HOW LIKELY IS IT?

AN INTRODUCTION TO PROBABILITY FOR YOUNG PEOPLE

by

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Preliminary version. Comments welcome.

TO DAVID, MY STAR PUPIL.

How likely is it?

An introduction to probability.

Probability is a branch of mathematics which helps people figure out how likely it is that some events will occur. For example, if you toss a fair coin, how likely is it that it will come up heads? Probability assigns the fraction $\frac{1}{2}$ to this. That means for every two tosses of the coin, you can expect heads once.. This doesn't happen for every two tosses, but on the average heads will come up half the time. In probability we study the mathematical consequences of more complicated situations. Suppose, for instance, that a coin is tossed four times. What is the probability that heads will come up exactly twice? The answer is not $\frac{1}{2}$ (it's actually $\frac{3}{8}$) although it might seem as if it should be. The answers given by probability are not based on ESP ; instead, they are based on some basic assumptions and some clever ideas for how to count things and do calculations. The predictions made by probability are very important in making decisions in business, science, and other professions.

Usually, probability is not taught until the end of high school or college. However, there is no reason why younger students cannot learn much of the material, now that everyone has access to calculators. If possible, you should get a calculator that has a "factorial" key and an exponential key. But you should not think that probability is only calculating things. It is also a system of ideas about how to think about how likely things are.

What is involved in studying probability is learning new ways to count and also thinking about how to translate words into mathematics. There are many examples worked out in the book, and many exercises. You might try to work out the examples yourself, and compare your answer and method with the one given. Then try the exercises. If you get stuck, try to find an example that might be similar, but easier. Some of the exercises are very easy, but a few are very hard, and you should not be discouraged if there are some that you cannot do.

In this book it is assumed that you know fractions and decimals. If you have experience with computer languages, that will be very useful, because you might already have an idea of what a function is, and what it means to assign a letter to represent a number. Also, there are some computer exercises included.

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1.The idea of probability.

In this chapter we will learn how to find a number, a fraction or decimal between 0 and 1, which represents how likely it is that a certain event will happen. For example, if the event occurs 4 out of 5 times, we will say that the probability is 4/5.

1.1. Sets

In mathematics ordinary words are used in a very precise way. We will need to give exact meanings to a few words before we can say what we mean by the probability that something will happen. By a set we'll mean a collection of objects of any sort, for example, sets of numbers, sets of toys, sets of symbols. Sets are described by writing the objects inside curly brackets: {assorted stuff}.

Examples:

1. {A, B, C} is the set consisting of the first three letters of the alphabet.
2. {0,1,2,3,4} is the set of the first 5 whole numbers.
3. {0,2,4,6,8} is the set of the first 5 even numbers.
4. {Washington, Adams, Jefferson, Madison} is the set of the first four U.S. presidents.

Exercises:

Write the following sets.

1. The set of the last four letters of the alphabet.
2. The set of the first three whole numbers.
3. The set of all the different U.S. coins now in use.
4. The set of all even numbers less than 20.
5. The set of all vowels.
6. The set all odd numbers between 10 and 30.

To save the trouble of writing and saying a given set each time we refer to it, we'll write a letter to stand for the set. This is called naming the set, just as people are named. For instance, let's write S for the set {0,1,2,3}. We describe this naming process by writing the equation

$$S = \{0,1,2,3\}.$$

Then we can use the letter S instead of writing the set. For example, we can say

S contains four numbers.

instead of saying that {0,1,2,3} contains four numbers. The objects belonging to a set are called the <u>elements</u> of the set. If S is again the set {0,1,2,3} we can write

S has four elements.

1.2 Sample spaces.

We shall use sets to describe possible things that can happen. Suppose that a coin is tossed. It comes up either heads or tails. The set of all possible outcomes is called the <u>sample space</u>. We can write the sample space in the coin tossing situation as {H,T}, where H stands for heads coming up, and T stands for tails coming up. If we write S for the sample space, then

$$S = \{H,T\}.$$

The elements of a sample space are called the <u>outcomes</u>. In the example above the outcomes are H and T.

