta^{B, \Lambda^{0} \cdot} \otimes E$ the Bochner Laplacian defined by $\nabla^{B, \Lambda^{0,} \bullet \otimes E}$.

## Theorem 1.4.7.

$$
\begin{align*}
D^{2}=\Delta^{B, \Lambda^{0} \cdot \bullet} \otimes E
\end{align*} \frac{r^{X}}{4}+{ }^{c}\left(R^{E}+\frac{1}{2} \operatorname{Tr}\left[R^{T^{(1,0)} X}\right]\right) .
$$

Proof. Let $R^{\text {det }}$ be the curvature of the holomorphic Hermitian connection on $\operatorname{det}\left(T^{(1,0)} X\right)$. Then

$$
\begin{equation*}
R^{\mathrm{det}}=\operatorname{Tr}\left[R^{T^{(1,0)} X}\right] \tag{1.4.30}
\end{equation*}
$$

Theorem 1.3.7 and relations (1.2.51), (1.4.17) and (1.4.30) entail (1.4.29).
Remark 1.4.8. If $(X, \Theta)$ is Kähler, then $\nabla^{B, E}$ coincides with $\nabla^{\Lambda\left(T^{*(0,1)} X\right) \otimes E}$, the connection on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ induced by the holomorphic Hermitian connections $\nabla^{T^{(1,0)} X}$ and $\nabla^{E}$. Moreover, $r^{X}=2 R^{\operatorname{det}}\left(w_{i}, \bar{w}_{i}\right)$. (1.4.29) reads

$$
\begin{align*}
D^{2}= & \Delta^{\Lambda\left(T^{*(0,1)} X\right) \otimes E}-R^{E}\left(w_{j}, \bar{w}_{j}\right) \\
& +2\left(R^{E}+\frac{1}{2} \operatorname{Tr}\left[R^{T^{(1,0)} X}\right]\right)\left(w_{i}, \bar{w}_{j}\right) \bar{w}^{j} \wedge i_{\bar{w}_{i}} \tag{1.4.31}
\end{align*}
$$

### 1.4.3 Bochner-Kodaira-Nakano formula

Let $\Theta$ be the real $(1,1)$-form associated to $g^{T X}$ as in (1.2.49). We define the Lefschetz operator $L=(\Theta \wedge) \otimes 1$ on $\Lambda^{\bullet \bullet}\left(T^{*} X\right) \otimes E$ and its adjoint $\Lambda=i(\Theta)$ with respect to the Hermitian product $\langle\cdot, \cdot\rangle_{\Lambda} \bullet \bullet \otimes E$ induced by $g^{T X}$ and $h^{E}$. For $\left\{w_{j}\right\}_{j=1}^{n}$ a local orthonormal frame of $T^{(1,0)} X$, we have

$$
\begin{equation*}
L=\sqrt{-1} w^{j} \wedge \bar{w}^{j} \wedge, \quad \Lambda=-\sqrt{-1} i_{\bar{w}_{j}} i_{w_{j}} . \tag{1.4.32}
\end{equation*}
$$

Let us define the formal adjoints $\left(\nabla^{E}\right)^{1,0 *}$ of $\left(\nabla^{E}\right)^{1,0}$ and $\left(\nabla^{E}\right)^{0,1 *}=\bar{\partial}^{E, *}$ of $\left(\nabla^{E}\right)^{0,1}=\bar{\partial}^{E}$ with respect to (1.3.14) as in Lemma 1.4.4. We use next the supercommutator defined in (1.3.30), and we apply it on $\Omega^{\bullet \bullet}(X, E)$ endowed with natural $\mathbb{Z}_{2}$-grading induced by the parity of degree.

Definition 1.4.9. The holomorphic and anti-holomorphic Kodaira Laplacians are defined by:

$$
\begin{align*}
& \bar{\square}^{E}=\left[\left(\nabla^{E}\right)^{1,0},\left(\nabla^{E}\right)^{1,0 *}\right]  \tag{1.4.33}\\
& \square^{E}=\left[\bar{\partial}^{E}, \bar{\partial}^{E, *}\right] .
\end{align*}
$$

The Hermitian torsion operator is defined by

$$
\begin{equation*}
\mathcal{T}:=[\Lambda, \partial \Theta]=[i(\Theta), \partial \Theta] . \tag{1.4.34}
\end{equation*}
$$

Let us express now $\mathcal{T}$ in terms of the torsion $T$ of the connection $\widetilde{\nabla}^{T X}$.
Lemma 1.4.10. We have

$$
\begin{equation*}
\mathcal{T}=\frac{1}{2}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle\left[2 w^{k} \wedge \bar{w}^{m} \wedge i_{\bar{w}_{j}}-2 \delta_{j m} w^{k}-w^{j} \wedge w^{k} \wedge i_{w_{m}}\right] \tag{1.4.35}
\end{equation*}
$$

Proof. From (1.2.48), (1.2.54) and (1.4.34), we obtain

$$
\begin{align*}
\mathcal{T}=\frac{\sqrt{-1}}{2}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle\left\{\left[\Lambda, \omega^{j}\right] \wedge \omega^{k} \wedge \bar{w}^{m}\right. \\
\left.+\omega^{j} \wedge\left[\Lambda, \omega^{k}\right] \wedge \bar{w}^{m}+\omega^{j} \wedge \omega^{k} \wedge\left[\Lambda, \bar{w}^{m}\right]\right\} \tag{1.4.36}
\end{align*}
$$

By the formula (1.4.32) for $\Lambda$, we easily get

$$
\begin{equation*}
\left[\Lambda, \omega^{j}\right]=-\sqrt{-1} i_{\bar{w}_{j}}, \quad\left[\Lambda, \bar{\omega}^{m}\right]=\sqrt{-1} i_{w_{m}} \tag{1.4.37}
\end{equation*}
$$

Now, (1.4.36), (1.4.37) together with $T\left(w_{j}, w_{k}\right)=-T\left(w_{k}, w_{j}\right)$ imply the desired relation (1.4.35).

We have the following generalization of the usual Kähler identities in the presence of torsion.

## Theorem 1.4.11 (generalized Kähler identities).

$$
\begin{align*}
{\left[\bar{\partial}^{E, *}, L\right] } & =\sqrt{-1}\left(\left(\nabla^{E}\right)^{1,0}+\mathcal{T}\right)  \tag{1.4.38a}\\
{\left[\left(\nabla^{E}\right)^{1,0 *}, L\right] } & =-\sqrt{-1}\left(\bar{\partial}^{E}+\overline{\mathcal{T}}\right)  \tag{1.4.38b}\\
{\left[\Lambda, \bar{\partial}^{E}\right] } & =-\sqrt{-1}\left(\left(\nabla^{E}\right)^{1,0 *}+\mathcal{T}^{*}\right)  \tag{1.4.38c}\\
{\left[\Lambda,\left(\nabla^{E}\right)^{1,0}\right] } & =\sqrt{-1}\left(\bar{\partial}^{E, *}+\overline{\mathcal{T}}^{*}\right) \tag{1.4.38d}
\end{align*}
$$

Proof. Remark that the third and forth formulas are the adjoints of the first two. Thus it suffices to prove (1.4.38a), (1.4.38b). Using (1.4.9) we find

$$
\begin{align*}
{\left[\bar{\partial}^{E, *}, L\right]=\left[-i \bar{w}_{i} \widetilde{\nabla}_{w_{i}}^{T X}, L\right] } & -\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{k}\right\rangle\left[i_{\bar{w}_{j}}, L\right] \\
& +\frac{1}{2}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle\left[\bar{w}^{m} \wedge i_{\bar{w}_{k}} i_{\bar{w}_{j}}, L\right] \tag{1.4.39}
\end{align*}
$$

By (1.4.32),

$$
\begin{equation*}
\left[i_{\bar{w}_{j}}, L\right]=-\sqrt{-1} w^{j} \wedge, \quad\left[i_{w_{j}}, L\right]=\sqrt{-1} \bar{w}^{j} \wedge \tag{1.4.40}
\end{equation*}
$$

By (1.2.52), $\widetilde{\nabla}_{w_{i}}^{T X} L=L \widetilde{\nabla}_{w_{i}}^{T X}$ so from (1.4.40)

$$
\begin{equation*}
\left[-i_{\bar{w}_{j}} \widetilde{\nabla}_{w_{j}}^{T X}, L\right]=-\left[i_{\bar{w}_{j}}, L\right] \widetilde{\nabla}_{w_{j}}^{T X}=\sqrt{-1} w^{j} \wedge \widetilde{\nabla}_{w_{j}}^{T X} \tag{1.4.41}
\end{equation*}
$$

By (1.4.40), we infer

$$
\begin{align*}
{\left[\bar{w}^{m} \wedge i_{\bar{w}_{k}} i_{\bar{w}_{j}}, L\right] } & =\bar{w}^{m} \wedge\left(\left[i_{\bar{w}_{k}}, L\right] i_{\bar{w}_{j}}+i_{\bar{w}_{k}}\left[i_{\bar{w}_{j}}, L\right]\right) \\
& =-\sqrt{-1} \bar{w}^{m} \wedge\left(\omega^{k} \wedge i_{\bar{w}_{j}}+i i_{\bar{w}_{k}} \omega^{j}\right) \tag{1.4.42}
\end{align*}
$$

Relations (1.4.39)-(1.4.42) yield finally

$$
\begin{align*}
{\left[\bar{\partial}^{E, *}, L\right]=\sqrt{-1} w^{j} \wedge \widetilde{\nabla}_{w_{j}}^{T X} } & +\sqrt{-1}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{k}\right\rangle w^{j} \\
& +\sqrt{-1}\left\langle T\left(w_{j}, w_{k}\right), \bar{w}_{m}\right\rangle w^{k} \wedge \bar{w}^{m} \wedge i_{\bar{w}_{j}} \tag{1.4.43}
\end{align*}
$$

Adding (1.4.8) and (1.4.35) shows that $\sqrt{-1}\left(\left(\nabla^{E}\right)^{1,0}+\mathcal{T}\right)$ equals the right-hand side of (1.4.43), hence (1.4.38a) holds.

Formula (1.4.38b) can be proved along similar lines as (1.4.38a). Alternatively, as the computation is local, we can choose a local holomorphic frame of $E$ and using (1.4.40), we reduce the proof to the case of a trivial line bundle $E$. But then (1.4.38b) follows from (1.4.38a) by conjugation.

## Theorem 1.4.12 (Bochner-Kodaira-Nakano formula).

$$
\begin{equation*}
\square^{E}=\bar{\square}^{E}+\left[\sqrt{-1} R^{E}, \Lambda\right]+\left[\left(\nabla^{E}\right)^{1,0}, \mathcal{T}^{*}\right]-\left[\left(\nabla^{E}\right)^{0,1}, \overline{\mathcal{T}}^{*}\right] \tag{1.4.44}
\end{equation*}
$$

Proof. From (1.4.38d) we deduce that $\bar{\partial}^{E, *}=-\sqrt{-1}\left[\Lambda,\left(\nabla^{E}\right)^{1,0}\right]-\overline{\mathcal{T}}^{*}$. Thus

$$
\begin{equation*}
\square^{E}=\left[\bar{\partial}^{E}, \bar{\partial}^{E, *}\right]=-\sqrt{-1}\left[\bar{\partial}^{E},\left[\Lambda,\left(\nabla^{E}\right)^{1,0}\right]\right]-\left[\bar{\partial}^{E}, \overline{\mathcal{T}}^{*}\right] \tag{1.4.45}
\end{equation*}
$$

The Jacobi identity (1.3.31) implies

$$
\begin{equation*}
\left[\bar{\partial}^{E},\left[\Lambda,\left(\nabla^{E}\right)^{1,0}\right]\right]=\left[\Lambda,\left[\left(\nabla^{E}\right)^{1,0}, \bar{\partial}^{E}\right]\right]+\left[\left(\nabla^{E}\right)^{1,0},\left[\bar{\partial}^{E}, \Lambda\right]\right] . \tag{1.4.46}
\end{equation*}
$$

Since $\left(\bar{\partial}^{E}\right)^{2}=0,\left(\left(\nabla^{E}\right)^{1,0}\right)^{2}=0$, we have

$$
\begin{equation*}
R^{E}=\left(\nabla^{E}\right)^{2}=\left[\left(\nabla^{E}\right)^{1,0}, \bar{\partial}^{E}\right] \tag{1.4.47}
\end{equation*}
$$

Using the expression of $\left[\bar{\partial}^{E}, \Lambda\right]$ given in (1.4.38c) we find

$$
\begin{equation*}
\left[\left(\nabla^{E}\right)^{1,0},\left[\bar{\partial}^{E}, \Lambda\right]\right]=\sqrt{-1}\left[\left(\nabla^{E}\right)^{1,0},\left(\nabla^{E}\right)^{1,0 *}\right]+\sqrt{-1}\left[\left(\nabla^{E}\right)^{1,0}, \mathcal{T}^{*}\right] \tag{1.4.48}
\end{equation*}
$$

Taking into account the definition of $\bar{\square}^{E}$ (cf. (1.4.33)), we conclude (1.4.44) from (1.4.45)-(1.4.48).

Corollary 1.4.13. Assume that $\left(X, g^{T X}\right)$ is Kähler. Then

$$
\begin{align*}
& \square^{E}=\bar{\square}^{E}+\left[\sqrt{-1} R^{E}, \Lambda\right],  \tag{1.4.49a}\\
& \Delta=2 \square=2 \bar{\square} . \tag{1.4.49b}
\end{align*}
$$

Here $\bar{\square}:=\bar{\square}^{\mathbb{C}}=\partial \partial^{*}+\partial^{*} \partial ; \square:=\square^{\mathbb{C}}$ are usual $\partial$-Laplacian and $\bar{\partial}$-Laplacian, $\Delta=d d^{*}+d^{*} d$ is the Bochner Laplacian on $\Lambda\left(T^{*} X\right)$ and $d^{*}$ is the adjoint of $d$.

Therefore, the Hodge decomposition holds for the de Rham cohomology group $H^{\bullet}(X, \mathbb{C})$ :
(a) $H^{j}(X, \mathbb{C}) \cong \oplus_{p+q=j} H^{q}\left(X, \mathscr{O}_{X}^{p}\right) \cong \oplus_{p+q=j} H^{p, q}(X)$,
(b) $H^{p, q}(X) \cong \overline{H^{q, p}(X)}$.

We denote here by $H^{p, q}(X):=H^{p, q}(X, \mathbb{C})$ the Dolbeault cohomology groups.
Proof. Indeed, by Theorem 1.2.8, $g^{T X}$ is Kähler if and only if $\mathcal{T}=0$, so (1.4.49a) follows trivially from (1.4.44). By taking $E=\mathbb{C}$ with a trivial metric, we obtain $\square=\bar{\square}$. Moreover

$$
\begin{equation*}
\Delta=\left[d, d^{*}\right]=\left[\partial+\bar{\partial}, \partial^{*}+\bar{\partial}^{*}\right]=\square+\bar{\square}+\left[\partial, \bar{\partial}^{*}\right]+\left[\bar{\partial}, \partial^{*}\right], \tag{1.4.50}
\end{equation*}
$$

and the two latter brackets vanish (Problem 1.6). By the real analogue of Theorem 1.4.1 (Hodge theory), $H^{\bullet}(X, \mathbb{C}) \simeq \operatorname{Ker}(\Delta)$. This completes the proof.

Theorem 1.4.14 (Nakano's inequality). For any $s \in \Omega_{0}^{\bullet \bullet \bullet}(X, E)$,

$$
\begin{align*}
\frac{3}{2}\left\langle\square^{E} s, s\right\rangle \geqslant\langle & {\left.\left[\sqrt{-1} R^{E}, \Lambda\right] s, s\right\rangle } \\
& \quad-\frac{1}{2}\left(\|\mathcal{T} s\|_{L^{2}}^{2}+\left\|\mathcal{T}^{*} s\right\|_{L^{2}}^{2}+\|\overline{\mathcal{T}} s\|_{L^{2}}^{2}+\left\|\overline{\mathcal{T}}^{*} s\right\|_{L^{2}}^{2}\right) \tag{1.4.51}
\end{align*}
$$

If $\left(X, g^{T X}\right)$ is Kähler, then

$$
\begin{equation*}
\left\langle\square^{E} s, s\right\rangle \geq\left\langle\left[\sqrt{-1} R^{E}, \Lambda\right] s, s\right\rangle \tag{1.4.52}
\end{equation*}
$$

Proof. Let $s \in \Omega_{0}^{\bullet \bullet \bullet}(X, E)$. Since

$$
\begin{align*}
& \left\langle\square^{E} s, s\right\rangle=\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}  \tag{1.4.53}\\
& \left\langle\bar{\square}^{E} s, s\right\rangle=\left\|\left(\nabla^{E}\right)^{1,0} s\right\|_{L^{2}}^{2}+\left\|\left(\nabla^{E}\right)^{1,0 *} s\right\|_{L^{2}}^{2}
\end{align*}
$$

we deduce from (1.4.44) that

$$
\begin{align*}
& \left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}=\left\|\left(\nabla^{E}\right)^{1,0} s\right\|_{L^{2}}^{2}+\left\|\left(\nabla^{E}\right)^{1,0 *} s\right\|_{L^{2}}^{2}  \tag{1.4.54}\\
& \quad+\left\langle\left[\sqrt{-1} R^{E}, \Lambda\right] s, s\right\rangle+\left\langle\left[\left(\nabla^{E}\right)^{1,0}, \mathcal{T}^{*}\right] s, s\right\rangle-\left\langle\left[\bar{\partial}^{E}, \overline{\mathcal{T}}^{*}\right] s, s\right\rangle .
\end{align*}
$$

By the Cauchy-Schwarz inequality, we find

$$
\begin{aligned}
\left|\left\langle\left[\left(\nabla^{E}\right)^{1,0}, \mathcal{T}^{*}\right] s, s\right\rangle\right| & \leqslant \frac{1}{2}\left(\left\|\left(\nabla^{E}\right)^{1,0} s\right\|_{L^{2}}^{2}+\left\|\left(\nabla^{E}\right)^{1,0 *} s\right\|_{L^{2}}^{2}+\|\mathcal{T} s\|_{L^{2}}^{2}+\left\|\mathcal{T}^{*} s\right\|_{L^{2}}^{2}\right) \\
\left|\left\langle\left[\bar{\partial}^{E}, \overline{\mathcal{T}}^{*}\right] s, s\right\rangle\right| & \leqslant \frac{1}{2}\left(\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}+\|\overline{\mathcal{T}}\|_{L^{2}}^{2}+\left\|\overline{\mathcal{T}}^{*} s\right\|_{L^{2}}^{2}\right)
\end{aligned}
$$

Therefore

$$
\begin{align*}
& \frac{3}{2}\left(\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *}{ }_{s}\right\|_{L^{2}}^{2}\right) \geqslant \frac{1}{2}\left(\left\|\left(\nabla^{E}\right)^{1,0} s\right\|_{L^{2}}^{2}+\left\|\left(\nabla^{E}\right)^{1,0 *} s\right\|_{L^{2}}^{2}\right) \\
& \quad+\left\langle\left[\sqrt{-1} R^{E}, \Lambda\right] s, s\right\rangle-\frac{1}{2}\left(\|\mathcal{T} s\|_{L^{2}}^{2}+\left\|\mathcal{T}^{*} s\right\|_{L^{2}}^{2}+\|\overline{\mathcal{T}} s\|_{L^{2}}^{2}+\left\|\overline{\mathcal{T}}^{*} s\right\|_{L^{2}}^{2}\right) \tag{1.4.55}
\end{align*}
$$

whereby the conclusion.
For the purpose of proving vanishing theorems and the spectral gap for forms of bidegree $(0, q)$ with values in a positive bundle (especially on non-compact manifolds or with boundary), we derive sometimes another form of the Bochner-Kodaira-Nakano formula. Set $\widetilde{E}=E \otimes K_{X}^{*}$ where

$$
K_{X}^{*}=\Lambda^{n}\left(T^{(1,0)} X\right)=\operatorname{det}\left(T^{(1,0)} X\right)
$$

