Lecture 4 – Connexion, Holonomy and Covariant Derivatives **by L. Ni**

A fiber bundle is a triple (E, F, M) with a projection map $p : E \to M$ such that p is regular with $p^{-1}(x)$ being diffeomorphic to F such that for any point $p \in M$, there exists U_α and $\varphi : U_\alpha \times F \to p^{-1}(U_\alpha)$ such that $\varphi(x, f) \in p^{-1}(x)$. We say it has a *structure group* G, if the transition functions $T_{\alpha\beta}(x)$ (where $\varphi_\beta^{-1} \circ \varphi_\alpha(x, f) = (x, T_{\alpha\beta}(x)(f))$ is in G. For example $(TM, \mathbb{R}^n, M)$ is the tangent bundle with the transition group being $GL(\mathbb{R}^n)$. If M is endowed with a Riemannian metric $(TM, \mathbb{R}^n, M)$ can have a structure group of $\mathsf{O}(n)$. In general there are vector bundles of any dimension k. Of course we can define its tensor bundles and exterior product bundles (the structure group will be the corresponding representation of $GL(\mathbb{R}^n)$ or $\mathsf{O}(n)$ acting on the corresponding tensor products). In general the concept admits a lot more cases. One example is $(\widetilde{M}, F, M)$ with $\widetilde{M}$ being a covering of M, F being the discrete set of isolated points can be viewed as a very special case. For this the structure group is $\sigma(\pi_1(M))$, the monodromy group of the covering (with $\sigma : \pi_1(M) \to$ the permutation group of the discrete sets F). Another is the *principle bundle*, with F being a Lie group G and the the structure group being G acting by the left multiplication.

A connexion of (E, F, M) is for any piece-wisely smooth path $\gamma : (0, 1) \to M$, there exists $\varphi_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ such that it satisfies that φ depends on γ smoothly, and $\varphi_{\gamma_1 \circ \gamma_2} = \varphi_{\gamma_1} \circ \varphi_{\gamma_2}$ and $\varphi_{\gamma^{-1}} = (\varphi_\gamma)^{-1}$. Such φ_γ is called the *parallel transport* along γ. In general φ is in $\mathrm{Diff}(F)$, but in specific cases it lies inside the structure group G correspondingly.

These concepts can be pushed further into the case of that (E, M) being topological spaces and p being continuous and having path-lifting and covering homotopy properties (so-called *Serre fibration*). There one has correspondingly the *homotopy connexion*, which is a homotopy equivalence of the fibers $p^{-1}(\gamma(0))$ and $p^{-1}(\gamma(1))$. The example of covering spaces is one where the homotopy connexion is the monodromy.

Let $\Omega(x_0, M)$ be the *loop spaces* at x_0. Then φ_γ is a homomorphism $\varphi : \Omega(x_0, M) \to \mathrm{Diff}(F_{x_0})$ (or G). Then the image (denoted by H_{x_0}) is called the *holonomy group*. For covering spaces, this is just the monodromy group. Most often emphasizes are given to the image of the connected component of the trivial loop $\gamma(t) \equiv x_0$, namely the loops which are homotopically trivial. This is called the relative holonomy group, denoted by $H^0_{x_0}$. It is easy to see that for a different choice of the base point x_1, if γ is a path from x_0 to x_1, then $H_{x_0} = \varphi_{\gamma^{-1}} H_{x_1} \varphi_\gamma$.