Examples:

5. If a die is tossed once, find the sample space.

Solution. The outcomes are the numbers from 1 to 6, each of which might come up. Therefore the sample space is {1,2,3,4,5,6}.

6. Suppose a ball is picked from a jar containing some black balls and some white balls. Find the sample space.

Solution. The sample space is {B,W}, where B stands for the outcome of picking a black ball, and W for that of a white ball.

Exercises:

7. Suppose a 6-sided die with the numbers 2,4,6,8,10,12 is tossed. What is the sample space?

8. If a ball is picked from a jar containing black, white, and red balls, what is the sample space?

1.3. **Ordered Pairs.**

How can we describe all the possible outcomes when a coin is tossed more than once? For this we'll use the notation of ordered pairs or ordered triples , etc. Suppose a coin is tossed twice. We'll write (H,T) for the outcome that heads came up first and then tails. (Notice that we are using round brackets for ordered pairs, while for sets we use curly brackets.) Then (T,H) will stand for the outcome that the first toss was a tail and the second a head. If both tosses are heads we'll write (H,H) for that outcome. The pairs we write down are called ordered pairs because the order matters, that is (H,T) means something different from (T,H). In contrast, when we write sets, the order is not important, so that {T,H} does mean the same as {H,T}.

Examples:

7. A coin is tossed twice. It lands tails both times. Write an expression for this outcome

Solution. This outcome is expressed by the ordered pair (T,T).

8. A die is tossed three times. It comes up 3, then 4, then 1. What is this outcome?

Solution. The outcome can be expressed as the ordered triple (3,4,1).

9. Two students are chosen to be in a school play. First Jane is chosen, then Fred. Write an ordered pair for this outcome.

Solution. This outcome is (Jane, Fred).

Exercises:

9. A die which is tossed twice comes up 5 then 2. Write this as an ordered pair.

10. Two white balls and then one black ball are picked from a jar. Write this outcome as an ordered triple.

11. First Jack, then Jill, go tumbling down a hill. Express this as an ordered pair.

1.4. Events.

Now we can write the sample space of all possible outcomes in more complicated situations. Suppose that a coin is tossed twice. What can happen? There are four possibilities: One possibility is that heads come up both times, that is, (H,H). A second possibility is that tails comes up both times, (T,T). The other two possibilities are heads first, then tails, that is, (H,T) or tails first and then heads, which we write (T,H). Then the sample space of all possible outcomes is

$$\{ (H,H),(T,T),(H,T),(T,H) \}.$$

If the coin is tossed three times then there are many more possiblities. We can write the 8 possible outcomes as

$$\{ (H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,T,T), (T,T,H), (T,H,T), (T,H,T) \}$$

Obviously, writing down the sample space can get very complicated. In the next chapter we shall discuss ways that this counting can be made easier.

We will be interested in knowing how likely it is that some of the outcomes in a sample space will occur. For instance, how likely is it that if you toss a coin three times that you will get exactly one head? We define an <u>event</u> as a set of outcomes. An event contains some, but not necessarily all, of the elements of the sample space. For example, the event of getting exactly one head in three tosses is the set

$$\{ (H,T,T), (T,H,T), (T,T,H) \}.$$

(A collection of some, but not necessarily all, of the elements of a set S is called a <u>subset</u> of S. An event is a subset of the sample space.)

Examples.

10. A die is tossed twice. What is the event that the sum of the two tosses is 6?

Solution. Written as a set of outcomes, this event is

$$\{ (1,5), (2,4), (3,3), (4,2), (5,1) \}.$$

12. A coin is tossed 5 times. What is the event that the coin lands on the same side all 5 times ?

Solution. This event is

$$\{ (H,H,H,H,H) , (T,T,T,T,T) \}.$$

Exercises:

12. A coin is tossed three times. Write the event that tails appear exactly once as a set of outcomes.

13. A die is tossed twice. Write a set for the event that the sum of the two tosses is five.

14. A coin is tossed four times. What is the event that heads appear at least once?

15. Two balls are picked from a jar containing some black and some white balls. Write a set for the event that no white ball is picked.