Since $K_{X} \otimes K_{X}^{*} \cong \mathbb{C}$, there exists a natural isometry

$$
\begin{align*}
& \Psi=\sim: \Lambda^{0, q}\left(T^{*} X\right) \otimes E \longrightarrow \Lambda^{n, q}\left(T^{*} X\right) \otimes \widetilde{E}  \tag{1.4.56}\\
& \Psi s=\widetilde{s}=\left(w^{1} \wedge \cdots \wedge w^{n} \wedge s\right) \otimes\left(w_{1} \wedge \cdots \wedge w_{n}\right)
\end{align*}
$$

where $\left\{w_{j}\right\}_{j=1}^{n}$ a local orthonormal frame of $T^{(1,0)} X$.
Theorem 1.4.15. For any $s \in \Omega^{0, \bullet}(X, E)$, we have

$$
\begin{align*}
\square^{E} s= & \Psi^{-1} \bar{\square}^{\tilde{E}} \Psi s+R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i \bar{w}_{j} s  \tag{1.4.57}\\
& +\Psi^{-1}\left(\nabla^{\tilde{E}}\right)^{1,0} \mathcal{T}^{*} \Psi s-\left[\bar{\partial}^{E}, \Psi^{-1} \overline{\mathcal{T}}^{*} \Psi\right] s
\end{align*}
$$

Proof. We apply (1.4.44) for $\widetilde{s}$ :

$$
\begin{equation*}
\square^{\widetilde{E}} \widetilde{s}=\bar{\square}^{\widetilde{E}} \widetilde{s}+\left[\sqrt{-1} R^{\widetilde{E}}, \Lambda\right] \widetilde{s}+\left[\left(\nabla^{\widetilde{E}}\right)^{1,0}, \mathcal{T}^{*}\right] \widetilde{s}-\left[\bar{\partial}^{\widetilde{E}}, \overline{\mathcal{T}}^{*}\right] \widetilde{s} \tag{1.4.58}
\end{equation*}
$$

Since $K_{X}^{*}$ is a holomorphic bundle,

$$
\begin{equation*}
\bar{\partial}^{\widetilde{E}} \widetilde{s}=\left(\bar{\partial}^{E} s\right)^{\sim}, \quad \bar{\partial}^{\widetilde{E}, *} \widetilde{s}=\left(\bar{\partial}^{E, *} s\right)^{\sim}, \quad \square^{\widetilde{E}} \widetilde{s}=\left(\square^{E} s\right)^{\sim} \tag{1.4.59}
\end{equation*}
$$

Hence $\Psi^{-1} \square^{\tilde{E}} \widetilde{s}=\square^{E}$ s. Likewise

$$
\begin{align*}
& \Psi^{-1}\left[\bar{\partial}^{\widetilde{E}}, \overline{\mathcal{T}}^{*}\right] \widetilde{s}=\left[\bar{\partial}^{E}, \Psi^{-1} \overline{\mathcal{T}}^{*} \Psi\right] s \\
& \Psi^{-1}\left[\left(\nabla^{\widetilde{E}}\right)^{1,0}, \mathcal{T}^{*}\right] \widetilde{s}=\Psi^{-1}\left(\nabla^{\widetilde{E}}\right)^{1,0} \mathcal{T}^{*} \widetilde{s}  \tag{1.4.60}\\
& \Psi^{-1} \bar{\square}^{\widetilde{E}} \Psi s=\Psi^{-1}\left(\nabla^{\widetilde{E}}\right)^{1,0}\left(\nabla^{\widetilde{E}}\right)^{1,0 *} \Psi s
\end{align*}
$$

By (1.4.37) we have

$$
\begin{equation*}
\left[\sqrt{-1} R^{\tilde{E}}, \Lambda\right]=R^{\tilde{E}}\left(w_{j}, \bar{w}_{k}\right)\left(w^{j} \wedge i_{w_{k}}-i_{\bar{w}_{j}} \bar{w}^{k} \wedge\right) \tag{1.4.61}
\end{equation*}
$$

thus

$$
\begin{equation*}
\Psi^{-1}\left[\sqrt{-1} R^{\widetilde{E}}, \Lambda\right] \widetilde{s}=R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i_{\bar{w}_{j}} s \tag{1.4.62}
\end{equation*}
$$

From(1.4.59), (1.4.60) and (1.4.62), we obtain (1.4.57).
Remark 1.4.16. Assume now that $g^{T X}$ is Kähler. Then $\mathcal{T}=0$, and $\widetilde{\nabla}^{T X}$ on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ is induced by the holomorphic Hermitian connections $\nabla^{T^{(1,0)} X}$, $\nabla^{E}$. On $\Omega^{0, \bullet}(X, E)$, set $\Delta^{0, \bullet}=-\sum_{i}\left(\widetilde{\nabla}_{w_{i}}^{T X} \widetilde{\nabla}_{\overline{w_{i}}}^{T X}-\widetilde{\nabla}_{\nabla_{w_{i}}^{T X} \bar{w}_{i}}^{T X}\right)$. From (1.4.8) and (1.4.10), for $s \in \Omega^{0, \bullet}(X, E)$, we obtain $\Psi^{-1} \square^{\widetilde{E}} \Psi s=\Delta^{0, \bullet} s$. We infer from (1.4.57):

$$
\begin{equation*}
\square^{E} s=\Delta^{0, \bullet} s+R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i_{\bar{w}_{j}} s \quad \text { for } s \in \Omega^{0, \bullet}(X, E) \tag{1.4.63}
\end{equation*}
$$

Corollary 1.4.17. For any $s \in \Omega_{0}^{0, q}(X, E)$,

$$
\begin{align*}
\frac{3}{2}\left(\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *}{ }_{s}\right\|_{L^{2}}^{2}\right) & \geqslant\left\langle R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i_{\bar{w}_{j}} s, s\right\rangle \\
& -\frac{1}{2}\left(\left\|\mathcal{T}^{*} \widetilde{s}\right\|_{L^{2}}^{2}+\|\overline{\mathcal{T}} \widetilde{s}\|_{L^{2}}^{2}+\left\|\overline{\mathcal{T}}^{*} \widetilde{s}\right\|_{L^{2}}^{2}\right) \tag{1.4.64}
\end{align*}
$$

Proof. By applying (1.4.59) and (1.4.62) to (1.4.51) with $\widetilde{s} \in \Omega^{n, q}(X, \widetilde{E})$, we obtain (1.4.64). Alternatively, we can repeat the proof of Theorem 1.4 .14 by replacing (1.4.44) with (1.4.57).

### 1.4.4 Bochner-Kodaira-Nakano formula with boundary term

Keeping the same notations as before, let $M$ be a smooth, relatively compact domain in $X$. We set $M=\{x \in X: \varrho(x)<0\}$ where $\varrho \in \mathscr{C}^{\infty}(X)$ satisfies $|d \varrho|=1$ on $\partial M$. (This is always possible by replacing $\varrho$ by $\varrho /|d \varrho|$ near $\partial M$ and using a partition of unity argument.) Let $\bar{M}$ be the closure of $M$.

Let $-e_{\mathfrak{n}} \in T M$ be the metric dual of $d \varrho$. Then $e_{\mathfrak{n}}$ is the inward pointing unit normal at $\partial M$. We decompose $e_{\mathfrak{n}}$ as $e_{\mathfrak{n}}=e_{\mathfrak{n}}^{(1,0)}+e_{\mathfrak{n}}^{(0,1)} \in T^{(1,0)} X \oplus T^{(0,1)} X$. Then we have

$$
\begin{equation*}
e_{\mathfrak{n}}^{(1,0)}=-\bar{w}_{j}(\varrho) w_{j}, \quad e_{\mathfrak{n}}^{(0,1)}=-w_{j}(\varrho) \bar{w}_{j} . \tag{1.4.65}
\end{equation*}
$$

To simplify the notation in the rest of this section, for $s_{1}, s_{2} \in \Omega^{\bullet \bullet}(\bar{M}, E)$, we will denote by $\left\langle s_{1}, s_{2}\right\rangle$ the integral $\int_{M}\left\langle s_{1}, s_{2}\right\rangle_{\Lambda \bullet, \bullet \otimes E, x} d v_{X}(x)$.
Lemma 1.4.18. For $s_{1}, s_{2} \in \Omega^{\bullet \bullet}(\bar{M}, E)$, we have

$$
\begin{align*}
& \left\langle\bar{\partial}^{E} s_{1}, s_{2}\right\rangle-\left\langle s_{1}, \bar{\partial}^{E, *} s_{2}\right\rangle=\int_{\partial M}\left\langle\bar{\partial} \varrho \wedge s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet E E} d v_{\partial M} \\
& \left\langle s_{1},\left(\nabla^{E}\right)^{1,0} s_{2}\right\rangle-\left\langle\left(\nabla^{E}\right)^{1,0 *} s_{1}, s_{2}\right\rangle=\int_{\partial M}\left\langle s_{1}, \partial \varrho \wedge s_{2}\right\rangle_{\Lambda \cdot \bullet \otimes E} d v_{\partial M} \tag{1.4.66}
\end{align*}
$$

Proof. Let $\gamma$ be a 1-form on $X$. From (1.2.9), (1.2.10) and Stokes theorem (remark that $d v_{X}=d \varrho \wedge d v_{\partial M}$ on $\left.\partial M\right)$, we get

$$
\begin{equation*}
\int_{M} \operatorname{Tr}(\nabla \gamma) d v_{X}=-\int_{\partial M} \gamma\left(e_{\mathfrak{n}}\right) d v_{\partial M} \tag{1.4.67}
\end{equation*}
$$

In view of (1.4.15), (1.4.65) and (1.4.67), we get

$$
\begin{align*}
& \left\langle s_{1}, \bar{\partial}^{E} s_{2}\right\rangle-\left\langle\bar{\partial}^{E, *} s_{1}, s_{2}\right\rangle=\int_{\partial M} \alpha\left(e_{\mathfrak{n}}\right) d v_{\partial M} \\
& \quad=-\int_{\partial M}\left\langle i_{e_{\mathrm{n}}^{(0,1)}} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \otimes E} d v_{\partial M}=\int_{\partial M}\left\langle s_{1}, \bar{\partial} \varrho \wedge s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet E} d v_{\partial M} \tag{1.4.68}
\end{align*}
$$

Similarly, from (1.4.10), (1.4.16) and (1.4.67), we obtain

$$
\begin{align*}
& \left\langle s_{1},\left(\nabla^{E}\right)^{1,0} s_{2}\right\rangle-\left\langle\left(\nabla^{E}\right)^{1,0 *} s_{1}, s_{2}\right\rangle \\
& \quad=-\int_{\partial M}\left\langle i_{e_{\mathrm{n}}^{(1,0)}} s_{1}, s_{2}\right\rangle_{\Lambda \bullet \bullet \otimes E} d v_{\partial M}=\int_{\partial M}\left\langle s_{1}, \partial \varrho \wedge s_{2}\right\rangle_{\Lambda \bullet \bullet \bullet E} d v_{\partial M} \tag{1.4.69}
\end{align*}
$$

The proof of Lemma 1.4.18 is complete.
Let $\bar{\partial}_{H}^{E, *}$ be the Hilbert space adjoint of $\bar{\partial}^{E}$ on $M$. By definition, $s \in$ $\operatorname{Dom}\left(\bar{\partial}_{H}^{E, *}\right)$ if and only if there exists $s_{1} \in L^{2}\left(M, \Lambda^{\bullet \bullet \bullet}\left(T^{*} X\right) \otimes E\right)$ such that for any $s_{2} \in \operatorname{Dom}\left(\bar{\partial}^{E}\right),\left\langle s, \bar{\partial}^{E} s_{2}\right\rangle=\left\langle s_{1}, s_{2}\right\rangle$ and then $\bar{\partial}_{H}^{E, *} s=s_{1}$. Let us set

$$
\begin{equation*}
B^{0, q}(M, E)=\left\{s \in \Omega^{0, q}(\bar{M}, E): i_{e_{\mathrm{n}}^{(0,1)}} s=0 \text { on } \partial M\right\} . \tag{1.4.70}
\end{equation*}
$$

Proposition 1.4.19. We have $B^{0, q}(M, E)=\operatorname{Dom}\left(\bar{\partial}_{H}^{E, *}\right) \cap \Omega^{0, q}(\bar{M}, E)$ and $\bar{\partial}_{H}^{E, *}=$ $\bar{\partial}^{E, *}$ on $B^{0, q}(M, E)$.

Proof. For $s_{1} \in \operatorname{Dom}\left(\bar{\partial}_{H}^{E, *}\right) \cap \Omega^{0, q}(\bar{M}, E), s_{2} \in \Omega^{0, q-1}(\bar{M}, E)$,

$$
\left\langle\bar{\partial}_{H}^{E, *} s_{1}, s_{2}\right\rangle=\left\langle s_{1}, \bar{\partial}^{E} s_{2}\right\rangle=\left\langle\bar{\partial}^{E, *} s_{1}, s_{2}\right\rangle-\int_{\partial M}\left\langle i_{\left.e_{\mathrm{n}}^{(0,1)} s_{1}, s_{2}\right\rangle_{\Lambda \cdot \bullet \otimes E} d v_{\partial M} . . . .}\right.
$$

If $s_{2} \in \Omega_{0}^{0, q-1}(M, E)$, the boundary term vanishes, thus $\left\langle\bar{\partial}_{H}^{E, *} s_{1}, s_{2}\right\rangle=\left\langle\bar{\partial}^{E, *} s_{1}, s_{2}\right\rangle$. Since $\Omega_{0}^{0, q-1}(M, E)$ is dense in $L_{0, q-1}^{2}(M, E)$, it follows that $\bar{\partial}_{H}^{E, *} s_{1}=\bar{\partial}^{E, *} s_{1}$. This implies that the boundary term vanishes for all $s_{2} \in \Omega^{0, q-1}(\bar{M}, E)$, so $i_{e_{\mathrm{n}}^{(0,1)}} s_{1}=0$ on $\partial M$.
Definition 1.4.20. The Levi form of $\partial M$ is the restriction of $\partial \bar{\partial} \varrho$ to the holomorphic tangent bundle of $\partial M$. For $s \in \Omega^{0, q}(\bar{M}, E)$, at $y \in \partial M$, set

$$
\begin{equation*}
\mathscr{L}_{\varrho}(s, s)=(\partial \bar{\partial} \varrho)\left(w_{k}, \bar{w}_{j}\right)\left\langle\bar{w}^{j} \wedge i_{\bar{w}_{k}} s, s\right\rangle_{\Lambda \bullet \bullet \otimes E, y} \tag{1.4.71}
\end{equation*}
$$

Theorem 1.4.21. For any $s \in B^{0, \bullet}(M, E)$, we have

$$
\begin{align*}
&\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}=\left\|\left(\nabla^{\widetilde{E}}\right)^{1,0 *} \widetilde{s}^{\prime}\right\|_{L^{2}}^{2}+\left\langle R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i_{\bar{w}_{j}} s, s\right\rangle \\
&-\left\langle\bar{\partial}^{E} s, \Psi^{-1} \overline{\mathcal{T}} \widetilde{s}\right\rangle-\left\langle\Psi^{-1} \overline{\mathcal{T}}^{*} \widetilde{s}, \bar{\partial}^{E, *} s\right\rangle+\left\langle\mathcal{T}^{*} \widetilde{s},\left(\nabla^{E}\right)^{1,0 *} \widetilde{s}\right\rangle \\
&+\int_{\partial M} \mathscr{L}_{\varrho}(s, s) d v_{\partial M} \tag{1.4.72}
\end{align*}
$$

Proof. Since $s \in B^{0, \bullet}(M, E)=\operatorname{Dom}\left(\bar{\partial}_{H}^{E, *}\right) \cap \Omega^{0, q}(\bar{M}, E)$, by (1.4.66), we have

$$
\begin{align*}
& \left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}=\left\langle\bar{\partial}^{E} \bar{\partial}^{E, *} s, s\right\rangle, \\
& \left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}=\left\langle\bar{\partial}^{E, *} \bar{\partial}^{E} s, s\right\rangle+\int_{\partial M}\left\langle\bar{\partial}^{E} s, \bar{\partial} \varrho \wedge s\right\rangle_{\Lambda \bullet \bullet \otimes E} d v_{\partial M}, \\
& \left\langle\left[\bar{\partial}^{E}, \Psi^{-1} \overline{\mathcal{T}}^{*} \Psi\right] s, s\right\rangle=\left\langle\bar{\partial}^{E} s, \Psi^{-1} \overline{\mathcal{T}} \widetilde{s}\right\rangle+\left\langle\Psi^{-1} \overline{\mathcal{T}}^{*} \widetilde{s}, \bar{\partial}^{E, *} s\right\rangle,  \tag{1.4.73}\\
& \left\langle\left(\nabla^{\widetilde{E}}\right)^{1,0} \mathcal{T}^{*} \widetilde{s}, \widetilde{s}\right\rangle=\left\langle\mathcal{T}^{*} \widetilde{s},\left(\nabla^{\widetilde{E}}\right)^{1,0 * \widetilde{s}\rangle}+\int_{\partial M}\left\langle\partial \varrho \wedge \mathcal{T}^{*} \widetilde{s}, \widetilde{s}\right\rangle_{\Lambda \bullet \bullet \bullet E} d v_{\partial M},\right. \\
& \left\langle\bar{\square}^{\widetilde{E}} \widetilde{s}, \widetilde{s}\right\rangle=\left\|\left(\nabla^{\widetilde{E}}\right)^{1,0 *} \widetilde{s}\right\|_{L^{2}}^{2}+\int_{\partial M}\left\langle\partial \varrho \wedge\left(\nabla^{\widetilde{E}}\right)^{1,0 * \widetilde{s}, \widetilde{s}\rangle_{\Lambda \bullet \bullet \otimes E} d v_{\partial M} .}\right.
\end{align*}
$$

Thus (1.4.57), (1.4.73) yield

$$
\begin{gather*}
\left\|\bar{\partial}^{E}{ }_{s}\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}=\left\|\left(\nabla^{\widetilde{E}}\right)^{1,0 *} \widetilde{s}\right\|_{L^{2}}^{2}+\left\langle R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i_{\bar{w}_{j}} s, s\right\rangle \\
\quad-\left\langle\bar{\partial}^{E} s, \Psi^{-1} \overline{\mathcal{T}} \widetilde{s}\right\rangle-\left\langle\Psi^{-1} \overline{\mathcal{T}}^{*} \widetilde{s}, \bar{\partial}^{E, *} s\right\rangle+\left\langle\mathcal{T}^{*} \widetilde{s},\left(\nabla^{E}\right)^{1,0 *} \widetilde{s}\right\rangle  \tag{1.4.74}\\
+\int_{\partial M}\left(\left\langle\bar{\partial}^{E} s, \bar{\partial} \varrho \wedge s\right\rangle_{\Lambda \bullet \bullet} \otimes E+\left\langle\partial \varrho \wedge\left(\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}\right) \widetilde{s}, \widetilde{s}\right\rangle_{\Lambda \bullet \bullet} \otimes E\right) d v_{\partial M}
\end{gather*}
$$

To conclude our theorem, we need to compute the last two terms in (1.4.74). By (1.4.59) and (1.4.65), we infer

$$
\begin{align*}
\left\langle\bar{\partial}^{E} s, \bar{\partial} \varrho\right. & \wedge s\rangle_{\Lambda \bullet \bullet \bullet E E}+\left\langle\partial \varrho \wedge\left(\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}\right) \widetilde{s}, \widetilde{s}\right\rangle_{\Lambda \bullet \bullet \otimes E} \\
& =\left\langle-i_{e_{\mathrm{n}}^{(0,1)}} \bar{\partial}^{E} s+\Psi^{-1} \partial \varrho \wedge\left(\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}\right) \Psi s, s\right\rangle_{\Lambda \bullet \bullet \bullet \otimes E} \tag{1.4.75}
\end{align*}
$$

Recall that on $T X \otimes_{\mathbb{R}} \mathbb{C}$, we denote also by $\langle$,$\rangle the \mathbb{C}$-bilinear form induced by $g^{T X}$. As in the proof of Lemma 1.4.4, we denote by $\widetilde{\nabla}^{T X}$ the connection on $\Lambda^{\bullet \bullet}\left(T^{*} X\right) \otimes E$ induced by $\nabla^{E}$ and $\widetilde{\nabla}^{T X}$. From (1.4.35), we get

$$
\begin{equation*}
\mathcal{T}^{*}=\frac{1}{2}\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{m}\right\rangle\left[2 \bar{w}^{j} \wedge i_{\bar{w}_{m}} \wedge i_{w_{k}}-2 \delta_{j m} i_{w_{k}}-w^{m} \wedge i_{w_{k}} i_{w_{j}}\right] \tag{1.4.76}
\end{equation*}
$$