A *covariant derivative* at point p is a map $\nabla : T_pM \times \mathcal{T}_pM \to T_pM$ satisfying axioms: (i) $\nabla_{\alpha\xi+\beta\eta} Y = \alpha\nabla_\xi Y + \beta\nabla_\eta Y$; (ii) linear in the second component; (iii) $\nabla_\xi(fY) = (\xi f)Y + f\nabla_\xi Y$. This is also called an *affine connection*. The covariant derivative is a concept more linear than the Lie derivative since for smooth vectors X, Y and function f, $\nabla_{fX} Y = f\nabla_X Y$, a property fails to hold for the Lie derivative. A *global affine connection* is the one defined for all $p \in M$ satisfying that if X, Y are smooth $\nabla_X Y$ is smooth. Once M is endowed with a global affine connection we can define the *covariant derivative along a curve* $c(t) : (a, b) \to M$ (even along a smooth map $\phi, N \to M$) for a *vector field* $X(t)$ along $c(t)$ by $\frac{D}{dt} X(c(t)) = \nabla_{\dot{c}(t)} X$, if X is defined globally near $c(t)$. This leads to the connexion defined above via the concept of *parallel transport* along $c(t)$. For any $X_{x_0} \in T_{x_0}M$ and a curve $\gamma(t)$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, $X(t) \in T_{\gamma(t)}M$ can be constructed by solving the ODE $\frac{D}{dt} X(t) = 0$. Then define $\varphi_\gamma(X(0)) = X(1)$. We extends the definition to $\varphi_\gamma^{t_1,t_2} : T_{\gamma(t_1)}M \to T_{\gamma(t_2)}M$ as $\varphi_\gamma^{t_1,t_2}(\xi) = X(t_2)$ if X is parallel with $X(t_1) = \xi$. Note that all this discussion make sense for a smooth vector bundle (E, M) of rank k. A basic result, which asserts that a connexion on a vector bundle (with linear structure group) is equivalent to an affine connection, is

following:

Lemma 0.1.

$$\frac{D}{dt}X(t)\bigg|_{t_0} = \lim_{t\to t_0} \frac{\varphi_\gamma^{t,t_0}(X(t)) - X(t_0)}{t - t_0}.$$

As before we can extends the covariant derivative to the whole tensor spaces $\mathcal{T}_s^r(M)$ and show that it preserves the type and commutes with the contraction. Once there exists an affine connection one can define the geodesics by requiring the curve $c(t)$ satisfies $\frac{D\dot{c}}{dt} = 0$. Note that the concept of geodesic is a little nonlinear, only makes sense for affine connections on TM.

On a Riemannian manifold, there exists a canonical affine connection called *Levi-Civita* connection ∇. A Levi-Civita satisfies two more requirements. (i) It is torsion free (namely for any smooth vector fields X, Y, $[X, Y] = \nabla_X Y - \nabla_Y X$); (ii) and it is compatible with the metric (namely $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Y Z\rangle$). The induced connexion of a metric compatible connection is the parallel transports in $\mathsf{O}(n)$.

Lemma 0.2. *Let ∇ be a torsion free connection. For $\omega \in \Omega^k(M)$,*

$$d\omega(X_0, X_1, \cdots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \cdots, \hat{X}_i, \cdots, X_k). \tag{1}$$

Proof. Note that the right hand side above, using that the covariant derivative is commuting with the contraction, can be written as

$$\sum_{i=0}^{k} (-1)^i \left[X_i(\omega(X_0, \cdots, \hat{X}_i, \cdots X_k) - \sum_j \omega(X_0, \cdots \hat{X}_i, \cdots, \nabla_{X_i} X_j, \cdots, X_k) \right]$$

where the second term the summation is for $j \neq i, 0 \leq j \leq k$ and the $\nabla_{X_i} X_j$ can appear before the i-th term.

On the other hand the Corollary 1 of last lecture gives that another expression whose first summand matches the first above. The second summand can be written as

$$\begin{aligned} & \sum_{0\leq i<j\leq k} (-1)^{i+j} \omega(\nabla_{X_i} X_j - \nabla_{X_j} X_i, X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k) \\ = & \sum_{i=0}^{k} (-1)^{i-1} \sum_{j>i} \omega(X_0, \cdots, \hat{X}_i, \cdots, \nabla_{X_i} X_j, \cdots, X_k) \\ & + \sum_{i=0}^{k} (-1)^{j-1} \sum_{j>i} \omega(X_0, \cdots, \nabla_{X_j} X_i, \cdots, \hat{X}_j, \cdots, X_k). \end{aligned}$$