1.5. **Probability of an event.**

Now we can give a precise meaning to the probability of an event occuring. We want to say what we mean by an event occurring, say, 4 out of 5 times. We think of an event as a subset of the sample space, that is, as a set of outcomes: { (H,T,T) , (T,H,T), (T,T,H) }. So it contains a certain number of outcomes. For example, the event of getting exactly one head in three tosses of a coin contains three outcomes. The sample space for three coin tosses contains eight outcomes (The number of outcomes in the sample space is always as least as big as the number of outcomes in any event, since any outcome is in the sample space.)

Let's assume now that all the outcomes in the sample space are equally likely to happen. Then the probability of the event is defined as the number of outcomes in the event divided by the number of outcomes in the sample space. The probability of any event is a number bigger than or equal to 0 and less than or equal to one. If an event is impossible, its probability is zero. If an event must occur,its probability is one. As a fraction, the numerator of the probability is the number of times the event will occur, and the denominator is the total number of possible outcomes.

Examples.

13. A coin is tossed once. What is the probability that it will come up heads?

Solution: The sample space is {H,T}, which contains two outcomes. The event of coming up heads is {H}, containing one outcome. So the probability is 1/2.

14. A die is tossed once. What is the probability that the number that comes up will be less than three?

Solution: The sample space is {1,2,3,4,5,6},which contains six elements The event of coming up less than three is {1,2}, which contains two elements. Therefore the probability is 2/6, which is equal to 1/3.

15. Suppose a coin is tossed three times. What is the probability of getting exactly two heads?

Solution: The sample space (see (3) above) contains 8 elements. The event of getting exactly two heads is

{ (H,H,T), (T,H,H), (H,T,H) } ,

which contains 3 elements. Therefore the probability is 3/8.

16. A die is tossed twice. What is the probability that the sum of the two tosses will be either either seven or eleven?

Solution: First we will write down the whole sample space:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4.6)

(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)

(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

which contains 6 rows and 6 columns and therefore 36 events. The event of getting a sum of seven or eleven is

{ (1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (5,6), (6,5)} .

Therefore the probability is 8/36 or 2/9.

Exercises:

15. What is the probability of getting exactly one head if a coin is tossed twice?

16. A die is tossed once. What is the probability that the number which appears will be bigger than two?

17. If a coin is tossed three times, what is the probablity that the same side will come up all three times?

18. A die is tossed twice. What is the probability that the sum is 8?
(Hint: the sample space contains 36 elements.)

19. A die is tossed twice. What is the probability that a 5 will appear on at least one of the rolls?

20. If a die is tossed twice, what is the probability that the sum will be be larger than nine?

Experiments:

1. Toss two dice 36 times. (This is easier than tossing one die twice 36 times.) Make a note of the number of times that the sum is 8. How does this compare with your answer to Exercise 18?

2. Toss two dice 36 times. Note how many times a 5 has appeared on at least one of the die. Compare this with your answer for Exercise 19.

Study and discussion questions.

1. What is a sample space? Give an example.
2. What are the outcomes in a sample space. What is an event? Give examples.
3. What is the probability of an event?
4. What does it mean if the probability of an event is 1?

Chapter 1 Worksheet

1. Find the sample space if a coin is tossed twice.

2. A ball is picked from a jar containing red, white, and blue tennis balls. What is the sample space ?

3. A coin is tossed twice. What is the probability of getting heads at least once?.

4. Write the set of all positive whole numbers which are multiples of 10 and are less than 100.

5. Write the sample space if a jelly bean is picked from a jar containing red, green, yellow and black jelly beans.

6. A die is tossed twice. What is the probability that the sum is either 2 or 12?

7 A coin is tossed 5 times. What is the event that heads come up exactly once?

8. A 12 sided die has the numbers 1 through 12 . What is the probability that the number which comes up will be less than 4?