By (1.4.10) and (1.4.76), we obtain

$$
\begin{equation*}
\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}=-i_{w_{j}} \widetilde{\nabla}_{\overline{w_{j}}}^{T X}+\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{m}\right\rangle \bar{w}^{j} \wedge i_{\bar{w}_{m}} \wedge i_{w_{k}} \tag{1.4.77}
\end{equation*}
$$

Thus from (1.4.7), (1.4.65), (1.4.77) and $i_{e_{\mathrm{n}}^{(0,1)}} s=0$ on $\partial M$, we have on $\partial M$,

$$
\begin{align*}
& -i_{e_{\mathrm{n}}^{(0,1)}} \bar{\partial}^{E}{ }_{s}+\Psi^{-1} \partial \varrho \wedge\left(\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}\right) \Psi s \\
& =\left\{-i_{e_{\mathrm{n}}^{(0,1)}} \bar{w}^{j} \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+\left\langle e_{\mathfrak{n}}^{(0,1)}, w_{j}\right\rangle \widetilde{\nabla}_{\bar{w}_{j}}^{T X}\right. \\
& \left.+\left\langle T\left(\bar{w}_{j}, \bar{w}_{k}\right), w_{m}\right\rangle\left(-\frac{1}{2} i e_{e_{\mathfrak{n}}^{(0,1)}} \bar{w}^{j} \wedge \bar{w}^{k} \wedge i_{\bar{w}_{m}}-\left\langle e_{\mathfrak{n}}^{(0,1)}, w_{k}\right\rangle \bar{w}^{j} \wedge i_{\bar{w}_{m}}\right)\right\} s \\
& =\left(-i_{e_{\mathfrak{n}}^{(0,1)}} \bar{w}^{j} \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+\left\langle e_{\mathfrak{n}}^{(0,1)}, w_{j}\right\rangle \widetilde{\nabla} \widetilde{w}_{j}^{T X}\right) s . \tag{1.4.78}
\end{align*}
$$

To compute the term in (1.4.78), we use again our boundary condition. Relations (1.4.65) and (1.4.70) yield

$$
\begin{equation*}
\left(i_{\bar{w}_{j}} s\right) w_{j}(\varrho)=-i_{e_{\mathrm{n}}^{(0,1)}} s=0 \quad \text { on } \partial M \tag{1.4.79}
\end{equation*}
$$

Especially, $\left(i_{\bar{w}_{j}} s\right) w_{j} \in T \partial M \otimes \Lambda^{0, \bullet}\left(T^{*} X\right) \otimes E$ on $\partial M$. Thus at $y \in \partial M$, we have

$$
\begin{equation*}
0=\left\langle\widetilde{\nabla}_{\bar{w}_{j}}^{T X}\left(i_{e_{\mathrm{n}}^{(0,1)}} s\right), i_{\bar{w}_{j}} s\right\rangle_{\Lambda \bullet \bullet \bullet \otimes E, y}=\left\langle\bar{w}^{j} \widetilde{\nabla}_{\bar{w}_{j}}{ }_{j}^{X}\left(i_{e_{\mathrm{n}}^{(0,1)}} s\right), s\right\rangle_{\Lambda \bullet \bullet \bullet E, y} \tag{1.4.80}
\end{equation*}
$$

Now

$$
\begin{equation*}
\bar{w}^{j} \widetilde{\nabla}_{\bar{w}_{j}}^{T X} i_{e_{\mathbf{n}}^{(0,1)}}=-i_{e_{\mathrm{n}}^{(0,1)}} \bar{w}^{j} \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+\left\langle e_{\mathfrak{n}}^{(0,1)}, w_{j}\right\rangle \widetilde{\nabla}_{\bar{w}_{j}}^{T X}+\left\langle\widetilde{\nabla}_{\bar{w}_{j}}^{T X} e_{\mathfrak{n}}^{(0,1)}, w_{k}\right\rangle \bar{w}^{j} i_{\bar{w}_{k}} \tag{1.4.81}
\end{equation*}
$$

Moreover, from (1.4.7) and (1.4.65),

$$
\left.\begin{array}{rl}
\left\langle\widetilde{\nabla} \frac{T X}{\bar{w}_{j}} e_{\mathfrak{n}}^{(0,1)}, w_{k}\right\rangle=\bar{w}_{j}\left\langle e_{\mathfrak{n}}^{(0,1)}, w_{k}\right\rangle- & \left\langle e_{\mathfrak{n}}^{(0,1)}, \widetilde{\nabla}_{\bar{w}_{j}}^{T X} w_{k}\right\rangle \\
=-\bar{w}_{j}\left(\partial \varrho, w_{k}\right)+\left(\partial \varrho, \widetilde{\nabla} \bar{w}_{j} X\right. \\
\left.w_{k}\right) \tag{1.4.82}
\end{array}\right)
$$

Using (1.4.78), (1.4.80), (1.4.81) and (1.4.82), we get at $y \in \partial M$,

$$
\begin{align*}
\left\langle-i_{e_{\mathrm{n}}^{(0,1)}} \bar{\partial}^{E} s+\Psi^{-1} \partial \varrho\right. & \left.\wedge\left(\left(\nabla^{\widetilde{E}}\right)^{1,0 *}+\mathcal{T}^{*}\right) \Psi s, s\right\rangle_{\Lambda \bullet \bullet \bullet E, y} \\
& =(\partial \bar{\partial} \varrho)\left(w_{k}, \bar{w}_{j}\right)\left\langle\bar{w}^{j} i_{\bar{w}_{k}} s, s\right\rangle_{\Lambda \bullet \bullet \bullet E, y}=\mathscr{L}_{\varrho}(s, s) . \tag{1.4.83}
\end{align*}
$$

Finally, (1.4.74), (1.4.75) and (1.4.83) imply (1.4.72).
Similarly to Corollary 1.4.17, we obtain:
Corollary 1.4.22. For any $s \in B^{0, q}(M, E)$,

$$
\begin{align*}
\frac{3}{2}\left(\left\|\bar{\partial}^{E} s\right\|_{L^{2}}^{2}\right. & \left.+\left\|\bar{\partial}^{E, *} s\right\|_{L^{2}}^{2}\right) \geqslant \frac{1}{2}\left\|\left(\nabla^{\widetilde{E}}\right)^{1,0 *} \widetilde{s}\right\|_{L^{2}}^{2}+\left\langle R^{E \otimes K_{X}^{*}}\left(w_{j}, \bar{w}_{k}\right) \bar{w}^{k} \wedge i \bar{w}_{j} s, s\right\rangle \\
& +\int_{\partial M} \mathscr{L}_{\varrho}(s, s) d v \partial M-\frac{1}{2}\left(\left\|\mathcal{T}^{*} \widetilde{s}\right\|_{L^{2}}^{2}+\|\overline{\mathcal{T}} \widetilde{s}\|_{L^{2}}^{2}+\left\|\overline{\mathcal{T}}^{*} \widetilde{s}\right\|_{L^{2}}^{2}\right) . \tag{1.4.84}
\end{align*}
$$

Our proof of the Bochner-Kodaira-Nakano formula with boundary term (1.4.72) and of the estimate (1.4.84) takes a different route as the usual proof, which consists in integrating by parts starting with the left-hand side and deriving at the end also the Bochner-Kodaira-Nakano formula without boundary. We integrate here directly (1.4.57) and we can easily identify the boundary term. It is remarkable that the curvature and the torsion do not contribute to the boundary integral.

### 1.5 Spectral gap

As a direct application of the Lichnerowicz formula and Bochner-Kodaira-Nakano formula, we obtain various vanishing theorems and exhibit the spectral gap for the modified Dirac operators. The spectral gap property will play an essential role in our approach to the Bergman kernel.

This section is organized as follows. In Section 1.5.1, we obtain the vanishing theorems and the spectral gap property for the Kodaira Laplacian. In Section 1.5.2 we establish the spectral gap property for a modified Dirac operator on symplectic manifolds.

### 1.5.1 Vanishing theorem and spectral gap

Lemma 1.5.1 ( $\partial \bar{\partial}$-Lemma). Let $\varphi$ be a smooth, real, $d$-exact, $(q, q)$-form on a compact Kähler manifold $M$; then there exists a smooth, real, $(q-1, q-1)$-form $\rho$ on $M$ such that

$$
\begin{equation*}
\varphi=\sqrt{-1} \partial \bar{\partial} \rho \tag{1.5.1}
\end{equation*}
$$

Proof. Let $\bar{\partial}^{*}, \partial^{*}, d^{*}$ be the adjoint of $\bar{\partial}, \partial, d$ associated to the Kähler metric $g^{T M}$. From (1.4.49b) and (1.4.50), $\operatorname{Ker}(\square)=\operatorname{Ker}(d) \cap \operatorname{Ker}\left(d^{*}\right)$.

As $\varphi$ is $d$-exact, $\varphi$ is orthogonal to $\operatorname{Ker}\left(d^{*}\right)$, thus $\varphi$ is orthogonal to $\operatorname{Ker}(\square)$. By Hodge theory (Theorem 1.4.1) for $E=\Lambda^{q}\left(T^{*(1,0)} M\right)$, there exists a $(q, q)$-form $\varphi_{1}$ such that

$$
\begin{equation*}
\varphi=2 \square \varphi_{1}=\left(d d^{*}+d^{*} d\right) \varphi_{1} \tag{1.5.2}
\end{equation*}
$$

Again using $\varphi$ is $d$-exact and $\operatorname{Im}(d) \cap \operatorname{Im}\left(d^{*}\right)=0$, we get $\varphi=d d^{*} \varphi_{1}$.
Let $\psi^{q-1, q}$ (resp. $\psi^{q, q-1}$ ) be the $(q-1, q)$ (resp. $(q, q-1)$ )-component of $d^{*} \varphi_{1}$. As $\varphi$ is a $(q, q)$-form, we get

$$
\begin{equation*}
\varphi=\partial \psi^{q-1, q}+\bar{\partial} \psi^{q, q-1}, \quad \bar{\partial} \psi^{q-1, q}=0, \quad \partial \psi^{q, q-1}=0 \tag{1.5.3}
\end{equation*}
$$

(If $q=1$, we get directly (1.5.3) from the $d$-exactness of $\varphi$ ).
We claim that if $\theta$ is a $(q-1, q)$-form and $\bar{\partial} \theta=0$, then there exists a ( $q-1, q-1$ )-form $\eta$ such that

$$
\begin{equation*}
\partial \theta=\partial \bar{\partial} \eta \tag{1.5.4}
\end{equation*}
$$

By Hodge theory (Theorem 1.4.1) for $E=\Lambda^{q-1}\left(T^{*(1,0)} M\right)$, there exists a smooth ( $q-1, q-1$ )-form $\eta$ such that

$$
\begin{equation*}
\bar{\partial}^{*} \theta=\left(\bar{\partial}^{*} \bar{\partial}+\bar{\partial} \bar{\partial}^{*}\right) \eta \tag{1.5.5}
\end{equation*}
$$

(1.4.5) shows that $\operatorname{Im}(\bar{\partial}) \cap \operatorname{Im}\left(\bar{\partial}^{*}\right)=0$. Thus we get

$$
\begin{equation*}
\bar{\partial}^{*}(\theta-\bar{\partial} \eta)=0, \quad \bar{\partial} \bar{\partial}^{*} \eta=0 \tag{1.5.6}
\end{equation*}
$$

But from $\bar{\partial}(\theta-\bar{\partial} \eta)=0,(1.4 .49 \mathrm{~b})$ and (1.5.6) we know

$$
\begin{equation*}
\theta-\bar{\partial} \eta \in \operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}\left(\bar{\partial}^{*}\right)=\operatorname{Ker}(\square)=\operatorname{Ker}(\partial) \cap \operatorname{Ker}\left(\partial^{*}\right) \tag{1.5.7}
\end{equation*}
$$

Thus we get (1.5.4) for $\theta$ and $\eta$.
For $\psi^{q, q-1}$, we will apply (1.5.4) for $\overline{\psi^{q, q-1}}$. Thus there exists $\rho$ such that (1.5.1) holds. As $\varphi$ is real, we can take $\rho$ as real.

For a holomorphic Hermitian line bundle $\left(F, h^{F}\right)$ on a complex manifold $M$, we will call the curvature $R^{F}$ associated to the holomorphic Hermitian connection $\nabla^{F}$ on $\left(F, h^{F}\right)$ simply the curvature $R^{F}$ associated to $h^{F}$.

The curvature $R^{F}$ is a $(1,1)$-form on $M$ and $\sqrt{-1} R^{F}$ is real. For any holomorphic local frame $s$ of $F$ on an open set $U$,

$$
\begin{equation*}
R^{F}(x)=\bar{\partial} \partial \log |s(x)|_{h^{F}}^{2} \quad \text { on } U . \tag{1.5.8}
\end{equation*}
$$

Definition 1.5.2. A holomorphic line bundle $F$ on a complex manifold $M$ is positive (resp. semi-positive) if there is a metric $h^{F}$ on $F$ with associated curvature $R^{F}$ such that $\sqrt{-1} R^{F}$ is a positive (resp. semi-positive) ( 1,1 )-form on M.F is negative if $F^{*}$ is positive.

Certainly, the notions of positivity (Definition 1.5.2), Griffiths positivity and Nakano positivity (Definition 1.1.6) are equivalent for holomorphic line bundles.

Proposition 1.5.3. Let $F$ be a holomorphic line bundle on a compact Kähler manifold $M$. If $\Omega$ is a real, closed $(1,1)$-form on $M$ with

$$
\begin{equation*}
[\Omega]=c_{1}(F) \in H^{2}(M, \mathbb{R}) \tag{1.5.9}
\end{equation*}
$$

then, up to multiplication by positive constants, there exists a unique metric $h^{F}$ on $F$ such that $\Omega=\frac{\sqrt{-1}}{2 \pi} R^{F}$, where $R^{F}$ is the curvature associated to $h^{F}$. Thus $F$ is positive if and only if its first Chern class may be represented by a positive form in $H^{2}(M, \mathbb{R})$.
Proof. Let $h_{0}^{F}$ be a Hermitian metric on $F$ and let $R_{0}^{F}$ be the curvature associated to $h_{0}^{F}$. Then by (1.5.9), $\Omega-\frac{\sqrt{-1}}{2 \pi} R_{0}^{F}$ is a real, $d$-exact, $(1,1)$-form on $M$. By Lemma 1.5.1, there exists a real function $\rho$ on $M$ such that

$$
\begin{equation*}
\Omega=\frac{\sqrt{-1}}{2 \pi} R_{0}^{F}+\frac{\sqrt{-1}}{2 \pi} \bar{\partial} \partial \rho \tag{1.5.10}
\end{equation*}
$$

From (1.5.8) and (1.5.10), we know $-2 \pi \sqrt{-1} \Omega$ is the curvature associated to the metric $e^{\rho} h_{0}^{F}$ on $F$.

Let $h_{1}^{F}$ be another metric on $F$ such that $\Omega=\frac{\sqrt{-1}}{2 \pi} R_{1}^{F}$. Then there is a real function $\rho_{1}$ such that $h_{1}^{F}=e^{\rho_{1}} h^{F}$. By (1.5.8), we have

$$
\begin{equation*}
\bar{\partial} \partial \rho_{1}=0 \tag{1.5.11}
\end{equation*}
$$

Taking the trace of both sides in (1.5.11) and using (1.4.49b), we get $\Delta \rho_{1}=0$. Thus $\rho_{1}$ is a constant function on $X$ (cf. Problem 1.9).

For a variant of Proposition 1.5.3 for singular Hermitian metrics, see Lemma 2.3.5.
Theorem 1.5.4. Let $X$ be a compact complex manifold of dimension $n$ and $F$ be a positive holomorphic line bundle on $X$. Then:
(a) (Kodaira vanishing theorem) $H^{q}\left(X, F \otimes K_{X}\right)=0$, if $q>0$.
(b) (Nakano vanishing theorem) $H^{r, q}(X, F)=0$, if $r+q>n$.

Proof. Let $h^{F}$ be a metric on $F$ with associated curvature $R^{F}$ such that $\omega=$ $\frac{\sqrt{-1}}{2 \pi} R^{F}$ is a positive $(1,1)$-form. Let $g^{T X}:=\omega(\cdot, J \cdot)$ be the associated Kähler metric on $T X$. Then the Hermitian torsion $\mathcal{T}=0$. Moreover, as $\omega=\sqrt{-1} w^{j} \wedge \bar{w}^{j}$, by (1.4.37), we have

$$
\begin{equation*}
[\omega, \Lambda]=w^{k} \wedge i_{w_{k}}-i_{\bar{w}_{k}} \bar{w}^{k} \wedge \tag{1.5.12}
\end{equation*}
$$

Thus for $s \in \Omega^{r, q}(X, F)$, we have

$$
\begin{equation*}
[\omega, \Lambda] s=(r+q-n) s \tag{1.5.13}
\end{equation*}
$$

Now the Nakano inequality (1.4.51) implies that if $s$ is harmonic, i.e., $\square^{F} s=0$, it follows that $s=0$ wherever $r+q>n$. By Hodge theory (Theorem 1.4.1) for the
holomorphic vector bundle $\Lambda^{r}\left(T^{*(1,0)} X\right) \otimes F$, we get (b). (a) is a particular case of (b) for $r=n$.

Now we will study the spectral gap property for Kodaira Laplacians.
Let $(X, J)$ be a compact complex manifold with complex structure $J$ and $\operatorname{dim}_{\mathbb{C}} X=n$. Consider a holomorphic Hermitian line bundle ( $L, h^{L}$ ) on $X$, and a holomorphic Hermitian vector bundle ( $E, h^{E}$ ) on $X$. Let $\nabla^{E}, \nabla^{L}$ be the holomorphic Hermitian (i.e., Chern) connections on $\left(E, h^{E}\right),\left(L, h^{L}\right)$ with curvatures $R^{E}$, $R^{L}$. Choose any Riemannian metric $g^{T X}$ on $T X$, compatible with the complex structure J. Set

$$
\begin{equation*}
\omega:=\frac{\sqrt{-1}}{2 \pi} R^{L}, \quad \Theta(\cdot, \cdot):=g^{T X}(J \cdot, \cdot) \tag{1.5.14}
\end{equation*}
$$

Then $\omega, \Theta$ are real (1,1)-forms on $X$, and $\omega$ is the Chern-Weil representative of the first Chern class $c_{1}(L)$ of $L$. Then the Riemannian volume form $d v_{X}$ of $\left(T X, g^{T X}\right)$ is $\Theta^{n} / n$ !.

We will identify the two-form $R^{L}$ with the Hermitian matrix

$$
\dot{R}^{L} \in \operatorname{End}\left(T^{(1,0)} X\right)
$$

such that for $W, Y \in T^{(1,0)} X$,

$$
\begin{equation*}
R^{L}(W, \bar{Y})=\left\langle\dot{R}^{L} W, \bar{Y}\right\rangle \tag{1.5.15}
\end{equation*}
$$

Let $\left\{w_{j}\right\}_{j=1}^{n}$ be a local orthonormal frame of $T^{(1,0)} X$ with dual frame $\left\{w^{j}\right\}_{j=1}^{n}$. Set

$$
\begin{equation*}
\omega_{d}=-\sum_{l, m} R^{L}\left(w_{l}, \bar{w}_{m}\right) \bar{w}^{m} \wedge i_{\bar{w}_{l}}, \quad \tau(x)=\sum_{j} R^{L}\left(w_{j}, \bar{w}_{j}\right) \tag{1.5.16}
\end{equation*}
$$

Then $\omega_{d} \in \operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right)\right)$ and $R^{L}$ acts as the derivative $\omega_{d}$ on $\Lambda\left(T^{*(0,1)} X\right)$. By (1.3.32), we have

$$
\begin{equation*}
{ }^{c}\left(R^{L}\right)=\frac{1}{2} \sum_{i j} R^{L}\left(e_{i}, e_{j}\right) c\left(e_{i}\right) c\left(e_{j}\right)=-2 \omega_{d}-\tau \tag{1.5.17}
\end{equation*}
$$

If we choose $\left\{w_{j}\right\}_{j=1}^{n}$ to be an orthonormal basis of $T^{(1,0)} X$ such that

$$
\begin{equation*}
\dot{R}^{L}(x)=\operatorname{diag}\left(a_{1}(x), \ldots, a_{n}(x)\right) \in \operatorname{End}\left(T_{x}^{(1,0)} X\right) \tag{1.5.18}
\end{equation*}
$$

then

$$
\begin{equation*}
\omega_{d}(x)=-\sum_{j} a_{j}(x) \bar{w}^{j} \wedge i_{\bar{w}_{j}}, \quad \tau(x)=\sum_{j} a_{j}(x) \tag{1.5.19}
\end{equation*}
$$

For $p \in \mathbb{N}$, we denote by $L^{p}:=L^{\otimes p}$. By replacing $E$ by $L^{p} \otimes E$ in (1.4.3), we get

$$
\begin{align*}
& D_{p}=\sqrt{2}\left(\bar{\partial}^{L^{p} \otimes E}+\bar{\partial}^{L^{p} \otimes E, *}\right),  \tag{1.5.20}\\
& \square_{p}=\bar{\partial}^{L^{p} \otimes E} \bar{\partial}^{L^{p} \otimes E, *}+\bar{\partial}^{L^{p} \otimes E, *} \bar{\partial}^{L^{p} \otimes E}
\end{align*}
$$

$D_{p}^{2}=2 \square_{p}$ preserves the $\mathbb{Z}$-grading on $\Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$.