Putting them together we have the claim. □

One may view the right hand side as a derivative d_∇ induced by the covariant derivative on the forms. We remark that equation (1) holds for any 1-forms ω implies that ∇ is torsion

free. Namely

$$\begin{aligned} 0 &= (d_\nabla - d)\alpha(X,Y) = \nabla_X \alpha(Y) - \nabla_Y \alpha(X) - d\alpha(X,Y) \\ &= X(\alpha(Y)) - \alpha(\nabla_X Y) - Y(\alpha(X)) + \alpha(\nabla_Y X) - X(\alpha(Y)) + Y(\alpha(X)) + \alpha([X,Y]) \\ &= \alpha([X,Y] - \nabla_X Y + \nabla_Y X). \end{aligned}$$

This implies the connection is torsion free. The lemma simply states that it coincides with the exterior derivative if the affine connection is torsion free.

The torsion of an affine connection is defined as $T(x,y) = -(\nabla_X Y - \nabla_Y X - [X,Y])$, with X, Y being the extension of $x, y \in T_pM$, which can be easily checked to be a tensor and skew-symmetric. The curvature tensor R is defined to be for any $x, y, z \in T_pM$, $\mathrm{R}_{x,y}z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$ with X, Y, Z being extensions of x, y, z. Generally for vector bundle (E, M) we may denote D as the connection and define similarly $\mathrm{R}_{x,y}s = -D_X D_Y s + D_Y D_X s + D_{[X,Y]} s$ and check that it does not depend on the extension of s. It can be checked that $\mathrm{R}_{x,y}s = -\mathrm{R}_{y,x}s$. Both can be defined for connexions of principal bundles (which probably goes back to Cartan, see the paper of Ambrose-Singer).

If ∇ is torsion free on TM then R satisfies the 1-*st Bianchi identity*:

$$\mathrm{R}_{x,y}z + \mathrm{R}_{y,z}x + \mathrm{R}_{z,x}y = 0. \tag{2}$$

In fact for X, Y, Z extensions of x, y, z we have

$$\begin{aligned} \mathrm{R}_{x,y}z + \mathrm{R}_{y,z}x + \mathrm{R}_{z,x}y &= -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z - \nabla_Y \nabla_Z X + \nabla_Z \nabla_Y X \\ &\quad + \nabla_{[Y,Z]} X - \nabla_Z \nabla_X Y + \nabla_X \nabla_Z Y + \nabla_{[Z,X]} Y \\ &= -\nabla_X [Y,Z] + \nabla_Y [X,Z] + \nabla_Z [Y,X] + \nabla_{[X,Y]} Z \\ &\quad + \nabla_{[Y,Z]} X + \nabla_{[Z,X]} Y \\ &= [[Y,Z],X] + [[Z,X],Y] + [[X,Y],Z] = 0. \end{aligned}$$

Note that this and the above lemma on the exterior derivative only make sense for affine connections on TM the tangent bundle.

The next two results however holds for general affine connections on vector bundles. Note that R can be viewed as a section of $\wedge^2(M) \otimes E^* \otimes E$. Then $D_x R_{y,z}$ can be defined if there exists ∇ on TM.

The curvature of a torsion free affine connection also satisfies the 2-*nd Bianchi idenity*:

$$D_x \mathrm{R}_{y,z} + D_y \mathrm{R}_{z,x} + D_z \mathrm{R}_{x,y} = 0. \tag{3}$$