9. A student is picked from a group of 2 girls and 3 boys. What is the probability that the student picked is a girl?

10. A deck contains 52 cards, of which 4 are aces. One card is picked. How likely is it that the card picked is an ace?

2 Counting.

Maybe you think you learned how to count in kindergarten! When you first learned how to count you probably were taught how to arrange things so that you counted every object exactly once. Here we'll do the same thing, but for much complicated situations. To calculate probabilities, we will need to find the number of elements in big sample spaces, such as what you find when two or more coins are tossed. The numbers can get really big, so you're encouraged to use a calculator.

2.1 Functions.

A <u>function</u>, which we'll call f, is a rule which assigns to one number, call it n, another number, written f(n). The rule is usually described by writing an equation for f which describes the calculation of f(n). For example, the equation

$$f(n) = n + 2$$

says that the rule f assigns to any number the number plus two Therefore it assigns to 3 the number 5, it assigns to 7 the number 9, and it assigns to 20 the number 22. We write these as

$$f(3)=5, \quad f(7) = 9, \quad \text{and} \quad f(20) = 22.$$

We will use the usual mathematical shorthand and write 2n for two times the number n, 3n for three times the number n, and so forth. For instance, the equation

$$g(n) = 5n$$

means that g is the rule which assigns to a number five times that number. Then

$$g(1) = 5, \quad g(3) = 15, \text{ and } g(20) = 100.$$

Examples.

1. If $f(n) = n^2$, find f(4).

Solution: $f(4) = 4^2 = 16$.

2. If $f(n) = n + (n-1) + ... + 2 + 1$, find f(5).

Solution: The dot dot dot ,written "..." , means add all the other terms that belong there.

For instance, the next term is n-2, and the one after that is n-3. Then f(5) = 5 + 4 + 3 + 2 + 1 = 15.

3. Write an equation for the function which assigns to every whole number the next whole number.

Solution: $f(n) = n + 1$.

4 If $g(n) = \frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{2} + \frac{1}{1}$, find g(7).

Solution: We have $g(7) = \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1}$. Let's use a calculator for this and express the answer in terms of a decimal. Up to 4 decimal places the answer is 2.5928.

Exercises.

1 Suppose $f(n) = n^2 + 1$. Find f(7).

2. If $f(n) = n + (n-1) + (n-2)$, find f(10) .

3. For $f(n) = 4n + 5$, find f(2).

4. Write an equation for the rule which assigns to each whole number the whole number before it.

5. Write an equation for the rule which assigns to each whole number the product of it with the next whole number.

6. For $g(n) = \frac{2}{n} + \frac{2}{(n-1)} + \ldots \frac{2}{2} + \frac{2}{1}$, find g(5) as a decimal.

Computer exercise:

Write a program to calculate

$$h(n) = (\frac{1}{n})^2 + (\frac{1}{n-1})^2 \ldots (\frac{1}{2})^2 + (\frac{1}{1})^2 .$$

Try very large numbers for n. It is a fact, but very hard to see why, that h(n) is never bigger than 2.

2.2. **Basic rule of counting.**

We know that there are six possibilities if we toss a die once. We shall see that if we toss it twice there are 6 times 6, or 36, possible outcomes. If we toss a coin and then a die, there are 2 times 6 or 12 possible outcomes.

The basic rule of counting says that the outcomes multiply :

Suppose two experiments are to be done. If the first experiment has the number k possible outcomes and the second experiment has the number n possible outcomes, then the total number of possible outcomes when both experiments are performed is the product k times n.

To see why this rule works, we need some systematic way of writing down all the possible outcomes. Suppose that the first experiment has 5 possible outcomes (so that $k = 5$) that we label A1, A2, A3, A4, A5. Suppose that the second experiment has 3 possible outcomes (so that $n = 3$), which we label B1, B2, B3. Then we may write down all the possible outcomes by first making a row of those where A1 comes up, then a row where A2 comes up, and so forth:

(A1, B1)	(A1, B2)	(A1, B3)
(A2, B1)	(A2, B2)	(A2, B3)
(A3, B1)	(A3, B2)	(A3, B3)
(A4, B1)	(A4, B2)	(A4, B3)
(A5, B1)	(A5, B2)	(A5, B3)

Since there are 5 rows and 3 columns, there are a total of 5 times 3 ordered pairs.