We make the following basic assumption in the rest of this section.
Assumption: $\sqrt{-1} R^{L}$ is a positive $(1,1)$-form on $X$, equivalently, for any $0 \neq Y \in$ $T^{(1,0)} X$, we have

$$
\begin{equation*}
R^{L}(Y, \bar{Y})>0 \tag{1.5.21}
\end{equation*}
$$

In the notation of (1.5.15)-(1.5.18), the condition (1.5.21) is equivalent to:
$\dot{R}^{L} \in \operatorname{End}\left(T^{(1,0)} X\right)$ is positive-definite, i.e., $a_{j}(x)>0$ for any $x \in X, 1 \leqslant j \leqslant n$.

Theorem 1.5.5. There exist $C_{0}, C_{L}>0$ such that for any $p \in \mathbb{N}$ and any $s \in$ $\Omega^{0,>0}\left(X, L^{p} \otimes E\right)=\bigoplus_{q \geqslant 1} \Omega^{0, q}\left(X, L^{p} \otimes E\right)$,

$$
\begin{equation*}
\left\|D_{p} s\right\|_{L^{2}}^{2} \geqslant\left(2 C_{0} p-C_{L}\right)\|s\|_{L^{2}}^{2} . \tag{1.5.23}
\end{equation*}
$$

The spectrum $\operatorname{Spec}\left(\square_{p}\right)$, of the Kodaira Laplacian $\square_{p}$, is contained in the set $\{0\} \cup] p C_{0}-\frac{1}{2} C_{L},+\infty[$.

Proof. By (1.4.64) and (1.5.16), we get for any $s \in \Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$,

$$
\begin{equation*}
\left\|D_{p} s\right\|_{L^{2}}^{2}=2\left(\left\|\bar{\partial}^{L^{p} \otimes E} s\right\|_{L^{2}}^{2}+\left\|\bar{\partial}^{L^{p} \otimes E, *} s\right\|_{L^{2}}^{2}\right) \geqslant \frac{4}{3}\left\langle-\omega_{d} s, s\right\rangle p-C\|s\|_{L^{2}}^{2} \tag{1.5.24}
\end{equation*}
$$

Hence (1.5.18) and (1.5.22) yield (1.5.23). If $s \in \mathscr{C}^{\infty}\left(X, L^{p} \otimes E\right)$ is an eigensection of $D_{p}^{2}$ with $D_{p}^{2} s=\lambda s$ and $\lambda \neq 0$, then $0 \neq D_{p} s \in \Omega^{0,1}\left(X, L^{p} \otimes E\right)$, and $D_{p}^{2} D_{p} s=$ $\lambda D_{p} s$. Thus $\lambda \geqslant 2 C_{0} p-C_{L}$. This finishes the last part of Theorem 1.5.5.

By Theorems 1.4.1, 1.5.5, we conclude:
Theorem 1.5.6 (Kodaira-Serre vanishing theorem). If $L$ is a positive line bundle, then there exists $p_{0}>0$ such that for any $p \geqslant p_{0}$,

$$
\begin{equation*}
H^{q}\left(X, L^{p} \otimes E\right)=0 \quad \text { for any } q>0 \tag{1.5.25}
\end{equation*}
$$

### 1.5.2 Spectral gap of modified Dirac operators

Let $(X, J)$ be a compact manifold with almost complex structure $J$ and $\operatorname{dim}_{\mathbb{R}} X=$ $2 n$. Let $\left(L, h^{L}\right)$ be a Hermitian line bundle on $X$, and let $\left(E, h^{E}\right)$ be a Hermitian vector bundle on $X$. Let $\nabla^{E}, \nabla^{L}$ be Hermitian connections on $\left(E, h^{E}\right),\left(L, h^{L}\right)$. Let $R^{L}=\left(\nabla^{L}\right)^{2}, R^{E}=\left(\nabla^{E}\right)^{2}$ be the curvatures of $\nabla^{L}, \nabla^{E}$. Let $g^{T X}$ be any Riemannian metric on $T X$ compatible with the almost complex structure $J$. We use the notation from (1.5.14)-(1.5.19) now.
Assumption: (1.5.21) holds for $R^{L}$.
Set

$$
\begin{equation*}
\mu_{0}=\inf _{u \in T_{x}^{(1,0)} X, x \in X} R_{x}^{L}(u, \bar{u}) /|u|_{g^{T X}}^{2}=\inf _{x \in X, j} a_{j}(x)>0 \tag{1.5.26}
\end{equation*}
$$

Let $\nabla^{\text {det }}$ be a Hermitian connection on $\operatorname{det}\left(T^{(1,0)} X\right)$ with curvature $R^{\text {det }}$. We denote by $D_{p}^{c}, D_{ \pm, p}^{c}$ the spin ${ }^{c}$ Dirac operator defined in (1.3.15) associated to $L^{p} \otimes E$ and $\nabla^{\text {det }}$. For $A \in \Lambda^{3}\left(T^{*} X\right)$, by (1.3.34), set

$$
\begin{equation*}
D_{p}^{c, A}=D_{p}^{c}+{ }^{c} A, \quad D_{ \pm, p}^{c, A}=D_{ \pm, p}^{c}+{ }^{c} A \tag{1.5.27}
\end{equation*}
$$

Theorem 1.5.7. There exists $C_{L}>0$ such that for any $p \in \mathbb{N}$ and any $s \in$ $\Omega^{0,>0}\left(X, L^{p} \otimes E\right)=\bigoplus_{q \geqslant 1} \Omega^{0, q}\left(X, L^{p} \otimes E\right)$,

$$
\begin{equation*}
\left\|D_{p}^{c, A} s\right\|_{L^{2}}^{2} \geqslant\left(2 \mu_{0} p-C_{L}\right)\|s\|_{L^{2}}^{2} \tag{1.5.28}
\end{equation*}
$$

Especially, for $p$ large enough,

$$
\begin{equation*}
\operatorname{Ker}\left(D_{-, p}^{c, A}\right)=0 \tag{1.5.29}
\end{equation*}
$$

Proof. At first, we claim that there exists a constant $C>0$ such that for any $p \in \mathbb{N}, s \in \mathscr{C}^{\infty}\left(X, L^{p} \otimes E\right)$, we have

$$
\begin{equation*}
\left\|\nabla^{L^{p} \otimes E} s\right\|_{L^{2}}^{2}-p\langle\tau s, s\rangle \geqslant-C\|s\|_{L^{2}}^{2} . \tag{1.5.30}
\end{equation*}
$$

For $s \in \mathscr{C}^{\infty}\left(X, L^{p} \otimes E\right)$, by Lemma 1.3.4, Theorem 1.3.5, (1.3.20) and (1.5.17), we get

$$
\begin{align*}
\left\|D_{p}^{c} s\right\|_{L^{2}}^{2}= & \left\langle\left(D_{p}^{c}\right)^{2} s, s\right\rangle=\left\|\nabla^{\mathrm{Cl}} s\right\|_{L^{2}}^{2}-p\langle\tau s, s\rangle \\
& +\left\langle\left(\frac{r^{X}}{4}+\frac{1}{2}\left(R^{F}+\frac{1}{2} R^{\mathrm{det}}\right)\left(e_{i}, e_{j}\right) c\left(e_{i}\right) c\left(e_{j}\right)\right) s, s\right\rangle . \tag{1.5.31}
\end{align*}
$$

From (1.3.8), for $s \in \mathscr{C}^{\infty}\left(X, L^{p} \otimes E\right)$, the following identity holds:

$$
\begin{equation*}
\nabla^{\mathrm{Cl}} s=\nabla^{L^{p} \otimes E} s+\frac{1}{2}\left(\nabla^{\mathrm{det}}-\nabla^{\mathrm{det}_{1}}\right) s-\frac{1}{2}\left\langle S \bar{w}_{l}, \bar{w}_{m}\right\rangle \bar{w}^{l} \wedge \bar{w}^{m} \wedge s \tag{1.5.32}
\end{equation*}
$$

From (1.5.31) and (1.5.32), we know there exists $C>0$, which does not depend on $p$, such that

$$
\begin{equation*}
0 \leqslant\left\|\left(\nabla^{L^{p} \otimes E}+\frac{1}{2}\left(\nabla^{\operatorname{det}}-\nabla^{\operatorname{det}_{1}}\right)\right) s\right\|_{L^{2}}^{2}-p\langle\tau s, s\rangle+C\|s\|_{L^{2}}^{2} \tag{1.5.33}
\end{equation*}
$$

But $\left(\nabla^{\text {det }}-\nabla^{\text {det }_{1}}\right)$ is a purely imaginary 1-form, thus $\nabla_{1}^{E}=\nabla^{E}-\frac{1}{2}\left(\nabla^{\text {det }^{\prime}}-\nabla^{\text {det }_{1}}\right)$ is a Hermitian connection on $E$. Applying $\nabla_{1}^{E}$ on $E$ for (1.5.33), we get (1.5.30).

Relations (1.3.35), (1.5.17) imply that for $s \in \Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$,

$$
\begin{align*}
\left\|D_{p}^{c, A} s\right\|_{L^{2}}^{2}= & \left\|\nabla^{A} s\right\|_{L^{2}}^{2}-p\langle\tau s, s\rangle-2 p\left\langle\omega_{d} s, s\right\rangle \\
& +\left\langle\left(\frac{r^{X}}{4}+{ }^{c}\left(R^{E}+\frac{1}{2} R^{\mathrm{det}}\right)+{ }^{c}(d A)-2|A|^{2}\right) s, s\right\rangle . \tag{1.5.34}
\end{align*}
$$

Now we apply (1.5.30) for $E$ replaced by $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ with the Hermitian connection $\nabla^{A}$ in (1.3.33). Then we know that the sum of the first two terms of
(1.5.34) is bounded below by $-C\|s\|_{L^{2}}^{2}$. For $s \in \Omega^{0,>0}\left(X, L^{p} \otimes E\right)$ the third term of (1.5.34), $-2 p\left(\omega_{d} s, s\right)$ is bounded below by $2 \mu_{0} p\|s\|_{L^{2}}^{2}$, by (1.5.19) and (1.5.26), while the norm of the remaining terms of (1.5.34) is bounded by $C\|s\|_{L^{2}}^{2}$. Hence we obtain (1.5.28). The proof of Theorem 1.5.7 is completed.

Theorem 1.5.8. There exists $C_{L}>0$ such that for $p \in \mathbb{N}$, the spectrum of $\left(D_{p}^{c, A}\right)^{2}$ verifies

$$
\left.\operatorname{Spec}\left(\left(D_{p}^{c, A}\right)^{2}\right) \subset\{0\} \cup\right] 2 p \mu_{0}-C_{L},+\infty[.
$$

Proof. The operator $D_{p}^{c, A}$ changes the parity of $\Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$, so Theorem 1.5.7 shows that $\left(D_{p}^{c, A}\right)^{2}$ is invertible on $\Omega^{0, \text { odd }}\left(X, L^{p} \otimes E\right)$ for $p$ large enough and its spectrum is in $] 2 \mu_{0} p-C_{L},+\infty[$.

Now, if $s \in \Omega^{0, \text { even }}\left(X, L^{p} \otimes E\right)$ is an eigensection of $\left(D_{p}^{c, A}\right)^{2}$ with $\left(D_{p}^{c, A}\right)^{2} s=$ $\lambda s$ and $\lambda \neq 0$, then $D_{p}^{c, A} s \neq 0$ and

$$
\begin{equation*}
\left(D_{p}^{c, A}\right)^{2} D_{p}^{c, A} s=\lambda D_{p}^{c, A} s \tag{1.5.35}
\end{equation*}
$$

As $D_{p}^{c, A} s \in \Omega^{0, o d d}\left(X, L^{p} \otimes E\right)$, Theorem 1.5.7 yields $\lambda>2 \mu_{0} p-C_{L}$. The proof of Theorem 1.5.8 is complete.

Remark 1.5.9. From Theorems 1.4.5, 1.5.7, 1.5.8, we get another proof of Theorem 1.5.5.

### 1.6 Asymptotic of the heat kernel

This section is organized as follows. In Section 1.6.1, we explain the main result, Theorem 1.6.1, the asymptotic of the heat kernel. In the rest of this section, we prove Theorem 1.6.1. In Section 1.6.2, we explain that our problem is local. In Section 1.6.3, we do the rescaling operation on coordinates and compute the limit operators. In Section 1.6.4, we obtain the uniform estimate of the heat kernel. Finally, in Section 1.6.5, we prove Theorem 1.6.1.

### 1.6.1 Statement of the result

Let $(X, J)$ be a compact complex manifold with complex structure $J$ and $\operatorname{dim}_{\mathbb{C}} X=$ $n$. Let $\left(L, h^{L}\right)$ be a holomorphic Hermitian line bundle on $X$, and $\left(E, h^{E}\right)$ be a holomorphic Hermitian vector bundle on $X$. Let $\nabla^{E}, \nabla^{L}$ be the holomorphic Hermitian (i.e., Chern) connections on $\left(E, h^{E}\right),\left(L, h^{L}\right)$. Let $R^{L}, R^{E}$ be the curvatures of $\nabla^{L}, \nabla^{E}$. Let $g^{T X}$ be any Riemannian metric on $T X$ compatible with $J$. We use the notation in Section 1.5.1, especially $D_{p}$ was defined in (1.5.20).

For $p \in \mathbb{N}$, we write

$$
\begin{equation*}
E_{p}^{j}:=\Lambda^{j}\left(T^{*(0,1)} X\right) \otimes L^{p} \otimes E, \quad E_{p}=\oplus_{j} E_{p}^{j} \tag{1.6.1}
\end{equation*}
$$

We will denote by $\nabla^{B, E_{p}}$ the connection on $E_{p}$ defined by (1.4.27).

By (1.4.29), $D_{p}^{2}=2 \square_{p}$ is a second order elliptic differential operator with principal symbol $\sigma\left(D_{p}^{2}\right)(\xi)=|\xi|^{2}$ for $\xi \in T_{x}^{*} X, x \in X$. The heat operator $e^{-u D_{p}^{2}}$ is well defined for $u>0$. Let $\exp \left(-u D_{p}^{2}\right)\left(x, x^{\prime}\right),\left(x, x^{\prime} \in X\right)$ be the smooth kernel of the heat operator $\exp \left(-u D_{p}^{2}\right)$ with respect to the Riemannian volume form $d v_{X}\left(x^{\prime}\right)$. Then

$$
\begin{equation*}
\exp \left(-u D_{p}^{2}\right)\left(x, x^{\prime}\right) \in\left(E_{p}\right)_{x} \otimes\left(E_{p}\right)_{x^{\prime}}^{*} \tag{1.6.2}
\end{equation*}
$$

Especially

$$
\begin{equation*}
\exp \left(-u D_{p}^{2}\right)(x, x) \in \operatorname{End}\left(E_{p}\right)_{x}=\operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x} \tag{1.6.3}
\end{equation*}
$$

where we use the canonical identification $\operatorname{End}\left(L^{p}\right)=\mathbb{C}$ for any line bundle $L$ on $X$. Since $D_{p}^{2}$ preserves the $\mathbb{Z}$-grading of the Dolbeault complex $\Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$, we get from (D.1.7), that $\exp \left(-u D_{p}^{2}\right)\left(x, x^{\prime}\right) \in \bigoplus_{j}\left(\left(E_{p}^{j}\right)_{x} \otimes\left(E_{p}^{j}\right)_{x^{\prime}}^{*}\right)$, especially $\exp \left(-u D_{p}^{2}\right)(x, x) \in \bigoplus_{j} \operatorname{End}\left(\Lambda^{j}\left(T^{*(0,1)} X\right) \otimes E\right)_{x}$.

We will denote by det the determinant on $T^{(1,0)} X$. The following result is the main result of this section, and the rest of the section is devoted to its proof.
Theorem 1.6.1. For each $u>0$ fixed and any $k \in \mathbb{N}$ we have as $p \rightarrow \infty$

$$
\begin{align*}
& \exp \left(-\frac{u}{p} D_{p}^{2}\right)(x, x)=(2 \pi)^{-n} \frac{\operatorname{det}\left(\dot{R}^{L}\right) \exp \left(2 u \omega_{d}\right)}{\operatorname{det}\left(1-\exp \left(-2 u \dot{R}^{L}\right)\right)} \otimes \operatorname{Id}_{E} p^{n}+o\left(p^{n}\right) \\
& =\prod_{j=1}^{n} \frac{a_{j}(x)\left(1+\left(e^{-2 u a_{j}(x)}-1\right) \bar{w}^{j} \wedge i \bar{w}_{j}\right)}{2 \pi\left(1-e^{-2 u a_{j}(x)}\right)} \otimes \operatorname{Id}_{E} p^{n}+o\left(p^{n}\right) \tag{1.6.4}
\end{align*}
$$

in the $\mathscr{C}^{k}$-norm on $\mathscr{C}^{\infty}\left(X, \operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)\right)$. Here we use the convention that if an eigenvalue $a_{j}(x)(c f .(1.5 .18))$ of $\dot{R}_{x}^{L}$ is zero, then its contribution for $\operatorname{det}\left(\dot{R}_{x}^{L}\right) / \operatorname{det}\left(1-\exp \left(-2 u \dot{R}_{x}^{L}\right)\right)$ is $1 /(2 u)$. Finally, the convergence in (1.6.4) is uniform as $u$ varies in any compact subset of $\mathbb{R}_{+}^{*}$.

### 1.6.2 Localization of the problem

Let inj ${ }^{X}$ be the injectivity radius of ( $X, g^{T X}$ ), and $\left.\varepsilon \in\right] 0, \operatorname{inj}^{X} / 4[$.
As $X$ is compact, there exist $\left\{x_{i}\right\}_{i=1}^{N_{0}}$ such that $\left\{U_{x_{i}}=B^{X}\left(x_{i}, \varepsilon\right)\right\}_{i=1}^{N_{0}}$ is a covering of $X$. Now we use the normal coordinates as in Section 1.2.1. On $U_{x_{i}}$, we identify $E_{Z}, L_{Z}, \Lambda\left(T_{Z}^{*(0,1)} X\right)$ to $E_{x_{i}}, L_{x_{i}}, \Lambda\left(T_{x_{i}}^{*(0,1)} X\right)$ by parallel transport with respect to the connections $\nabla^{E}, \nabla^{L}, \nabla^{B, \Lambda^{0} \bullet}$ along the curve $[0,1] \ni u \rightarrow u Z$. This induces a trivialization of $E_{p}$ on $U_{x_{i}}$. Let $\left\{e_{i}\right\}_{i}$ be an orthonormal basis of $T_{x_{i}} X$. Denote by $\nabla_{U}$ the ordinary differentiation operator on $T_{x_{i}} X$ in the direction $U$.