The proof (we only present for the case $E = TM$) is quite similar to the above. Note that $\nabla_x R_{y,z}$ is defined independent of the extensions X, Y, Z and W. Moreover, since the covariant derivative commutes with the contraction, if we choose the extension so that $[X,Y] = [Y,Z] = [X,Z] = 0$, we have that

$$\begin{aligned} (D_X \mathrm{R}_{Y,Z})W &= D_X(\mathrm{R}_{Y,Z}W) - \mathrm{R}_{Y,Z}(D_X W) - \mathrm{R}_{\nabla_X Y, Z} W - \mathrm{R}_{Y, \nabla_X Z} W \\ &= D_X(-D_Y D_Z W + D_Z D_Y W) - (-D_Y D_Z D_X W + D_Z D_Y D_X W) \\ &\quad - \mathrm{R}_{\nabla_X Y, Z} W - \mathrm{R}_{Y, \nabla_X Z} W. \end{aligned}$$

The claimed result follows by permuting X, Y, Z and using the torsion vanishing assumption. There exists a more general formulation of the second Bianchi idenity via an exterior

derivative d_D defined on forms valued in vector bundles after defining affine connections on vector bundles.

Next is a useful commutator formula. First for any $T \in \mathcal{T}^r_s(M)$ we define invariantly the second derivative

$$\begin{aligned}
\nabla^2_{Y,X} T(\cdot) &\doteqdot \nabla^2 T(\cdot, X, Y) = \nabla_Y(\nabla T)(\cdot, X) \\
&= Y(\nabla T(\cdot, X)) - \nabla T(\nabla_Y(\cdot), X) - \nabla T(\cdot, \nabla_Y X) \\
&= Y(\nabla_X T(\cdot)) - \nabla_X T(\nabla_Y(\cdot)) - \nabla_{\nabla_Y X} T(\cdot) \\
&= \nabla_Y \nabla_X T(\cdot) - \nabla_{\nabla_Y X} T(\cdot).
\end{aligned}$$

Lemma 0.3. *Let $T \in \mathcal{T}^r_s(M)$ and ∇ be a torsion free affine connection. Then*

$$\nabla^2_{Y,X} T - \nabla^2_{X,Y} T = \mathrm{R}_{X,Y} \circ T. \tag{4}$$

Here $\mathrm{R}_{X,Y}\circ$ means that $\mathrm{R}_{X,Y} : T_pM \to T_pM$ is extended to the tensor product as an algebraic derivation.

Proof. First for $W^* \in \mathcal{T}^0_1$, $(\nabla_X W^*)(Z) = X(W^*(Z)) - W^*(\nabla_X Z)$, Hence

$$\begin{aligned}
(\nabla_Y \nabla_X W^*)(Z) &= Y((\nabla_X W^*)(Z)) - (\nabla_X W^*)(\nabla_Y Z) \\
&= YX(W^*(Z)) - Y(W^*(\nabla_X Z)) - X(W^*(\nabla_Y Z)) + W^*(\nabla_X \nabla_Y Z).
\end{aligned}$$

Hence if extends X, Y with $[X, Y] = 0$ we have that

$$(\nabla_Y \nabla_X W^*)(Z) - (\nabla_X \nabla_Y W^*)(Z) = -W^*(R_{X,Y} Z) = [-(R_{X,Y})^*(W^*)]\,(Z).$$

Namely $\nabla^2_{Y,X} W^* - \nabla^2_{X,Y} W^* = \nabla_Y \nabla_X W^* - \nabla_X \nabla_Y W^* = -(R_{X,Y})^*(W^*)$. This proves the result for the special case $T \in \mathcal{T}^0_1$. The general case follows by a similar argument. Precisely