Examples.

5 If an 8-sided die is tossed and then a 12-sided one, how many different outcomes are possible?

Solution: There are 8 possible outcomes from the first experiment and 12 possible outcomes from the second, so there are 8 times 12 or 96 possible outcomes.

6. If an alphabet, such as ours, has 26 letters, how many different two letter words can there

be? (You can assume that any two letter combination is possible, even if you cannot pronounce it!)

Solution: There are 26 possible outcomes in choosing the first letter, and 26 possible outcomes om choosing the second letter. Therefore there are 26 times 26 or 676 possible outcomes.

7. Write down the sample space of all possible outcomes when a coin is tossed and then a six-sided die.

Solution. We'll write first the row of all outcomes when the coin comes up heads, then a row of outcomes when the coin comes up tails.

(H,1) (H,2), (H,3), (H,4), (H,5), (H,6)

(T,1), (T,2), (T,3), (T,4), (T,5), (T,6)

8. Two students, one boy and one girl, are to be chosen from a group of 5 boys and 8 girls. How many possible choices are there?

Solution: We'll think of choosing the boy as the first experiment, and choosing the girl as the second experiment. There are 5 possible outcomes for the first, and 8 possible outcomes for the second. Therefore there are 5 times 8 or 40 outcomes in the sample space.

Exercises:

7. A six sided die is tossed and then an eight sided one is tossed. How many different outcomes are possible?

8. How many two letter combinations can be made using only the first ten letters of the alphabet?

9. Write down the sample space of outcomes when a fruit is picked from a bowl containing only apples and oranges and then another fruit is picked from a bowl containing bananas, pears, and lemons.

10. If an experiment with 15 possible outcomes is done twice, what is the size of the sample

space?

11. How many ways can 2 students be picked from a group of 10 fifth graders and 8 sixth graders if one student is to be from firth grade and one from sixth grade?

12. How many different 2-place letter number combinations are there if the letter must be put first?

2.3. The general rule of counting.

Let us now think about how to count the possiblities when more than two experiments are done. For instance, suppose that a die is tossed three times. What is the size of the sample space? The answer is 6 times 6 times 6 (written 6^3) or 216. To see this, we think of performing two experiments. The first is that of tossing a die twice. We know from before that there are 36 outcomes in the sample space. The second experiment is tossing a die once, which has 6 outcomes. By the counting rule for two experiments, we know that the the total number of outcomes is 36 times 6 or 216.

We can think this way to figure out the number of outcomes for any three experiments. We group together the first two and think of them as a new experiment. Then we just have to think about two experiments. For example, if we toss a 6 sided die, then an 8 sided die, then a 10 sided die, there are 6 times 8 or 48 possible outcomes for the first two tosses, and 10 possible outcomes for the third toss. Therefore there are a total of 48 times 10 or 480 outcomes.

The general rule of counting is the following.

If a number of experiments are done, then the total number of possible outcomes is the product of all the outcomes of each of the experiments.

Examples.

9. If a coin is tossed 5 times, what is the size of the sample space?

Solution: Each coin tossing has 2 possible outcomes. There, the total number of outcomes is 2 times 2 times 2 times 2 times 2 (= 2^5) or 32.

10. How many 3 letter initials are possible? (Assume that all letters can be used.)

Solution: The choice of the first letter of the initial is an experiment with 26 possible outcomes. The same is true of the choice of the second and third letters. Therefore there are 26 times 26 times 26 (= 26^3) or 17,576 possible initials.

11. How many different 6-place license plates are possible if the first three places can be any letters and the last three places can be any numbers.