Let $\left\{\varphi_{i}\right\}$ be a partition of unity subordinate to $\left\{U_{x_{i}}\right\}$. For $l \in \mathbb{N}$, we define a Sobolev norm on the $l$ th Sobolev space $\boldsymbol{H}^{l}\left(X, E_{p}\right)$ by

$$
\begin{equation*}
\|s\|_{\boldsymbol{H}^{l}(p)}^{2}=\sum_{i} \sum_{k=0}^{l} \sum_{i_{1} \ldots i_{k}=1}^{2 n}\left\|\nabla_{e_{i_{1}}} \ldots \nabla_{e_{i_{k}}}\left(\varphi_{i} s\right)\right\|_{L^{2}}^{2} . \tag{1.6.5}
\end{equation*}
$$

Lemma 1.6.2. For any $m \in \mathbb{N}$, there exists $C_{m}^{\prime}>0$ such that for any $s \in$ $\boldsymbol{H}^{2 m+2}\left(X, E_{p}\right), p \in \mathbb{N}^{*}$,

$$
\begin{equation*}
\|s\|_{\boldsymbol{H}^{2 m+2}(p)} \leqslant C_{m}^{\prime} p^{4 m+4} \sum_{j=0}^{m+1} p^{-4 j}\left\|D_{p}^{2 j} s\right\|_{L^{2}} \tag{1.6.6}
\end{equation*}
$$

Proof. Let $\tilde{e}_{i}(Z)$ be the parallel transport of $e_{i}$ with respect to $\nabla^{T X}$ along the curve $[0,1] \ni u \rightarrow u Z$. Then $\left\{\widetilde{e}_{i}\right\}_{i}$ is an orthonormal frame on $T X$. Let $\Gamma^{E}$, $\Gamma^{L}, \Gamma^{B, \Lambda^{0, \bullet}}$ be the corresponding connection forms of $\nabla^{E}, \nabla^{L}$ and $\nabla^{B, \Lambda^{0} \bullet}$ with respect to any fixed frame for $E, L, \Lambda\left(T^{*(0,1)} X\right)$ which is parallel along the curve $[0,1] \ni u \rightarrow u Z$ under the trivialization on $U_{x_{i}}$. On $U_{x_{i}}$, we have

$$
\begin{equation*}
D_{p}=c\left(\widetilde{e}_{j}\right)\left(\nabla_{\widetilde{e}_{j}}+p \Gamma^{L}\left(\widetilde{e}_{j}\right)+\Gamma^{B, \Lambda^{0,}}\left(\widetilde{e}_{j}\right)+\Gamma^{E}\left(\widetilde{e}_{j}\right)\right) . \tag{1.6.7}
\end{equation*}
$$

By Theorem A.1.7, (1.6.7), there exists $C>0$ (independent on $p$ ) such that for any $p \geqslant 1, s \in \boldsymbol{H}^{2}\left(X, E_{p}\right)$, we have $\|s\|_{\boldsymbol{H}^{1}(p)}^{2} \leqslant C\left(\|s\|_{\boldsymbol{H}^{2}(p)}+\|s\|_{L^{2}}\right)\|s\|_{L^{2}}$, and

$$
\begin{equation*}
\|s\|_{\boldsymbol{H}^{2}(p)} \leqslant C\left(\left\|D_{p}^{2} s\right\|_{L^{2}}+p^{2}\|s\|_{L^{2}}\right) \tag{1.6.8}
\end{equation*}
$$

Let $Q$ be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in $U_{x_{i}}$. Then

$$
\begin{equation*}
\left[D_{p}, Q\right]=p\left[c\left(\widetilde{e}_{j}\right) \Gamma^{L}\left(\widetilde{e}_{j}\right), Q\right]+\left[c\left(\widetilde{e}_{j}\right)\left(\nabla_{\widetilde{e}_{j}}+\Gamma^{B, \Lambda^{0, \bullet}}\left(\widetilde{e}_{j}\right)+\Gamma^{E}\left(\widetilde{e}_{j}\right)\right), Q\right] \tag{1.6.9}
\end{equation*}
$$

which are differential operators of order $m-1, m$ respectively. By (1.6.8), (1.6.9),

$$
\begin{align*}
\|Q s\|_{\boldsymbol{H}^{2}(p)} & \leqslant C\left(\left\|D_{p}^{2} Q s\right\|_{L^{2}}+p^{2}\|Q s\|_{L^{2}}\right) \\
& \leqslant C\left(\left\|Q D_{p}^{2} s\right\|_{L^{2}}+p^{2}\|Q s\|_{L^{2}}+p^{2}\|s\|_{\boldsymbol{H}^{2 m+1}(p)}\right) \tag{1.6.10}
\end{align*}
$$

Using (1.6.10), for $m \in \mathbb{N}$, there exists $C_{m}^{\prime}>0$ such that for $p \geqslant 1$,

$$
\begin{equation*}
\|s\|_{\boldsymbol{H}^{2 m+2}(p)} \leqslant C_{m}^{\prime}\left(\left\|D_{p}^{2} s\right\|_{\boldsymbol{H}^{2 m}(p)}+p^{2}\|s\|_{\boldsymbol{H}^{2 m+1}(p)}\right) . \tag{1.6.11}
\end{equation*}
$$

From (1.6.11), we get (1.6.6).
Let $f: \mathbb{R} \rightarrow[0,1]$ be a smooth even function such that

$$
f(v)=\left\{\begin{array}{lll}
1 & \text { for } & |v| \leqslant \varepsilon / 2  \tag{1.6.12}\\
0 & \text { for } & |v| \geqslant \varepsilon
\end{array}\right.
$$

Definition 1.6.3. For $u>0, \varsigma \geqslant 1, a \in \mathbb{C}$, set

$$
\begin{align*}
& \mathbf{F}_{u}(a)=\int_{-\infty}^{+\infty} e^{i v a} \exp \left(-\frac{v^{2}}{2}\right) f(\sqrt{u} v) \frac{d v}{\sqrt{2 \pi}} \\
& \mathbf{G}_{u}(a)=\int_{-\infty}^{+\infty} e^{i v a} \exp \left(-\frac{v^{2}}{2}\right)(1-f(\sqrt{u} v)) \frac{d v}{\sqrt{2 \pi}}  \tag{1.6.13}\\
& \mathbf{H}_{u, \varsigma}(a)=\int_{-\infty}^{+\infty} e^{i v a} \exp \left(-\frac{v^{2}}{2 u}\right)(1-f(\sqrt{\varsigma} v)) \frac{d v}{\sqrt{2 \pi u}}
\end{align*}
$$

The functions $\mathbf{F}_{u}(a), \mathbf{G}_{u}(a)$ are even holomorphic functions. The restrictions of $\mathbf{F}_{u}, \mathbf{G}_{u}$ to $\mathbb{R}$ lie in the Schwartz space $\mathcal{S}(\mathbb{R})$. Clearly,

$$
\begin{equation*}
\mathbf{G}_{u}(v a)=\mathbf{H}_{v^{2}, \frac{u}{v^{2}}}(a), \quad \mathbf{F}_{u}\left(v D_{p}\right)+\mathbf{G}_{u}\left(v D_{p}\right)=\exp \left(-\frac{v^{2}}{2} D_{p}^{2}\right) \tag{1.6.14}
\end{equation*}
$$

Let $\mathbf{F}_{u}\left(v D_{p}\right)\left(x, x^{\prime}\right), \mathbf{G}_{u}\left(v D_{p}\right)\left(x, x^{\prime}\right)\left(x, x^{\prime} \in X\right)$ be the smooth kernels associated to $\mathbf{F}_{u}\left(v D_{p}\right), \mathbf{G}_{u}\left(v D_{p}\right)$, calculated with respect to the volume form $d v_{X}\left(x^{\prime}\right)$.
Proposition 1.6.4. For any $m \in \mathbb{N}, u_{0}>0, \varepsilon>0$, there exists $C>0$ such that for any $x, x^{\prime} \in X, p \in \mathbb{N}^{*}, u>u_{0}$,

$$
\begin{equation*}
\left|\mathbf{G}_{\frac{u}{p}}\left(\sqrt{u / p} D_{p}\right)\left(x, x^{\prime}\right)\right|_{\mathscr{C}^{m}} \leqslant C p^{3 m+8 n+8} \exp \left(-\frac{\varepsilon^{2} p}{16 u}\right) \tag{1.6.15}
\end{equation*}
$$

Here the $\mathscr{C}^{m}$ norm is induced by $\nabla^{L}, \nabla^{E}, \nabla^{B, \Lambda^{0, \bullet}}$ and $h^{L}, h^{E}, g^{T X}$.
Proof. Due to the obvious relation $i^{m} a^{m} e^{i v a}=\frac{\partial^{m}}{\partial v^{m}}\left(e^{i v a}\right)$, we can integrate by parts in the expression of $a^{m} \mathbf{H}_{u, \varsigma}(a)$ given by (1.6.13) and obtain that for any $m \in \mathbb{N}$ there exists $C_{m}>0$ (which depends on $\varepsilon$ ) such that for $u>0, \varsigma \geqslant 1$,

$$
\begin{equation*}
\sup _{a \in \mathbb{R}}|a|^{m}\left|\mathbf{H}_{u, \varsigma}(a)\right| \leqslant C_{m} \varsigma^{\frac{m}{2}} \exp \left(-\frac{\varepsilon^{2}}{16 u \varsigma}\right) \tag{1.6.16}
\end{equation*}
$$

Here we use that $z^{k} \exp \left(-z^{2}\right)$ is bounded on $\mathbb{R}_{+}$.
Let $Q$ be a differential operator of order $m \in \mathbb{N}$ with scalar principal symbol and with compact support in $U_{x_{i}}$. From

$$
\left\langle D_{p}^{m^{\prime}} \mathbf{H}_{\frac{u}{p}, 1}\left(D_{p}\right) Q s, s^{\prime}\right\rangle=\left\langle s, Q^{*} \mathbf{H}_{\frac{u}{p}, 1}\left(D_{p}\right) D_{p}^{m^{\prime}} s^{\prime}\right\rangle
$$

(C.2.5) (or Theorem D.1.3, or using the Fourier transform as in (1.6.16) and the boundedness of the wave operator $e^{i u D_{p}}$ in $L^{2}$-norm implied by (D.2.16)), (1.6.6) and (1.6.16), we know that for $m, m^{\prime} \in \mathbb{N}$, there exists $C_{m, m^{\prime}}>0$ such that for $p \geqslant 1, u>u_{0}>0$,

$$
\begin{equation*}
\left\|D_{p}^{m^{\prime}} \mathbf{H}_{\frac{u}{p}, 1}\left(D_{p}\right) Q s\right\|_{L^{2}} \leqslant C_{m, m^{\prime}} p^{2 m+2} \exp \left(-\frac{\varepsilon^{2} p}{16 u}\right)\|s\|_{L^{2}} \tag{1.6.17}
\end{equation*}
$$

We deduce from (1.6.17) that if $P, Q$ are differential operators of order $m, m^{\prime}$ with compact support in $U_{x_{i}}, U_{x_{j}}$ respectively, then there exists $C>0$ such that for $p \geqslant 1, u \geqslant u_{0}$,

$$
\begin{equation*}
\left\|P \mathbf{H}_{\frac{u}{p}, 1}\left(D_{p}\right) Q s\right\|_{L^{2}} \leqslant C p^{2 m+2 m^{\prime}+4} \exp \left(-\frac{\varepsilon^{2} p}{16 u}\right)\|s\|_{L^{2}} \tag{1.6.18}
\end{equation*}
$$

By using the Sobolev inequality and (1.6.14) on $U_{x_{i}} \times U_{x_{j}}$, we conclude Proposition 1.6.4.

Using (1.6.13) and the finite propagation speed, Theorem D.2.1 and (D.2.17), it is clear that for $x, x^{\prime} \in X, \mathbf{F}_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}} D_{p}\right)\left(x, x^{\prime}\right)$ only depends on the restriction of $D_{p}$ to $B^{X}(x, \varepsilon)$, and is zero if $d\left(x, x^{\prime}\right) \geqslant \varepsilon$.

### 1.6.3 Rescaling of the operator $D_{p}^{2}$

Let $\rho: \mathbb{R} \rightarrow[0,1]$ be a smooth even function such that

$$
\begin{equation*}
\rho(v)=1 \text { if }|v|<2 ; \quad \rho(v)=0 \text { if }|v|>4 . \tag{1.6.19}
\end{equation*}
$$

Let $\Phi_{E}$ be the smooth self-adjoint section of $\operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)$ on $X$ defined by
(compare (1.4.29)).
We fix $x_{0} \in X$. From now on, we identify $B^{T_{x_{0}} X}(0,4 \varepsilon)$ with $B^{X}\left(x_{0}, 4 \varepsilon\right)$ as in Section 1.2.1. For $Z \in B^{T_{x_{0}} X}(0,4 \varepsilon)$, we identify $E_{Z}, L_{Z}, \Lambda\left(T_{Z}^{*(0,1)} X\right)$ to $E_{x_{0}}, L_{x_{0}}, \Lambda\left(T_{x_{0}}^{*(0,1)} X\right)$ by parallel transport with respect to the connections $\nabla^{E}$, $\nabla^{L}, \nabla^{B, \Lambda^{0, \bullet}}$ along the curve $[0,1] \ni u \rightarrow u Z$. Thus on $B^{X}\left(x_{0}, 4 \varepsilon\right),\left(E, h^{E}\right)$, $\left(L, h^{L}\right),\left(\Lambda\left(T^{*(0,1)} X\right), h^{\Lambda^{0, \bullet}}\right), E_{p}$ are identified to the trivial Hermitian bundles $\left(E_{x_{0}}, h^{E_{x_{0}}}\right),\left(L_{x_{0}}, h^{L_{x_{0}}}\right),\left(\Lambda\left(T_{x_{0}}^{*(0,1)} X\right), h^{\Lambda_{x_{0}}^{0}}\right),\left(E_{p, x_{0}}, h^{E_{p, x_{0}}}\right)$. Let $\Gamma^{E}, \Gamma^{L}, \Gamma^{B, \Lambda^{0} \bullet}$ be the corresponding connection forms of $\nabla^{E}, \nabla^{L}$ and $\nabla^{B, \Lambda^{0} \bullet}$ on $B^{X}\left(x_{0}, 4 \varepsilon\right)$. Then $\Gamma^{E}, \Gamma^{L}, \Gamma^{B, \Lambda^{0} \bullet}$ are skew-adjoint with respect to $h^{E_{x_{0}}}, h^{L_{x_{0}}}, h^{\Lambda_{x_{0}}^{0, \bullet}}$.

Denote by $\nabla_{U}$ the ordinary differentiation operator on $T_{x_{0}} X$ in the direction $U$. From the above discussion,

$$
\begin{equation*}
\nabla^{E_{p, x_{0}}}=\nabla+\rho(|Z| / \varepsilon)\left(p \Gamma^{L}+\Gamma^{E}+\Gamma^{B, \Lambda^{0, \bullet}}\right)(Z) \tag{1.6.21}
\end{equation*}
$$

defines a Hermitian connection on $\left(E_{p, x_{0}}, h^{E_{p, x_{0}}}\right)$ on $\mathbb{R}^{2 n} \simeq T_{x_{0}} X$ where the identification is given by

$$
\begin{equation*}
\mathbb{R}^{2 n} \ni\left(Z_{1}, \ldots, Z_{2 n}\right) \longrightarrow \sum_{i} Z_{i} e_{i} \in T_{x_{0}} X \tag{1.6.22}
\end{equation*}
$$

Here $\left\{e_{i}\right\}_{i}$ is an orthonormal basis of $T_{x_{0}} X$.
Let $g^{T X_{0}}$ be a metric on $X_{0}:=\mathbb{R}^{2 n}$ which coincides with $g^{T X}$ on $B^{T_{x_{0}} X}(0,2 \varepsilon)$, and $g^{T_{x_{0}} X}$ outside $B^{T_{x_{0}} X}(0,4 \varepsilon)$. Let $d v_{X_{0}}$ be the Riemannian volume form of $\left(X_{0}, g^{T X_{0}}\right)$. Let $\Delta^{E_{p, x_{0}}}$ be the Bochner Laplacian associated to $\nabla^{E_{p, x_{0}}}$ and $d v_{X_{0}}$ on $X_{0}$. Set

$$
\begin{equation*}
L_{p, x_{0}}=\Delta^{E_{p, x_{0}}}-p \rho(|Z| / \varepsilon)\left(2 \omega_{d, Z}+\tau_{Z}\right)-\rho(|Z| / \varepsilon) \Phi_{E, Z} \tag{1.6.23}
\end{equation*}
$$

Then $L_{p}$ is a self-adjoint operator with respect to the scalar product (1.3.14) induced by $h^{E_{p, x_{0}}}, g^{T X_{0}}$. Moreover, $L_{p, x_{0}}$ coincides with $D_{p}^{2}$ on $B^{T X}(0,2 \varepsilon)$.

Let $d v_{T X}$ be the Riemannian volume form on $\left(T_{x_{0}} X, g^{T_{x_{0}} X}\right)$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$
\begin{equation*}
d v_{X_{0}}(Z)=\kappa(Z) d v_{T X}(Z) \tag{1.6.24}
\end{equation*}
$$

with $k(0)=1$.

Let $\exp \left(-u L_{p, x_{0}}\right)\left(Z, Z^{\prime}\right),\left(Z, Z^{\prime} \in \mathbb{R}^{2 n}\right)$ be the smooth kernel of the heat operator $\exp \left(-u L_{p, x_{0}}\right)$ on $X$ with respect to $d v_{X_{0}}\left(Z^{\prime}\right)$.
Lemma 1.6.5. Under the notation in Proposition 1.6.4, the following estimate holds uniformly on $x_{0} \in X$ :

$$
\begin{equation*}
\left|\exp \left(-\frac{u}{2 p} D_{p}^{2}\right)\left(x_{0}, x_{0}\right)-\exp \left(-\frac{u}{2 p} L_{p, x_{0}}\right)(0,0)\right| \leqslant C p^{8 n+8} \exp \left(-\frac{\varepsilon^{2} p}{16 u}\right) \tag{1.6.25}
\end{equation*}
$$

Proof. Let $\widetilde{\mathbf{F}}_{u}, \widetilde{\mathbf{G}}_{u}, \widetilde{\mathbf{H}}_{u, \varsigma}$ be the holomorphic functions on $\mathbb{C}$ such that

$$
\begin{equation*}
\widetilde{\mathbf{F}}_{u}\left(a^{2}\right)=\mathbf{F}_{u}(a), \quad \widetilde{\mathbf{G}}_{u}\left(a^{2}\right)=\mathbf{G}_{u}(a), \quad \widetilde{\mathbf{H}}_{u, \varsigma}\left(a^{2}\right)=\mathbf{H}_{u, \varsigma}(a) \tag{1.6.26}
\end{equation*}
$$

Then $\widetilde{\mathbf{G}}_{u}(u a)=\widetilde{\mathbf{H}}_{u, 1}(a)$ still verifies (1.6.16). And on $\mathbb{R}^{2 n}$, Lemma 1.6.2 still holds uniformly on $x_{0} \in X$, if we replace $D_{p}^{2}$ therein by $L_{p, x_{0}}$. Thus from the proof of Proposition 1.6.4, we still have (1.6.15) for $\widetilde{\mathbf{G}}_{u}\left(u L_{p, x_{0}}\right)$.