$$\begin{aligned}
&(\nabla_Y(\nabla_X T))(W_1^*, \cdots, W_r^*, X_1, \cdots, X_s) = YX(T(W_1^*, \cdots, W_r^*, X_1, \cdots, X_s)) \\
&- \sum_{i=1}^r Y(T(W_1^*, \cdots, \nabla_X W_i^*, \cdots, W_r^*, X_{\dots})) - \sum_{k=1}^s Y(T(W_{\dots}^*, X_1, \cdots, \nabla_X X_k, \cdots, X_s)) \\
&- \sum_{i=1}^r X(T(W_1^*, \cdots, \nabla_Y W_i^*, \cdots, W_r^*, X_{\dots})) - \sum_{k=1}^s X(T(W_{\dots}^*, X_1, \cdots, \nabla_Y X_k, \cdots, X_s)) \\
&+ \sum_{i \neq j} T(W_1^*, \cdots, \nabla_X W_i^*, \cdots, \nabla_Y W_j^*, \cdots, W_k^*, X_{\dots}) \\
&+ \sum_{k \neq l} T(W_{\dots}^*, X_1, \cdots, \nabla_X X_k, \cdots, \nabla_Y X_l, \cdots, X_s) \\
&+ \sum_{i=1}^r T(W_1^*, \cdots, \nabla_X \nabla_Y W_i^*, \cdots, W_r^*, X_{\dots}) + \sum_{k=1}^s T(W_{\dots}^*, X_1, \cdots, \nabla_X \nabla_Y X_k, \cdots, X_s)
\end{aligned}$$

The claimed result follows by subtracting from the above the same equation with X, Y swapped. □

Another proof can be done by assuming $T = X_1 \otimes \cdots \otimes X_r \otimes W_1^* \otimes \cdots \otimes W_s^*$ and applying $\nabla_Y \nabla_X - \nabla_X \nabla_Y$ into via the Leibniz rule. Once M is endowed with a metric we define

$\Delta T = \nabla^2_{e_i,e_i} T$ for an othonormal frame $\{e_i\}$. Note that this lemma applies to the affine connections on any rank k vector bundle $(E, \mathbb{R}^k, M)$ over the manifold M. To make it sensible a torsion free connection ∇ on TM and an affine connection $D : T_pM \times \Gamma(E) \to E_p$ is needed. The second derivative $D^2_{X,Y}T \doteqdot D_X D_Y T - D_{\nabla_X Y} T$ can be defined invariantly. The above result holds for sections of the tensor bundle $T^r_s(E)$. In this case $R_{x,y} \doteqdot D^2_{y,x} - D^2_{x,y}$. The same holds for the second Bianchi identity (namely true for general vector bundle).

For the Levi-Civita connection there is a Koszul formula uniquely determines the connection:

$$(5)\quad 2\langle Z, \nabla_Y X\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle.$$

The curvature tensor now satisfies further identities:

$$\langle \mathrm{R}_{x,y} z, w\rangle = -\langle \mathrm{R}_{x,y} w, z\rangle; \qquad \langle \mathrm{R}_{x,y} z, w\rangle = \langle \mathrm{R}_{z,w} x, y\rangle.$$

The proof of these two can be found almost on every book on Riemannian geometry. In fact even the first equation holds for a metric connection on general vector bundle. Namely $\langle \mathrm{R}_{x,y} Z, W\rangle = -\langle \mathrm{R}_{x,y} W, Z\rangle$ for Z, W sections of the vector bundle which is endowed with a metric and a metric compatible connection D.

For (M^n, g) a Riemannian manifold with $p \in M$, let γ be a closed path at p. We also use γ to denote the parallel transport along γ, which is an isometry of M_p, the tangent space at p. The Riemannian curvature tensor has a geometric interpretation via the holonomy. First for any $x, y, z, w \in M_p$, define

$$\langle \gamma(\mathrm{R}^q)_{x,y} z, w\rangle \doteqdot \langle \mathrm{R}_{\gamma^{-1}(x),\gamma^{-1}(y)} \gamma^{-1}(z), \gamma^{-1}(w)\rangle.$$

Here we also use γ to denote the parallel transport along γ from M_q to M_p.

Recall that H_p, the holonomy group at $p \in M$, is defined as the group consisting of all such $\gamma \in \mathsf{O}(M_p)$. A result of De Rham asserts that if the action of H_p on M_p is reducible then the universal cover of M splits accordingly into ΠM_i such that each factor M_i with $H_p(M_i)$ being one of the invariant subspaces. This suggests that we make the assumption that H_p acts irreducibly on M_p.