Solution: The answer is going to be a really big number! We know from example 2 that there are 17,576 possible three letter combinations. Now we have to figure out how many three number cominations there can be. For each place we have 10 possible outcomes, namely the numbers 0,1,2,3,4,5,6,7,8,9.

Therefore, there are 10 times 10 times 10 (= 10^3) or 1,000 possible outcomes.

The total number of outcomes is 17,576 times 1,000 or 17,576,000 possible outcomes!

12. A group of 3 students, consisting of one each from fifth, sixth, and seventh grades, is to be chosen. They are to be chosen from 10 possible fifth graders, 8 possible sixth graders, and 3 possible seventh graders. How many different groups could be chosen?

Solution. There are 10 possible outcomes from the choice of a fifth grader, 8 from the choice of the sixth grader, and 3 from the choice of a seventh grader. The total number of outcomes is 10 times 8 times 3 or 240.

Exercises.

13. What is the size of the sample space if a coin is tossed 6 times?

14. How many possibilities are there if a 6-sided die is tossed 3 times?

15. How many 4 letter words can be made using only the first 10 letters of the alphabet?

16. Suppose that an ice cream store offers 32 flavors of ice cream, 2 types of ice cream cones, and 3 different toppings to be put on the ice cream cone. How many choices are there for an

ice cream cone with one kind of ice cream and one topping?

17. How many different 7-place license plates are possible if the first 2 places are for letters and the last 5 are for numbers?

18. In Exercise 4, assume that 2 flavors (which may be either the same or different) will be used in each cone. How many choices are there in that case?

19. How many 7-place license plates are possible if only numbers are used? Do you see why license plates like these are not practical if a state has more than 10 million cars registered?

3. Permutations and combinations.

3.1 Permutations.

Suppose that we have the letters a, b, and c on three small squares of paper, and we want to count the number of different ways that we can arrange them in a row. These arrangements are called permutations. If we put the letter a first there are two possibilities:

a b c   and   a c b .

If b is first the possibilities are

b a c   and   b c a .

If c is first there are again two:

c a b   and   c b a .

There are a total of six outcomes. We can think of this in the following way. There are 3 choices for the first letter. After the first letter is chosen there are only 2 choices for the second letter. After choosing the second letter, we have only one choice for the third. Therefore, we can use the general counting rule we learned in the last section to conclude that there are 3 times 2 times 1 or 6 possible ways to arrange the letters.

We can use the same reasoning to think about the different ways of arranging a given number of objects. Suppose, for instance, that we want to find all the different ways of arranging 4 objects in a row. We have 4 choices for the first, then 3 for the second (since we've already used one) then only 2 for the third, and just 1 for the fourth. Then there are 4 times 3 times 2 times 1 or 24 different arrangements of the four objects.

There is a special mathematical function which solves this kind of problem. It is called the factorial function and is written n! Maybe you've seen this on a calculator that has scientific functions. Its definition is

$$n! = n(n-1)(n-2)...3 \cdot 2 \cdot 1 \ .$$

This means that you multiply together the number times the number minus one and so forth until you get down to multiplication times 1. For instance

$$3! = 3 \text{ times } 2 \text{ times } 1 = 3 \cdot 2 \cdot 1 = 6 \ .$$

$$4! = 4 \text{ times } 3 \text{ times } 2 \text{ times } 1 = 4 \cdot 3 \cdot 2 \cdot 1 = 24,$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120.$$

(We are using the dot notation for multiplication : $5 \cdot 4$ means 5 times 4.

Factorials get big very fast. Use your calculator to see how big 10! is.

By continuing the kind of reasoning that we have been doing, you can see that the number of different ways of arranging a number , called n, objects in a row is n!.

Examples.

1. How many different ways can 5 letters be arranged?

Solution. There are 5! or 120 different arrangements.

2. In how many different batting orders can a team of nine baseball players be arranged ?

Solution. There are 9! or 362,880 different batting orders.