Now by the finite propagation speed (Theorem D.2.1), we know that

$$
\mathbf{F}_{\frac{u}{p}}\left(\sqrt{\frac{u}{p}} D_{p}\right) \quad\left(x_{0}, \cdot\right)=\widetilde{\mathbf{F}}_{\frac{u}{p}}\left(\frac{u}{p} L_{p, x_{0}}\right)(0, \cdot)
$$

Thus, we get (1.6.25) by (1.6.14).
Let $S_{L}$ be a unit vector of $L_{x_{0}}$. Using $S_{L}$, we get an isometry $E_{p, x_{0}} \simeq$ $\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x_{0}}=: \mathbf{E}_{x_{0}}$. As the operator $L_{p, x_{0}}$ takes values in $\operatorname{End}\left(E_{p, x_{0}}\right)=$ $\operatorname{End}(\mathbf{E})_{x_{0}}$ (using the natural identification $\operatorname{End}\left(L^{p}\right) \simeq \mathbb{C}$, which does not depend on $S_{L}$ ), thus our formulas do not depend on the choice of $S_{L}$. Now, under this identification, we will consider $L_{p, x_{0}}$ acting on $\mathscr{C}^{\infty}\left(X_{0}, \mathbf{E}_{x_{0}}\right)$. For $s \in \mathscr{C}^{\infty}\left(\mathbb{R}^{2 n}, \mathbf{E}_{x_{0}}\right)$, $Z \in \mathbb{R}^{2 n}$ and $t=\frac{1}{\sqrt{p}}$, set

$$
\begin{align*}
& \left(S_{t} s\right)(Z)=s(Z / t) \\
& \nabla_{t}=S_{t}^{-1} t \kappa^{1 / 2} \nabla^{E_{p, x_{0}}} \kappa^{-1 / 2} S_{t}  \tag{1.6.27}\\
& L_{2}^{t}=S_{t}^{-1} \kappa^{1 / 2} t^{2} L_{p, x_{0}} \kappa^{-1 / 2} S_{t}
\end{align*}
$$

Put

$$
\begin{align*}
& \nabla_{0, \cdot}=\nabla \cdot+\frac{1}{2} R_{x_{0}}^{L}(Z, \cdot) \\
& L_{2}^{0}=-\sum_{i}\left(\nabla_{0, e_{i}}\right)^{2}-2 \omega_{d, x_{0}}-\tau_{x_{0}} \tag{1.6.28}
\end{align*}
$$

Lemma 1.6.6. When $t \rightarrow 0$, we have

$$
\begin{equation*}
\nabla_{t, \cdot}=\nabla_{0, \cdot}+\mathscr{O}(t), \quad L_{2}^{t}=L_{2}^{0}+\mathscr{O}(t) \tag{1.6.29}
\end{equation*}
$$

Proof. Let $g_{i j}(Z)=g_{Z}^{T X_{0}}\left(e_{i}, e_{j}\right)$, and let $\left(g^{i j}(Z)\right)_{i j}$ be the inverse of the matrix $\left(g_{i j}(Z)\right)_{i j}$. Let $\nabla_{e_{i}}^{T X_{0}} e_{j}=\Gamma_{i j}^{k}(Z) e_{k}$. By (1.3.19), we know that on $B(0,4 \varepsilon)$,

$$
\begin{equation*}
\Delta^{B, E_{p}}=-g^{i j}(t Z)\left(\nabla_{e_{i}}^{B, E_{p}} \nabla_{e_{j}}^{B, E_{p}}-\nabla_{\nabla_{e_{i}}^{T X} e_{j}}^{B, E_{p}}\right) \tag{1.6.30}
\end{equation*}
$$

From (1.5.17), (1.6.21), (1.6.23), (1.6.27) and (1.6.30), we get

$$
\begin{align*}
\nabla_{t, \cdot}= & \kappa^{1 / 2}(t Z)\left(\nabla .+\rho(|t Z| / \varepsilon)\left(t \Gamma_{t Z}^{B, \Lambda^{0, \bullet}}+\frac{1}{t} \Gamma_{t Z}^{L}+t \Gamma_{t Z}^{E}\right)\right) \kappa^{-1 / 2}(t Z) \\
L_{2}^{t}= & -g^{i j}(t Z)\left(\nabla_{t, e_{i}} \nabla_{t, e_{j}}-t \Gamma_{i j}^{k}(t Z) \nabla_{t, e_{k}}\right)  \tag{1.6.31}\\
& +\rho(|t Z| / \varepsilon)\left(-2 \omega_{d, t Z}-\tau_{t Z}+t^{2} \Phi_{E, t Z}\right)
\end{align*}
$$

Since $g^{i j}(0)=\delta_{i j},(1.2 .31)$ and (1.6.31) imply (1.6.29).

### 1.6.4 Uniform estimate on the heat kernel

Let $h^{\mathbf{E}_{x_{0}}}$ be the metric on $\mathbf{E}_{x_{0}}$ induced by $h_{x_{0}}^{\Lambda^{0, \bullet}}, h_{x_{0}}^{E}$. We also denote by $\langle\cdot, \cdot\rangle_{0, L^{2}}$ and $\|\cdot\|_{0, L^{2}}$ the scalar product and the $L^{2}$ norm on $\mathscr{C}^{\infty}\left(X_{0}, \mathbf{E}_{x_{0}}\right)$ induced by $g^{T X_{0}}, h^{\mathbf{E}_{x_{0}}}$ as in (1.3.14). For $s \in \mathscr{C}^{\infty}\left(T_{x_{0}} X, \mathbf{E}_{x_{0}}\right)$, set

$$
\begin{align*}
& \|s\|_{t, 0}^{2}:=\|s\|_{0}^{2}=\int_{\mathbb{R}^{2 n}}|s(Z)|_{h^{\mathbf{E}_{x_{0}}}}^{2} d v_{T X}(Z), \\
& \|s\|_{t, m}^{2}=\sum_{l=0}^{m} \sum_{i_{1}, \ldots, i_{l}=1}^{2 n}\left\|\nabla_{t, e_{i_{1}}} \cdots \nabla_{t, e_{i_{l}}} s\right\|_{t, 0}^{2} . \tag{1.6.32}
\end{align*}
$$

We denote by $\left\langle s^{\prime}, s\right\rangle_{t, 0}$ the inner product on $\mathscr{C}^{\infty}\left(X_{0}, \mathbf{E}_{x_{0}}\right)$ corresponding to $\|\cdot\|_{t, 0}^{2}$. Let $\boldsymbol{H}_{t}^{m}$ be the Sobolev space of order $m$ with norm $\|\cdot\|_{t, m}$. Let $\boldsymbol{H}_{t}^{-1}$ be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on $\boldsymbol{H}_{t}^{-1}$ defined by $\|s\|_{t,-1}=$ $\sup _{0 \neq s^{\prime} \in \boldsymbol{H}_{t}^{1}}\left|\left\langle s, s^{\prime}\right\rangle_{t, 0}\right| /\left\|s^{\prime}\right\|_{t, 1}$. If $A \in \mathscr{L}\left(\boldsymbol{H}_{t}^{m}, \boldsymbol{H}_{t}^{m^{\prime}}\right)\left(m, m^{\prime} \in \mathbb{Z}\right)$, we denote by $\|A\|_{t}^{m, m^{\prime}}$ the norm of $A$ with respect to the norms $\|\cdot\|_{t, m}$ and $\|\cdot\|_{t, m^{\prime}}$.

Since $L_{p, x_{0}}$ is formally self-adjoint with respect to $\|\cdot\|_{0, L^{2}}, L_{2}^{t}$ is also a formally self-adjoint elliptic operator with respect to $\|\cdot\|_{t, 0}^{2}$, and is a smooth family of operators with parameter $x_{0} \in X$.

Theorem 1.6.7. There exist constants $C_{1}, C_{2}, C_{3}>0$ such that for $\left.\left.t \in\right] 0,1\right]$ and any $s, s^{\prime} \in \mathscr{C}_{0}^{\infty}\left(\mathbb{R}^{2 n}, \mathbf{E}_{x_{0}}\right)$,

$$
\begin{align*}
& \left\langle L_{2}^{t} s, s\right\rangle_{t, 0} \geqslant C_{1}\|s\|_{t, 1}^{2}-C_{2}\|s\|_{t, 0}^{2}  \tag{1.6.33}\\
& \left|\left\langle L_{2}^{t} s, s^{\prime}\right\rangle_{t, 0}\right| \leqslant C_{3}\|s\|_{t, 1}\left\|s^{\prime}\right\|_{t, 1}
\end{align*}
$$

Proof. Now from (1.4.29) and (1.5.17),

$$
\begin{equation*}
\left\langle L_{p, x_{0}} s, s\right\rangle_{0, L^{2}}=\left\|\nabla^{E_{p, x_{0}}} s\right\|_{0, L^{2}}^{2}+\left\langle\rho\left(\frac{|Z|}{\varepsilon}\right)\left(-2 p \omega_{d}-p \tau+\Phi_{E}\right) s, s\right\rangle_{0, L^{2}} . \tag{1.6.34}
\end{equation*}
$$

From (1.6.24), (1.6.27), (1.6.32) and (1.6.34),

$$
\begin{equation*}
\left\langle L_{2}^{t} s, s\right\rangle_{t, 0}=\left\|\nabla_{t} s\right\|_{t, 0}^{2}-\left\langle\rho(|t Z| / \varepsilon)\left(-2 \omega_{d, t Z}-\tau_{t Z}+t^{2} \Phi_{E, t Z}\right) s, s\right\rangle_{t, 0} \tag{1.6.35}
\end{equation*}
$$

From (1.6.35), we get (1.6.33).


Figure 1.1.

Let $\Gamma$ be the oriented path in $\mathbb{C}$ defined by Figure 1.1.
Theorem 1.6.8. There exists $C>0$ such that for $t \in] 0,1], \lambda \in \Gamma$, and $x_{0} \in X$,

$$
\begin{align*}
& \left\|\left(\lambda-L_{2}^{t}\right)^{-1}\right\|_{t}^{0,0} \leqslant C \\
& \left\|\left(\lambda-L_{2}^{t}\right)^{-1}\right\|_{t}^{-1,1} \leqslant C\left(1+|\lambda|^{2}\right) \tag{1.6.36}
\end{align*}
$$

Proof. As $L_{2}^{t}$ is a self-adjoint differential operator, by (1.6.33), $\left(\lambda-L_{2}^{t}\right)^{-1}$ exists for $\lambda \in \Gamma$. The first inequality of (1.6.36) comes from our choice of $\Gamma$. Now, by (1.6.33), for $\lambda_{0} \in \mathbb{R}, \lambda_{0} \leqslant-2 C_{2},\left(\lambda_{0}-L_{2}^{t}\right)^{-1}$ exists, and we have $\left\|\left(\lambda_{0}-L_{2}^{t}\right)^{-1}\right\|_{t}^{-1,1} \leqslant \frac{1}{C_{1}}$. Then,

$$
\begin{equation*}
\left(\lambda-L_{2}^{t}\right)^{-1}=\left(\lambda_{0}-L_{2}^{t}\right)^{-1}-\left(\lambda-\lambda_{0}\right)\left(\lambda-L_{2}^{t}\right)^{-1}\left(\lambda_{0}-L_{2}^{t}\right)^{-1} \tag{1.6.37}
\end{equation*}
$$

Thus (1.6.37) imply for $\lambda \in \Gamma$

$$
\begin{equation*}
\left\|\left(\lambda-L_{2}^{t}\right)^{-1}\right\|_{t}^{-1,0} \leqslant \frac{1}{C_{1}}\left(1+\frac{1}{C}\left|\lambda-\lambda_{0}\right|\right) . \tag{1.6.38}
\end{equation*}
$$

Now we interchange the last two factors in (1.6.37), apply (1.6.38) and obtain

$$
\begin{align*}
\left\|\left(\lambda-L_{2}^{t}\right)^{-1}\right\|_{t}^{-1,1} & \leqslant \frac{1}{C_{1}}+\frac{\left|\lambda-\lambda_{0}\right|}{C_{1}{ }^{2}}\left(1+\frac{1}{C}\left|\lambda-\lambda_{0}\right|\right)  \tag{1.6.39}\\
& \leqslant C\left(1+|\lambda|^{2}\right)
\end{align*}
$$

The proof of our theorem is complete.
Proposition 1.6.9. Take $m \in \mathbb{N}^{*}$. There exists $C_{m}>0$ such that for $\left.\left.t \in\right] 0,1\right]$, $Q_{1}, \ldots, Q_{m} \in\left\{\nabla_{t, e_{i}}, Z_{i}\right\}_{i=1}^{2 n}$ and $s, s^{\prime} \in \mathscr{C}_{0}^{\infty}\left(\mathbb{R}^{2 n}, \mathbf{E}_{x_{0}}\right)$,

$$
\begin{equation*}
\left|\left\langle\left[Q_{1},\left[Q_{2}, \ldots,\left[Q_{m}, L_{2}^{t}\right] \ldots\right]\right] s, s^{\prime}\right\rangle_{t, 0}\right| \leqslant C_{m}\|s\|_{t, 1}\left\|s^{\prime}\right\|_{t, 1} . \tag{1.6.40}
\end{equation*}
$$

Proof. Note that $\left[\nabla_{t, e_{i}}, Z_{j}\right]=\delta_{i j}$. Thus by (1.6.31), we know that $\left[Z_{j}, L_{2}^{t}\right]$ verifies (1.6.40).

Let $R_{\rho}^{\Lambda^{0} \bullet}, R_{\rho}^{L}$ and $R_{\rho}^{E}$ be the curvatures of the connections $\nabla+\rho(|Z| / \varepsilon) \Gamma^{B, \Lambda^{0}, \bullet}$, $\nabla+\rho(|Z| / \varepsilon) \Gamma^{L}$ and $\nabla+\rho(|Z| / \varepsilon) \Gamma^{E}$. Then by (1.6.21), (1.6.27),

$$
\begin{equation*}
\left[\nabla_{t, e_{i}}, \nabla_{t, e_{j}}\right]=\left(R_{\rho}^{L}+t^{2} R_{\rho}^{\Lambda^{0, \bullet}}+t^{2} R_{\rho}^{E}\right)(t Z)\left(e_{i}, e_{j}\right) \tag{1.6.41}
\end{equation*}
$$

Thus from (1.6.31) and (1.6.41), we know that $\left[\nabla_{t, e_{k}}, L_{2}^{t}\right]$ has the same structure as $L_{2}^{t}$ for $\left.\left.t \in\right] 0,1\right]$, i.e., $\left[\nabla_{t, e_{k}}, L_{2}^{t}\right]$ is of the type

$$
\begin{equation*}
\sum_{i j} a_{i j}(t, t Z) \nabla_{t, e_{i}} \nabla_{t, e_{j}}+\sum_{i} d_{i}(t, t Z) \nabla_{t, e_{i}}+c(t, t Z) \tag{1.6.42}
\end{equation*}
$$

and $a_{i j}(t, Z), d_{i}(t, Z), c(t, Z)$ and their derivatives on $Z$ are uniformly bounded for $Z \in \mathbb{R}^{2 n}, t \in[0,1]$; moreover, they are polynomials in $t$.

Let $\left(\nabla_{t, e_{i}}\right)^{*}$ be the adjoint of $\nabla_{t, e_{i}}$ with respect to $\langle\cdot, \cdot\rangle_{t, 0}$ (see (1.6.32)). Then

$$
\begin{equation*}
\left(\nabla_{t, e_{i}}\right)^{*}=-\nabla_{t, e_{i}}-t\left(\kappa^{-1} \nabla_{e_{i}} \kappa\right)(t Z) \tag{1.6.43}
\end{equation*}
$$

and the last term of (1.6.43) and its derivatives in $Z$ are uniformly bounded in $Z \in \mathbb{R}^{2 n}, t \in[0,1]$.

By (1.6.42) and (1.6.43), (1.6.40) is verified for $m=1$.
By iteration, we know that $\left[Q_{1},\left[Q_{2}, \ldots,\left[Q_{m}, L_{2}^{t}\right] \ldots\right]\right]$ has the same structure (1.6.42) as $L_{2}^{t}$. By (1.6.43), we get Proposition 1.6.9.

Theorem 1.6.10. For any $t \in] 0,1], \lambda \in \Gamma, m \in \mathbb{N}$, the resolvent $\left(\lambda-L_{2}^{t}\right)^{-1}$ maps $\boldsymbol{H}_{t}^{m}$ into $\boldsymbol{H}_{t}^{m+1}$. Moreover for any $\alpha \in \mathbb{N}^{2 n}$, there exist $N \in \mathbb{N}, C_{\alpha, m}>0$ such that for $t \in] 0,1], \lambda \in \Gamma, s \in \mathscr{C}_{0}^{\infty}\left(X_{0}, \mathbf{E}_{x_{0}}\right)$,

$$
\begin{equation*}
\left\|Z^{\alpha}\left(\lambda-L_{2}^{t}\right)^{-1} s\right\|_{t, m+1} \leqslant C_{\alpha, m}\left(1+|\lambda|^{2}\right)^{N} \sum_{\alpha^{\prime} \leqslant \alpha}\left\|Z^{\alpha^{\prime}} s\right\|_{t, m} \tag{1.6.44}
\end{equation*}
$$

Proof. For $Q_{1}, \ldots, Q_{m} \in\left\{\nabla_{t, e_{i}}\right\}_{i=1}^{2 n}, Q_{m+1}, \ldots, Q_{m+|\alpha|} \in\left\{Z_{i}\right\}_{i=1}^{2 n}$, we can express $Q_{1} \ldots Q_{m+|\alpha|}\left(\lambda-L_{2}^{t}\right)^{-1}$ as a linear combination of operators of the type

$$
\begin{equation*}
\left[Q_{1},\left[Q_{2}, \ldots\left[Q_{m^{\prime}},\left(\lambda-L_{2}^{t}\right)^{-1}\right] \ldots\right]\right] Q_{m^{\prime}+1} \ldots Q_{m+|\alpha|} \quad m^{\prime} \leqslant m+|\alpha| \tag{1.6.45}
\end{equation*}
$$

Let $\mathscr{R}_{t}$ be the family of operators $\mathscr{R}_{t}=\left\{\left[Q_{j_{1}},\left[Q_{j_{2}}, \ldots\left[Q_{j_{l}}, L_{2}^{t}\right] \ldots\right]\right]\right\}$. Clearly, any commutator $\left[Q_{1},\left[Q_{2}, \ldots\left[Q_{m^{\prime}},\left(\lambda-L_{2}^{t}\right)^{-1}\right] \ldots\right]\right]$ is a linear combination of operators of the form

$$
\begin{equation*}
\left(\lambda-L_{2}^{t}\right)^{-1} R_{1}\left(\lambda-L_{2}^{t}\right)^{-1} R_{2} \ldots R_{m^{\prime}}\left(\lambda-L_{2}^{t}\right)^{-1} \tag{1.6.46}
\end{equation*}
$$

with $R_{1}, \ldots, R_{m^{\prime}} \in \mathscr{R}_{t}$.

By Proposition 1.6.9, the norm $\left\|\|_{t}^{1,-1}\right.$ of the operators $R_{j} \in \mathscr{R}_{t}$ is uniformly bounded by $C$. By Theorem 1.6.8, we find that there exist $C>0$ and $N \in \mathbb{N}$ such that the norm $\|\cdot\|_{t}^{0,1}$ of operators (1.6.46) is dominated by $C\left(1+|\lambda|^{2}\right)^{N}$.

Let $e^{-u L_{2}^{t}}\left(Z, Z^{\prime}\right)$ be the smooth kernels of the operators $e^{-u L_{2}^{t}}$ with respect to $d v_{T X}\left(Z^{\prime}\right)$. Note that $L_{2}^{t}$ are families of differential operators with coefficients in $\operatorname{End}\left(\mathbf{E}_{x_{0}}\right)=\operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x_{0}}$. Let $\pi: T X \times_{X} T X \rightarrow X$ be the natural projection from the fiberwise product of $T X$ on $X$. Then we can view $e^{-u L_{2}^{t}}\left(Z, Z^{\prime}\right)$ as smooth sections of $\pi^{*}\left(\operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)\right)$ on $T X \times_{X} T X$. Let $\nabla^{\operatorname{End}(\mathbf{E})}$ be the connection on $\operatorname{End}\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)$ induced by $\nabla^{B, \Lambda^{0}, \bullet}$ and $\nabla^{E}$. Then $\nabla^{\operatorname{End}(\mathbf{E})}$ induces naturally a $\mathscr{C}^{m}$-norm for the parameter $x_{0} \in X$.
Theorem 1.6.11. Set $u>0$ fixed; then for any $m, m^{\prime} \in \mathbb{N}$, there exists $C>0$, such that for $t \in] 0,1], Z, Z^{\prime} \in T_{x_{0}} X,|Z|,\left|Z^{\prime}\right| \leqslant 1$,

$$
\begin{equation*}
\sup _{|\alpha|,\left|\alpha^{\prime}\right| \leqslant m}\left|\frac{\partial^{|\alpha|+\left|\alpha^{\prime}\right|}}{\partial Z^{\alpha} \partial Z^{\prime \alpha^{\prime}}} e^{-u L_{2}^{t}}\left(Z, Z^{\prime}\right)\right|_{\mathscr{C} m^{\prime}(X)} \leqslant C . \tag{1.6.47}
\end{equation*}
$$

Here $\mathscr{C}^{m^{\prime}}(X)$ is the $\mathscr{C}^{m^{\prime}}$ norm for the parameter $x_{0} \in X$.
Proof. By (1.6.33) and (1.6.36), (cf. also (C.2.5)), for any $k \in \mathbb{N}^{*}$,

$$
\begin{equation*}
e^{-u L_{2}^{t}}=\frac{(-1)^{k-1}(k-1)!}{2 \pi i u^{k-1}} \int_{\Gamma} e^{-u \lambda}\left(\lambda-L_{2}^{t}\right)^{-k} d \lambda \tag{1.6.48}
\end{equation*}
$$

For $m \in \mathbb{N}$, let $\mathcal{Q}^{m}$ be the set of operators $\left\{\nabla_{t, e_{i_{1}}} \ldots \nabla_{t, e_{i_{j}}}\right\}_{j \leqslant m}$. From Theorem 1.6.10, we deduce that if $Q \in \mathcal{Q}^{m}$, there are $M \in \mathbb{N}, C_{m}>0$ such that for any $\lambda \in \Gamma$,

$$
\begin{equation*}
\left\|Q\left(\lambda-L_{2}^{t}\right)^{-m}\right\|_{t}^{0,0} \leqslant C_{m}\left(1+|\lambda|^{2}\right)^{M} . \tag{1.6.49}
\end{equation*}
$$

Observe that if an operator $Q_{t}$ has the structure and properties after (1.6.42) and $\left\{a_{i j}(t, Z)\right\}$ is uniformly positive, then all the above arguments apply for $Q_{t}$. Next we study $L_{2}^{t *}$, the formal adjoint of $L_{2}^{t}$ with respect to (1.6.32). Then $L_{2}^{t *}$ has the same structure (1.6.31) as the operator $L_{2}^{t}$ (in fact, $L_{2}^{t *}=L_{2}^{t}$ ), especially,

$$
\begin{equation*}
\left\|Q\left(\lambda-L_{2}^{t *}\right)^{-m}\right\|_{t}^{0,0} \leqslant C_{m}\left(1+|\lambda|^{2}\right)^{M} \tag{1.6.50}
\end{equation*}
$$

After taking the adjoint of (1.6.50), we get

$$
\begin{equation*}
\left\|\left(\lambda-L_{2}^{t}\right)^{-m} Q\right\|_{t}^{0,0} \leqslant C_{m}\left(1+|\lambda|^{2}\right)^{M} \tag{1.6.51}
\end{equation*}
$$

From (1.6.48), (1.6.49) and (1.6.51), we have, for $Q, Q^{\prime} \in \mathcal{Q}^{m}$,

$$
\begin{equation*}
\left\|Q e^{-u L_{2}^{t}} Q^{\prime}\right\|_{t}^{0,0} \leqslant C_{m} \tag{1.6.52}
\end{equation*}
$$

Let $\left\|\|_{m}\right.$ be the usual Sobolev norm on $\mathscr{C}^{\infty}\left(\mathbb{R}^{2 n}, \mathbf{E}_{x_{0}}\right)$ induced by $h^{\mathbf{E}_{x_{0}}}=$ $h^{\Lambda\left(T_{x_{0}}^{*(0,1)} X\right) \otimes E_{x_{0}}}$ and the volume form $d v_{T X}(Z)$ as in (1.6.32).