The relative holonomy group H^0_p is a path-connected subgroup of $\mathsf{O}(M_p)$, hence it is Lie subgroup of $\mathsf{SO}(M_p)$. A basic result is that H^0_p is a closed sub-group of $\mathsf{SO}(n)$ (a result of Borel-Lichnerowicz). The basic theorem of Ambrose-Singer sets the ground in understanding the relation between the curvature and the holonomy group.

Theorem 0.1 (Ambrose-Singer)**.** *When γ varies among all (piece-wisely smooth) pathes from q to p, and x, y vary among all vectors in M_p, $\gamma(\mathrm{R}^q)_{x,y}$ generates the holonomy algebra, namely the Lie algebra $\mathfrak{h}$, of H^0_p.*

In fact the result was formulated and proved for general principal bundles (which includes the special case of Riemannian geometry where the associated principal bundle is the orthonormal frame bundle which is a principal $\mathsf{O}(n)$ bundle). But we shall focus on the Riemannian setting and prove the first half of the result in the next lecture.

A fundamental result of Berger states:

Theorem 0.2 (Berger)**.** *Assume that H^0_p acts irreducibly on M_p. Then either H^0_p acts transitively on $\mathbb{S}^{n-1}$ or (M, g) is a locally symmetric space of rank ≥ 2.*

Recall that (M, g) is locally symmetric if $\nabla \mathrm{R} = 0$, namely the curvature tensor is parallel. The condition $\nabla \mathrm{R} = 0$ is equivalent to that for any p and $q \neq p$, and any path γ from q to p, $\gamma(\mathrm{R}^q) = \mathrm{R}^p$. A locally symmetric space has rank ≥ 2 if for any p, there exists $W \subset M_p$ with $\dim(W) \geq 2$ and $\mathrm{R}_{x,y} = 0$ for any $x, y \in W$. Since there is a classification on subgroups of $\mathsf{SO}(n)$ which acts transitively on $\mathbb{S}^{n-1}$, the above result implies a complete classifications of the holonomy group H^0 for Riemannian manifolds.

If the honolomy group H^0 of (M, g) is not $\mathsf{SO}(n)$ we say that (M, g) has a *special honolomy.* The above result of Berger implies that besides the locally symmetric spaces, there are only finite possibilities for the special holonomy groups since there exists a complete classification of subgroups of $\mathsf{SO}(n)$ which does act transitively on $\mathbb{S}^{n-1}$. The study of manifolds with special holonomy is an important subject in Riemannian geometry. For a Riemannian manifold with special holonomy group H with Lie algebra $\mathfrak{h}$, and for any loop γ at p, the parallel transport $\gamma \in H$, via part of the Ambrose-Singer Theorem (proved in the next lecture) implies that $\gamma(\mathrm{R})_{x,y} \in \mathfrak{h}$ for any x, y (recalling $R(x \wedge y) = -R_{x,y}$), in particular we have that $\gamma(R)(\mathfrak{h}) \subset \mathfrak{h}$.

For the proof of Theorem 0.2 the following algebraic formulation holds the key. We call $S = \{V, \mathrm{R}, G\}$ a *holonomy system* if V is a Euclidean space of dimension n endowed with an inner product, G is a connected compact subgroup of $\mathsf{O}(n)$, and R is an algebraic curvature operator on V. Namely R is a $(3, 1)$-tensor which satisfies the 1st Bianchi identity and other symmetries, namely $\mathrm{R} \in S^2_B(\wedge^2 V)$. Furthermore we assume that for any x, y, $\mathrm{R}_{x,y} \in \mathfrak{g}$, the Lie algebra of G. The system S is called irreducible if G acts irreducibly on V. This formulation reducing the problem to a algebraic one is the first step in Simons' proof of Theorem 0.2, allows one to focus on algebraic issue and to apply algebraic methods. This can also be seen in several works (some joint) of Wilking including the celebrated work of Böhm-Wilking on Ricci flow of positively curved metrics (Ann. Math. 2006).