Observe that by (1.6.31), (1.6.32), for any $m \geqslant 0$, there exists $C_{m}>0$ such that for $s \in \mathscr{C}^{\infty}\left(X_{0}, \mathbf{E}_{x_{0}}\right), \operatorname{supp}(s) \subset B^{T_{x_{0}} X}(0,1)$,

$$
\begin{equation*}
\frac{1}{C_{m}}\|s\|_{t, m} \leqslant\|s\|_{m} \leqslant C_{m}\|s\|_{t, m} \tag{1.6.53}
\end{equation*}
$$

Now (1.6.52), (1.6.53) together with Sobolev's inequalities implies that if $Q, Q^{\prime} \in$ $\mathcal{Q}^{m}$,

$$
\begin{equation*}
\sup _{|Z|,\left|Z^{\prime}\right| \leqslant 1}\left|Q_{Z} Q_{Z^{\prime}}^{\prime} e^{-u L_{2}^{t}}\left(Z, Z^{\prime}\right)\right| \leqslant C \tag{1.6.54}
\end{equation*}
$$

Thus by (1.6.31), (1.6.54), we derive (1.6.47) for the case when $m^{\prime}=0$.
Finally, for $U$ a vector on $X$,

$$
\begin{equation*}
\nabla_{U}^{\pi^{*} \operatorname{End}(\mathbf{E})} e^{-u L_{2}^{t}}=\frac{(-1)^{k-1}(k-1)!}{2 \pi i u^{k-1}} \int_{\Gamma} e^{-u \lambda} \nabla_{U}^{\pi^{*} \operatorname{End}(\mathbf{E})}\left(\lambda-L_{2}^{t}\right)^{-k} d \lambda \tag{1.6.55}
\end{equation*}
$$

We use a similar formula as (1.6.46) for $\nabla_{U}^{\pi^{*} \operatorname{End}(\mathbf{E})}\left(\lambda-L_{2}^{t}\right)^{-k}$, where we replace $\mathscr{R}_{t}$ by $\left\{\nabla_{U}^{\pi^{*} \operatorname{End}(\mathbf{E})} L_{2}^{t}\right\}$. Moreover, we remark that $\nabla_{U}^{\pi^{*} \operatorname{End}(\mathbf{E})} L_{2}^{t}$ is a differential operator on $T_{x_{0}} X$ with the same structure as $L_{2}^{t}$. Then the above argument yields (1.6.47) for $m^{\prime} \geqslant 1$.

Theorem 1.6.12. There exists $C>0$ such that for $t \in[0,1]$,

$$
\begin{equation*}
\left\|\left(\left(\lambda-L_{2}^{t}\right)^{-1}-\left(\lambda-L_{2}^{0}\right)^{-1}\right) s\right\|_{0,0} \leqslant C t\left(1+|\lambda|^{4}\right) \sum_{|\alpha| \leqslant 3}\left\|Z^{\alpha} s\right\|_{0,0} \tag{1.6.56}
\end{equation*}
$$

Proof. Remark that by (1.6.31), (1.6.32), for $t \in[0,1], k \geqslant 1$,

$$
\begin{equation*}
\|s\|_{t, k} \leqslant C \sum_{|\alpha| \leqslant k}\left\|Z^{\alpha} s\right\|_{0, k} \tag{1.6.57}
\end{equation*}
$$

An application of Taylor expansion for (1.6.31) leads to the following equation, if $s, s^{\prime}$ have compact support:

$$
\begin{equation*}
\left|\left\langle\left(L_{2}^{t}-L_{2}^{0}\right) s, s^{\prime}\right\rangle_{0,0}\right| \leqslant C t\left\|s^{\prime}\right\|_{t, 1} \sum_{|\alpha| \leqslant 3}\left\|Z^{\alpha} s\right\|_{0,1} \tag{1.6.58}
\end{equation*}
$$

Thus we get

$$
\begin{equation*}
\left\|\left(L_{2}^{t}-L_{2}^{0}\right) s\right\|_{t,-1} \leqslant C t \sum_{|\alpha| \leqslant 3}\left\|Z^{\alpha} s\right\|_{0,1} \tag{1.6.59}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left(\lambda-L_{2}^{t}\right)^{-1}-\left(\lambda-L_{2}^{0}\right)^{-1}=\left(\lambda-L_{2}^{t}\right)^{-1}\left(L_{2}^{t}-L_{2}^{0}\right)\left(\lambda-L_{2}^{0}\right)^{-1} \tag{1.6.60}
\end{equation*}
$$

After taking the limit, we know that Theorems 1.6.8-1.6.10 still hold for $t=0$. From Theorem 1.6.10, (1.6.59) and (1.6.60), we infer (1.6.56).

Theorem 1.6.13. For $u>0$ fixed, there exists $C>0$, such that for $t \in] 0,1]$, $Z, Z^{\prime} \in T_{x_{0}} X,|Z|,\left|Z^{\prime}\right| \leqslant 1$,

$$
\begin{equation*}
\left|\left(e^{-u L_{2}^{t}}-e^{-u L_{2}^{0}}\right)\left(Z, Z^{\prime}\right)\right| \leqslant C t^{1 /(2 n+1)} \tag{1.6.61}
\end{equation*}
$$

Proof. Let $J_{x_{0}}^{0}$ be the vector space of square integrable sections of $\mathbf{E}_{x_{0}}$ over $\left\{Z \in T_{x_{0}} X,|Z| \leqslant 2\right\}$. If $s \in J_{x_{0}}^{0}$, put $\|s\|_{(1)}^{2}=\int_{|Z| \leqslant 2}|s|_{h^{\mathbf{E}}}^{2} d v_{T X}(Z)$. Let $\|A\|_{(1)}$ be the operator norm of $A \in \mathscr{L}\left(J_{x_{0}}^{0}\right)$ with respect to $\left\|\|_{(1)}\right.$. Let $u>0$ fixed. By (1.6.48) and (1.6.56), we get: There exists $C>0$, such that for $t \in] 0,1]$,

$$
\begin{align*}
\left\|\left(e^{-u L_{2}^{t}}-e^{-u L_{2}^{0}}\right)\right\|_{(1)} & \leqslant \frac{1}{2 \pi} \int_{\Gamma}\left|e^{-u \lambda}\right|\left\|\left(\lambda-L_{2}^{t}\right)^{-1}-\left(\lambda-L_{2}^{0}\right)^{-1}\right\|_{(1)} d \lambda  \tag{1.6.62}\\
& \leqslant C^{\prime} t \int_{\Gamma} e^{-u \operatorname{Re}(\lambda)}\left(1+|\lambda|^{4}\right) d \lambda \leqslant C t
\end{align*}
$$

Let $\phi: \mathbb{R}^{2 n} \rightarrow[0,1]$ be a smooth function with compact support, equal 1 near 0 , such that $\int_{T_{x_{0} X}} \phi(Z) d v_{T X}(Z)=1$. Take $\left.\left.\nu \in\right] 0,1\right]$. By the proof of Theorem 1.6.11, $e^{-u L_{2}^{0}}$ verifies the similar inequality as in (1.6.47). Thus by Theorem 1.6.11, there exists $C>0$ such that if $|Z|,\left|Z^{\prime}\right| \leqslant 1, U, U^{\prime} \in \mathbf{E}_{x_{0}}$,

$$
\begin{align*}
\mid\left\langle\left( e^{-u L_{2}^{t}}-\right.\right. & \left.\left.e^{-u L_{2}^{0}}\right)\left(Z, Z^{\prime}\right) U, U^{\prime}\right\rangle \\
& -\int_{T_{x_{0}} X \times T_{x_{0} X}}\left\langle\left(e^{-u L_{2}^{t}}-e^{-u L_{2}^{0}}\right)\left(Z-W, Z^{\prime}-W^{\prime}\right) U, U^{\prime}\right\rangle \\
& \left.\times \frac{1}{\nu^{4 n}} \phi(W / \nu) \phi\left(W^{\prime} / \nu\right) d v_{T X}(W) d v_{T X}\left(W^{\prime}\right)|\leqslant C \nu| U \| U^{\prime} \right\rvert\, \tag{1.6.63}
\end{align*}
$$

On the other hand, by (1.6.62),

$$
\begin{align*}
& \mid \int_{T_{x_{0}} X \times T_{x_{0}} X}\left\langle\left(e^{-u L_{2}^{t}}-e^{-u L_{2}^{0}}\right)\left(Z-W, Z^{\prime}-W^{\prime}\right) U, U^{\prime}\right\rangle \\
& \left.\quad \times \frac{1}{\nu^{4 n}} \phi(W / \nu) \phi\left(W^{\prime} / \nu\right) d v_{T X}(W) d v_{T X}\left(W^{\prime}\right)\left|\leqslant C t \frac{1}{\nu^{2 n}}\right| U| | U^{\prime} \right\rvert\, \tag{1.6.64}
\end{align*}
$$

By taking $\nu=t^{1 /(2 n+1)}$, we get (1.6.61).

### 1.6.5 Proof of Theorem 1.6.1

Note that in (1.6.24), $\kappa(0)=1$. Recall also that $t=1 / \sqrt{p}$. By (1.6.27), for $s \in$ $\mathscr{C}_{0}^{\infty}\left(X_{0}, E_{x_{0}}\right)$,

$$
\begin{align*}
& \left(e^{-u L_{2}^{t}} s\right)(Z)=\left(S_{t}^{-1} \kappa^{\frac{1}{2}} e^{-\frac{u}{p} L_{p}} \kappa^{-\frac{1}{2}} S_{t} s\right)(Z) \\
& =\kappa^{\frac{1}{2}}(t Z) \int_{\mathbb{R}^{2 n}} \exp \left(-\frac{u}{p} L_{p}\right)\left(t Z, Z^{\prime}\right)\left(S_{t} s\right)\left(Z^{\prime}\right) \kappa^{\frac{1}{2}}\left(Z^{\prime}\right) d v_{T X}\left(Z^{\prime}\right) \tag{1.6.65}
\end{align*}
$$

Thus, for $Z, Z^{\prime} \in T_{x_{0}} X$,

$$
\begin{equation*}
\exp \left(-\frac{u}{p} L_{p, x_{0}}\right)\left(Z, Z^{\prime}\right)=p^{n} e^{-u L_{2}^{t}}\left(Z / t, Z^{\prime} / t\right) \kappa^{-1 / 2}\left(Z^{\prime}\right) \kappa^{-1 / 2}(Z) \tag{1.6.66}
\end{equation*}
$$

By Theorem 1.6.13, (1.6.25), (1.6.66), we get that uniformly on $x_{0} \in X$, we have

$$
\begin{equation*}
\exp \left(-\frac{u}{p} D_{p}^{2}\right)\left(x_{0}, x_{0}\right)-p^{n} \exp \left(-u L_{2, x_{0}}^{0}\right)(0,0)=o\left(p^{n}\right) \tag{1.6.67}
\end{equation*}
$$

By (1.5.19), (1.6.28), (E.2.4) and (E.2.5), we get with the convention in Theorem 1.6.1,

$$
\begin{equation*}
\exp \left(-u L_{2}^{0}\right)(0,0)=\frac{1}{(2 \pi)^{n}} \frac{\operatorname{det}\left(\dot{R}_{x_{0}}^{L}\right) \exp \left(2 u \omega_{d, x_{0}}\right)}{\operatorname{det}\left(1-\exp \left(-2 u \dot{R}_{x_{0}}^{L}\right)\right)} \tag{1.6.68}
\end{equation*}
$$

Moreover, for any $j$ fixed,

$$
\begin{equation*}
\exp \left(-2 u a_{j}\left(x_{0}\right) \bar{w}^{j} \wedge i_{\bar{w}_{j}}\right)=1+\left(\exp \left(-2 u a_{j}\left(x_{0}\right)\right)-1\right) \bar{w}^{j} \wedge i_{\bar{w}_{j}} . \tag{1.6.69}
\end{equation*}
$$

From (1.6.67)-(1.6.69), we get (1.6.4).
If $u$ varies in a compact set of $\mathbb{R}_{+}^{*}$, the constant $C$ in (1.6.47) and (1.6.61) is uniformly bounded, so the convergence of (1.6.4) is uniform. The proof of Theorem 1.6.1 is complete.

### 1.7 Demailly's holomorphic Morse inequalities

We will use the notation of Section 1.6.1 and (1.5.14)-(1.5.19).
Let $X(q)$ be the set of points $x$ of $X$ such that $\sqrt{-1} R_{x}^{L}$ is non-degenerate and $\dot{R}_{x}^{L} \in \operatorname{End}\left(T_{x}^{(1,0)} X\right)$ has exactly $q$ negative eigenvalues. Set $X(\leqslant q)=\cup_{i=0}^{q} X(i)$, $X(\geqslant q)=\cup_{i=q}^{n} X(i)$.
Theorem 1.7.1. Let $X$ be a compact complex manifold with $\operatorname{dim} X=n$, and let $L, E$ be holomorphic vector bundles on $X, \operatorname{rk}(L)=1$. As $p \rightarrow \infty$, the following strong Morse inequalities hold for every $q=0,1, \ldots, n$ :

$$
\begin{equation*}
\sum_{j=0}^{q}(-1)^{q-j} \operatorname{dim} H^{j}\left(X, L^{p} \otimes E\right) \leqslant \operatorname{rk}(E) \frac{p^{n}}{n!} \int_{X(\leqslant q)}(-1)^{q}\left(\frac{\sqrt{-1}}{2 \pi} R^{L}\right)^{n}+o\left(p^{n}\right) \tag{1.7.1}
\end{equation*}
$$

with equality for $q=n$ (asymptotic Riemann-Roch-Hirzebruch formula).
In particular, we get the weak Morse inequalities

$$
\begin{equation*}
\operatorname{dim} H^{q}\left(X, L^{p} \otimes E\right) \leqslant \operatorname{rk}(E) \frac{p^{n}}{n!} \int_{X(q)}(-1)^{q}\left(\frac{\sqrt{-1}}{2 \pi} R^{L}\right)^{n}+o\left(p^{n}\right) \tag{1.7.2}
\end{equation*}
$$

Proof. For $0 \leqslant q \leqslant n$, set

$$
\begin{equation*}
B_{q}^{p}=\operatorname{dim} H^{q}\left(X, L^{p} \otimes E\right) \tag{1.7.3}
\end{equation*}
$$

Remark that the operator $D_{p}^{2}$ preserves the $\mathbb{Z}$-grading of the Dolbeault complex $\Omega^{0, \bullet}\left(X, L^{p} \otimes E\right)$. We will denote by $\operatorname{Tr}_{q}\left[e^{-\frac{u}{p} D_{p}^{2}}\right]$ the trace of $e^{-\frac{u}{p} D_{p}^{2}}$ acting on $\Omega^{0, q}\left(X, L^{p} \otimes E\right)$, then we have

$$
\begin{equation*}
\operatorname{Tr}_{q}\left[e^{-\frac{u}{p} D_{p}^{2}}\right]=\int_{X} \operatorname{Tr}_{q}\left[e^{-\frac{u}{p} D_{p}^{2}}(x, x)\right] d v_{X}(x) \tag{1.7.4}
\end{equation*}
$$

Lemma 1.7.2. For any $u>0, p \in \mathbb{N}^{*}, 0 \leqslant q \leqslant n$, we have

$$
\begin{equation*}
\sum_{j=0}^{q}(-1)^{q-j} B_{j}^{p} \leqslant \sum_{j=0}^{q}(-1)^{q-j} \operatorname{Tr}_{j}\left[\exp \left(-\frac{u}{p} D_{p}^{2}\right)\right] \tag{1.7.5}
\end{equation*}
$$

with equality for $q=n$.
Proof. If $\lambda$ is an eigenvalue of $D_{p}^{2}$, set $F_{j}^{\lambda}$ be the corresponding finite-dimensional eigenspace in $\Omega^{0, j}\left(X, L^{p} \otimes E\right)$. We claim that

$$
\begin{equation*}
\bar{\partial}^{L^{p} \otimes E}\left(F_{j}^{\lambda}\right) \subset F_{j+1}^{\lambda}, \quad \text { and } \quad \bar{\partial}^{L^{p} \otimes E, *}\left(F_{j+1}^{\lambda}\right) \subset F_{j}^{\lambda} \tag{1.7.6}
\end{equation*}
$$

In fact, if $s \in F_{j}^{\lambda}$, then $D_{p}^{2} s=\lambda s$. By (1.5.20), $\bar{\partial}^{L^{p} \otimes E}$ commutes with $D_{p}^{2}$, thus $D_{p}^{2} \bar{\partial}^{L^{p} \otimes E} s=\bar{\partial}^{L^{p} \otimes E} D_{p}^{2} s=\lambda \bar{\partial}^{L^{p} \otimes E} s$. In the same way, we get the second equation of (1.7.6).

Thus we have the complex

$$
\begin{equation*}
0 \longrightarrow F_{0}^{\lambda} \xrightarrow{\bar{\partial}^{L^{p} \otimes E}} F_{1}^{\lambda} \xrightarrow{\bar{\partial}^{L^{p} \otimes E}} \cdots \stackrel{\bar{\partial}^{L^{p} \otimes E}}{\longrightarrow} F_{n}^{\lambda} \longrightarrow 0 . \tag{1.7.7}
\end{equation*}
$$

If $\lambda=0$, then $F_{j}^{0} \simeq H^{j}\left(X, L^{p} \otimes E\right)$. If $\lambda>0$, we claim that the complex (1.7.7) is exact. In fact, if $\bar{\partial}^{L^{p} \otimes E} s=0$ and $s \in F_{j}^{\lambda}$, then by (1.5.20),

$$
\begin{equation*}
s=\lambda^{-1} D_{p}^{2} s=\lambda^{-1} \bar{\partial}^{L^{p} \otimes E}\left(\bar{\partial}^{L^{p} \otimes E, *} s\right) \tag{1.7.8}
\end{equation*}
$$

From (1.7.8), we know $s \in \bar{\partial}^{L^{p} \otimes E}\left(F_{j-1}^{\lambda}\right)$. Thus for $\lambda>0$, the complex (1.7.7) is exact and

$$
\begin{equation*}
\sum_{j=0}^{q}(-1)^{q-j} \operatorname{dim} F_{j}^{\lambda}=\operatorname{dim}\left(\bar{\partial}^{L^{p} \otimes E}\left(F_{q}^{\lambda}\right)\right) \geqslant 0 \tag{1.7.9}
\end{equation*}
$$

with equality when $q=n$. Now

$$
\begin{equation*}
\operatorname{Tr}_{j}\left[\exp \left(-\frac{u}{p} D_{p}^{2}\right)\right]=B_{j}^{p}+\sum_{\lambda>0} e^{-\frac{u}{p} \lambda} \operatorname{dim} F_{j}^{\lambda} \tag{1.7.10}
\end{equation*}
$$

(1.7.9) and (1.7.10) yield (1.7.5).

We denote by $\operatorname{Tr}_{\Lambda^{0, q}}$ the trace on $\Lambda^{q}\left(T^{*(0,1)} X\right)$. By (1.6.69), in the notation of (1.5.19),

$$
\begin{equation*}
\operatorname{Tr}_{\Lambda^{0, q}}\left[\exp \left(2 u \omega_{d}\right)\right]=\sum_{j_{1}<j_{2}<\cdots<j_{q}} \exp \left(-2 u \sum_{i=1}^{q} a_{j_{i}}(x)\right) \tag{1.7.11}
\end{equation*}
$$

Thus by the second equality in (1.6.4), $\frac{\left(\operatorname{det}\left(\dot{R}^{L} /(2 \pi)\right)\right)}{\operatorname{det}\left(1-\exp \left(-2 u \dot{R}^{L}\right)\right)} \operatorname{Tr}_{\Lambda^{0, q}}\left[\exp \left(2 u \omega_{d}\right)\right]$ is uniformly bounded for $x \in X, u>1,0 \leqslant q \leqslant n$, and for any $x_{0} \in X, 0 \leqslant q \leqslant n$,

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \frac{\operatorname{det}\left(\dot{R}^{L} /(2 \pi)\right) \operatorname{Tr}_{\Lambda^{0}, q}\left[\exp \left(2 u \omega_{d}\right)\right]}{\operatorname{det}\left(1-\exp \left(-2 u \dot{R}^{L}\right)\right)}\left(x_{0}\right)=1_{X(q)}(-1)^{q} \operatorname{det}\left(\frac{\dot{R}^{L}}{2 \pi}\right)\left(x_{0}\right) \tag{1.7.12}
\end{equation*}
$$

The function $1_{X(q)}$ is defined by 1 on $X(q), 0$ otherwise. From Theorem 1.6.1, (1.7.4) and (1.7.5), we have

$$
\begin{align*}
& \varlimsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^{q}(-1)^{q-j} B_{j}^{p} \\
& \quad \leqslant \operatorname{rk}(E) \int_{X} \frac{\operatorname{det}\left(\dot{R}^{L} /(2 \pi)\right) \sum_{j=0}^{q}(-1)^{q-j} \operatorname{Tr}_{\Lambda^{0, j}}\left[\exp \left(2 u \omega_{d}\right)\right]}{\operatorname{det}\left(1-\exp \left(-2 u \dot{R}^{L}\right)\right)} d v_{X}(x), \tag{1.7.13}
\end{align*}
$$

for any $q$ with $0 \leqslant q \leqslant n$ and any $u>0$. Using (1.7.12), (1.7.13) and dominate convergence, we get

$$
\begin{equation*}
\varlimsup_{p \rightarrow \infty} p^{-n} \sum_{j=0}^{q}(-1)^{q-j} B_{j}^{p} \leqslant(-1)^{q} \operatorname{rk}(E) \int_{\cup_{i=0}^{q} X(i)} \operatorname{det}\left(\frac{\dot{R}^{L}}{2 \pi}\right)(x) d v_{X}(x) \tag{1.7.14}
\end{equation*}
$$

But (1.5.18) entails

$$
\begin{equation*}
\operatorname{det}\left(\frac{\dot{R}^{L}}{2 \pi}\right)(x) d v_{X}(x)=\prod_{j} \frac{a_{j}(x)}{2 \pi} d v_{X}(x)=\left(\frac{\sqrt{-1}}{2 \pi} R^{L}\right)^{n} / n! \tag{1.7.15}
\end{equation*}
$$

Relations (1.7.14), (1.7.15) imply (1.7.1). For $q=n$, we apply (1.7.5) with equality, so we get (1.7.1) with equality. (1.7.2) follows by subtracting inequalities (1.7.1) for $q$ and $q+1$ (or directly from Theorem 1.6.1, (1.7.10) and (1.7.12)).

The proof of Theorem 1.7.1 is complete.

## Problems

Problem 1.1. Verify (1.3.2), (1.3.31) and (1.3.41). With the notation from (1.3.44) verify that

$$
\operatorname{Ker}\left(D^{c, A}\right)=\operatorname{Ker}\left(\left(D^{c, A}\right)^{2}\right)=\operatorname{Ker}\left(D_{+}^{c, A}\right) \oplus \operatorname{Ker}\left(D_{-}^{c, A}\right)
$$

Problem 1.2. In Section 1.3.2, we can always assume that $\nabla^{\mathrm{Cl}}$ on $\Lambda\left(T^{*(0,1)} X\right) \otimes E$ is induced by $\nabla^{T X}, \nabla^{\operatorname{det}_{1}}$ and a Hermitian connection $\nabla_{1}^{E}$ on $\left(E, h^{E}\right)$. (Hint: $\frac{1}{2}\left(\nabla^{\mathrm{det}}-\nabla^{\mathrm{det}_{1}}\right)$ is a purely imaginary 1 -form.)
Problem 1.3. In local coordinates $\left(x_{1}, \ldots, x_{n}\right)$ of a Riemannian manifold ( $X, g^{T X}$ ), we set $f_{j}=\frac{\partial}{\partial x_{j}}, g_{i j}(x)=\left\langle f_{i}, f_{j}\right\rangle_{g^{T X}}(x)$. Let $\left(g^{i j}(x)\right)$ be the inverse of the matrix $\left(g_{i j}(x)\right)$. Verify that in (1.3.19),

$$
\Delta^{F}=-\sum_{i j} g^{i j}(x)\left(\nabla_{f_{i}}^{F} \nabla_{f_{j}}^{F}-\nabla_{\nabla_{f_{i}}^{T X} f_{j}}^{F}\right)
$$

Problem 1.4. In the context of (1.4.5) show that

$$
\operatorname{Ker}(D)=\operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}\left(\bar{\partial}^{*}\right), \quad \operatorname{Im}(\bar{\partial}) \cap \operatorname{Im}\left(\bar{\partial}^{*}\right)=0
$$

Thus $\operatorname{Ker}(D), \operatorname{Im}(\bar{\partial})$ and $\operatorname{Im}\left(\bar{\partial}^{*}\right)$ are pairwise orthogonal.
Problem 1.5. Verify Remark 1.4.3 (cf. also [9, §2]). By Theorem A.3.2 for $k \in \mathbb{Z}$, and $D^{2}$ is elliptic, for $s$ a distribution with values in $E, D^{2} s=0$ in the sense of distributions implies that $s \in \Omega^{0, \bullet}(X, E)$ (cf. also [148, Chap. 3], [238, §7.4]). Using this fact, show that $\operatorname{Ker}(D) \subset \Omega^{0, \bullet}(X, E) \cap L_{0, \bullet}^{2}(X, E)$ is closed in $L_{0, \bullet}^{2}(X, E)$.

By the Schwartz kernel theorem, $P(x, y)$ is a distribution on $X \times X$ with values in $\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{x} \otimes\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{y}^{*}$. Prove first

$$
D_{x}^{2} P(x, y)=0, \quad D_{y}^{2} P(x, y)=0
$$

in the sense of distributions. Here we identify $\left(\Lambda\left(T^{*(0,1)} X\right) \otimes E\right)_{y}^{*}$ to $\left(\Lambda\left(T^{*(0,1)} X\right) \otimes\right.$ $E)_{y}$ by the Hermitian product $\langle\cdot, \cdot\rangle_{\Lambda^{0}, \bullet} \otimes E$. Now as $D_{x}^{2}+D_{y}^{2}$ is an elliptic operator on $X \times X,\left(D_{x}^{2}+D_{y}^{2}\right) P(x, y)=0$ in the sense of distributions implies $P(x, y)$ is $\mathscr{C}^{\infty}$ for $x, y \in X$.
Problem 1.6. Let $X$ be a Kähler manifold.
a) Show that $\left[\partial, \bar{\partial}^{*}\right]=0,\left[\bar{\partial}, \partial^{*}\right]=0$.
b) Show that $\Delta$ commutes with all operators $\partial, \bar{\partial}, \partial^{*}, \bar{\partial}^{*}, L, \Lambda$.
(Hint: Use Theorem 1.4.11 and (1.3.31).)
Problem 1.7. Verify first (1.5.8). Now let $(X, \omega, J)$ be a Kähler manifold. Let $K_{X}:=\operatorname{det}\left(T^{*(1,0)} X\right)$ be the canonical line bundle on $X$. Set

$$
\operatorname{Ric}_{\omega}=\operatorname{Ric}(J \cdot, \cdot)
$$

Using (1.2.55), verify that $R^{T X}$ is a (1, 1 )-form with values in $\operatorname{End}(T X)$. Using (1.2.5), verify that $\operatorname{Ric}_{\omega}=\sqrt{-1} R^{K_{X}^{*}}=\sqrt{-1} \operatorname{Tr}\left[R^{T^{(1,0)} X}\right]$.

Problem 1.8. We will use the homogeneous coordinate $\left(z_{0}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1}$ for $\mathbb{C P}^{n} \simeq\left(\mathbb{C}^{n+1} \backslash\{0\}\right) / \mathbb{C}^{*}$. Denote by $U_{i}=\left\{\left[z_{0}, \ldots, z_{n}\right] \in \mathbb{C P}^{n} ; z_{i} \neq 0\right\},(i=$ $0, \ldots, n)$, the open subsets of $\mathbb{C P}{ }^{n}$, and the coordinate charts are defined by $\phi_{i}$ : $U_{i} \simeq \mathbb{C}^{n}, \phi_{i}\left(\left[z_{0}, \ldots, z_{n}\right]\right)=\left(\frac{z_{0}}{z_{i}}, \ldots, \frac{\widehat{z_{i}}}{z_{i}}, \ldots, \frac{z_{n}}{z_{i}}\right)$. (A hat over a variable means that this variable is skipped.)

Let $\mathscr{O}(-1)$ be the tautological line bundle of $\mathbb{C P}^{n}$, i.e., $\mathscr{O}(-1)=\{([z], \lambda z) \in$ $\left.\mathbb{C P}^{n} \times \mathbb{C}^{n+1}, \lambda \in \mathbb{C}\right\}$. For any $\alpha=\left(\alpha_{0}, \ldots, \alpha_{n}\right) \in \mathbb{N}^{n+1}$, the map $\mathbb{C}^{n+1} \ni z \rightarrow$ $\prod_{j=0}^{n} z_{j}^{\alpha_{j}}$ is naturally identified with a holomorphic section of $\mathscr{O}(-|\alpha|)^{*}=\mathscr{O}(|\alpha|)$ on $\mathbb{C P}^{n}$; we denote it by $s_{\alpha}$.

Let $h^{\mathscr{O}(-1)}$ be the Hermitian metric on $\mathscr{O}(-1)$, as a subbundle of the trivial bundle $\mathbb{C}^{n+1}$ on $\mathbb{C P}^{n}$, induced by the standard metric on $\mathbb{C}^{n+1}$. Let $h^{\mathscr{O}(1)}$ be the Hermitian metric on $\mathscr{O}(1)$ induced by $h^{\mathscr{O}(-1)}$. Let $\omega_{F S}=\frac{\sqrt{-1}}{2 \pi} R^{\mathscr{O}(1)}$ be the $(1,1)$-form associated to $\left(\mathscr{O}(1), h^{\mathscr{O}(1)}\right)$ defined by (1.5.14).

On $U_{i}$, the trivialization of the line bundle $\mathscr{O}(1)$ is defined by $\mathscr{O}(1) \ni s \rightarrow$ $s / z_{i}$, here $z_{i}$ is considered as a holomorphic section of $\mathscr{O}(1)$.

We work now on $\mathbb{C}^{n}$ by using $\phi_{0}: U_{0} \rightarrow \mathbb{C}^{n}$. Verify that for $z \in \mathbb{C}^{n}$,

$$
\left|s_{(1,0, \ldots, 0)}\right|_{h \mathscr{O}(1)}^{2}(z)=\left(1+\sum_{j=1}^{n}\left|z_{j}\right|^{2}\right)^{-1}
$$

From (1.5.8), verify that for $z \in \mathbb{C}^{n}$,

$$
\omega_{F S}(z)=\frac{\sqrt{-1}}{2 \pi}\left(\frac{\sum_{j=1}^{n} d z_{j} d \bar{z}_{j}}{1+\sum_{j=1}^{n}\left|z_{j}\right|^{2}}-\frac{\sum_{j=1}^{n} \bar{z}_{j} d z_{j} \wedge \sum_{k=1}^{n} z_{k} d \bar{z}_{k}}{\left(1+\sum_{j=1}^{n}\left|z_{j}\right|^{2}\right)^{2}}\right)
$$

Thus $\omega_{F S}$ is a Kähler form on $\mathbb{C} \mathbb{P}^{n}$. $\omega_{F S}$ is called the Fubini-Study form, and the associated Riemannian metric $g_{F S}^{T \mathbb{P}^{n}}$ is the Fubini-Study metric on $\mathbb{C P}^{n}$.
Problem 1.9. Let $f$ be a harmonic function on a connected compact manifold $X$, i.e., $\Delta f=0$. Show that $f$ is constant on $X$. (Hint: $\int_{X}|d f|^{2} d v_{X}=\int_{X} \bar{f}(\Delta f) d v_{X}$ ). Problem 1.10. Consider a real (1,1)-form $\alpha \in \Omega^{1,1}(X)$. Let us choose the local orthonormal frame $\left\{w_{j}\right\}_{j=1}^{n}$ such that $\alpha=\sqrt{-1} \sum_{j=1}^{n} c_{j}(x) w^{j} \wedge \bar{w}^{j}$ at a given point $x \in X$. For any $u=\sum_{I, J} u_{I J} w^{I} \wedge \bar{w}^{J} \in \Omega^{\bullet \bullet}(X)$, from (1.4.37), (cf. (1.4.61)), we have

$$
[\alpha, \Lambda] u=\sum_{I, J}\left(\sum_{j \in I} c_{j}(x)+\sum_{j \in J} c_{j}(x)-\sum_{j=1}^{n} c_{j}(x)\right) u_{I J} w^{I} \wedge \bar{w}^{J}
$$

Problem 1.11. (a) (Nakano vanishing theorem) Let $X$ be a compact Kähler manifold and $\left(E, h^{E}\right)$ be a Nakano-positive vector bundle over $X$ (cf. Definition 1.1.6). Show that $H^{q}\left(X, E \otimes K_{X}\right)=0$ for any $q \geqslant 1$.
(b) On $T^{(1,0)} \mathbb{C} \mathbb{P}^{n}$ we consider the Fubini-Study metric. Show that $T^{(1,0)} \mathbb{C} \mathbb{P}^{n}$ $\otimes \mathscr{O}(p)$ is Nakano-positive for $p \geqslant 1$. Deduce that $T^{(1,0)} \mathbb{C} \mathbb{P}^{n} \otimes K_{\mathbb{C}}^{*} \mathbb{P}^{n}$ is Nakanopositive and that $H^{q}\left(\mathbb{C} \mathbb{P}^{n}, T^{(1,0)} \mathbb{C} \mathbb{P}^{n}\right)=0$ for $q \geqslant 1$.

Note: The case $q=1$ in (b) implies that the complex structure of $\mathbb{C} \mathbb{P}^{n}$ cannot be deformed (cf. [179, Ch.1, Th. $\gamma]$ ).
Problem 1.12. (a) Let $\left(E, h^{E}\right)$ be a holomorphic Hermitian vector bundle. Show that if $\left(E, h^{E}\right)$ is Nakano-positive, then $\left(E, h^{E}\right)$ is Griffiths-positive.
(b) Define the vector bundle $E=\mathbb{C} \mathbb{P}^{n} \times \mathbb{C}^{n+1} / \mathscr{O}(-1)$ over $\mathbb{C} \mathbb{P}^{n}$. Show that $E$ is Griffiths-positive but not Nakano-positive.

Note: The notion of Griffiths-positivity is more suitable for the study of ampleness than that of Nakano positivity. For more details see [79, Ch. VI], [217].
Problem 1.13. Verify that if $M$ is a weakly pseudoconvex domain (i.e., the Levi form is positive semi-definite), and $L$ is a positive line bundle on $\bar{M}$, then the spectral gap property for Kodaira Laplacian similar to Theorem 1.5.5 still holds. Problem 1.14. For $q=n$, prove directly (1.7.1) with equality (use Theorem 1.4.6).

### 1.8 Bibliographic notes

In Section 1.2 .1 we basically follow [15, $\S 1.2]$. For basic material concerning manifolds, vector bundles and Riemannian geometry we refer to [85], [252], [140] and [179]. The proof of Lemmas 1.2.3 and 1.2.4 appeared in [10, Appendix II].

A good references for Section 1.3 is [148, Appendix D]. Instead of referring to $[148$, Appendix D], $[160, \S 2]$ for a construction of the Clifford connection on $\Lambda\left(T^{*(0,1)} X\right)$, we define it here directly and verify its properties. The Atiyah-Singer index theorem was established in [11]. The Riemann-Roch-Hirzebruch theorem appears in Hirzebruch's Habilitation thesis [130] for an algebraic variety $X$. In [15, Chap. 4], the readers can find a heat kernel approach to the Atiyah-Singer index theorem.

Section 1.3.3 and Theorems 1.4.5, 1.4.7 are taken from [26], where Bismut used them to prove a local index theorem for modified Dirac operators.

The Kähler identities for Kähler manifolds were proved by A. Weil [251] using the primitive decomposition theorem. Ohsawa [187] used the approach of Weil for non-Kähler metrics and showed the existence of the Hermitian torsion operator satisfying the generalized Kähler identities. Theorem 1.4.11 and the Bochner-Kodaira-Nakano formula (1.4.44) were proved in this precise form by Demailly [73]. For (1.4.63) see also Kodaira-Morrow [179, Ch. 3, Th. 6.2].

Bochner-Kodaira-Nakano formulas with boundary term similar to (1.4.72) were proved by Andreotti-Vesentini [7, p. 113] and Griffiths [119, (7.14)]. Estimate (1.4.84) is a more geometric version of the famous Morrey-Kohn-Hörmander estimate $[143,131,108]$, which is essential in the solution of the $\bar{\partial}$-Neumann problem (cf. also Section 3.5).

Section 1.5. Theorems 1.5.7 and 1.5.8 are [160, Th. 1.1 and 2.5] if $A=0$. If $A=0$, Borthwick-Uribe [43] and Braverman [54] observed also (1.5.29). (1.5.23) was first proved by Bismut and Vasserot [35, Th. 1.1] by using the Bochner-Kodaira-Nakano formula [73, Th. 0.3].

Theorem 1.6.1 was first proved by Bismut in [25] by using probability theory. Demailly [74] and Bouche [48] gave a different approach. Our proof is new and is inspired by the analytic localization techniques of Bismut-Lebeau [33, §11]. Certainly, the argument here works well for the modified Dirac operator.

Theorem 1.7.1 represents Demailly's holomorphic Morse inequalities [72]. The proof in Section 1.7 is Bismut's heat kernel proof of Theorem 1.7.1.

Demailly's work [72] was influenced by Witten's seminal analytic proof of Morse inequalities [253] for a Morse function $f$ with isolated critical points on a compact manifold. In [24], Bismut gave a heat kernel proof of Morse inequalities and of the degenerate Morse inequalities. Subsequently, in [25], he adapted his heat kernel proof of Morse inequalities for Demailly's holomorphic Morse inequalities. Milnor's book [176] is the standard reference for the classical Morse theory. For the analytic proof of classical Morse inequalities, we refer our readers to the interesting recent book [263]. In the literature, there exists another type of holomorphic Morse inequalities [254, 175, 256], which relate the Dolbeault cohomology groups of the fixed point set $X^{G}$ of a compact connected Lie group $G$ acting on a compact Kähler manifold $X$ to the Dolbeault cohomology groups of $X$ itself.