3. Jack has 9 books that he wants to put on a bookshelf. Of these books, 4 are mysteries, 3 are adventures, and 2 are science fiction. How many different ways can these be arranged if the books of the same type are to be kept together?

Solution. This is a tough one! There are many choices, that is, experiments, to be made. Suppose he first decides which subject should come first, then second, then third. Fro example, he might arrange the subjects in this order:

adventure(3 books) mysteries(4 books) science fiction(2 books) .

Since there are three types of books, there are three choices for the first type of book, two choices for the second, and one for the third, which gives 3 times 2 times 1 or 6 choices for arranging the types of books. Once this has been chosen, there are 4! ways to arrange the mysteries, 3! ways to arrange the adventure books, and 2! ways to arrange the science fiction books. Multiplying together all these possible choices, we get $6 \cdot 4! \cdot 3! \cdot 2!$ possible choices, or 1,728 possible arrangements.

4. Suppose that Jack arranges his books in random order, that is, suppose that all ways of arranging the books are equally likely. What is the probability that all the books of the same

type will be kept together?

Solution: There are 9! or 362,880 different ways of arranging the nine books. This is the size of the sample space. The event that all the books of each type are kept together has 1,728 possible outcomes, as we saw in Example 4.

Therefore the probability is 1728/362880 or approximately .00476. That's not very likely!

5. A penny, nickel, dime, quarter, and half dollar are to be arranged in a row. How many different ways can this be done if the penny must be at the left end ?

Solution: If the penny is at the left end there are 4! or 24 ways to arrange the other 4 coins in the row.

Exercises:

1. In how many different ways can a group of 10 girls form a line?

2. How many different 6 letter combinations can be made from the letters abcdef ?

3. In a game, 4 pegs, one red, one green, one blue, and one white, are arranged in a row by one player, and the other player must guess the arrangement. How many possiblities are there?

4. Three families go together to a baseball game, and sit together on one row. The first two families have 4 people each, while the third has 5. Suppose that each family sits together. How many possible ways are there for the families to arrange themselves on the row? (Hint: look at Example 3 above.)

5. Twelve plates of different colors are to be displayed in a row a cabinet. Of the 12, 6 are large, 4 are medium-sized and 2 are small. Suppose that all the plates of one size should be next to each other. In how many ways can they be arranged?

6. Suppose that the twelve plates in Exercise 5 are arranged in random order, that is, suppose that all orders are equally likely. What is the probability that all the plates of a single size will be together ?

7. This is a tough one! Suppose in the game described in Exercise 3 all the arrangements are equally likely. What is the probability that the red peg will be next to both the white one and

the blue one?

3.2. Combinations.

Suppose that five students are to be chosen from your class to be the cast in a class play. . How many different possibilities are there for a 5 person cast? (Let's be fair and assume that all your classmates can act!) A similar question can be asked about choosing cards from a deck. How many different 5-card hands can be dealt from an ordinary 52-card deck?

We will see that there is a neat formula to answer both these questions and similar ones. To understand the formula , we will start with a simple example where we can write down everything Let us assume that 3 coins, a penny, a nickel, and a dime, are in a box. If 2 coins are chosen from the box, what are the possibilities? If we think about this in terms of choices, there are 3 possibilities for the first coin chosen, and 2 for the second:

PN PD NP ND DP DN,

giving $3 \cdot 2 = 6$ ways of choosing. The problem with this is that we have counted the combination of penny and nickel (PN) as different from the combination of nickel and penny (NP). In fact, we've counted all the combinations exactly 2 times each. To find the actual number of possiblities we need to divide by 2. Therefore, the number of combinations of 2 coins chosen from 3 is $\frac{3 \cdot 2}{2}$ or 3:

PN PD ND.

Now let us make the problem more complicated. Suppose that there are 5 coins, say a penny, nickel, dime, quarter, and half dollar. If 3 coins are to be chosen, what are all the possibilities. We will think about this in the same way as before, but without writing down all the possibilities. There are 5 choices for the first coin, 4 choices for the second, and 3 for the third, giving a total of $5 \cdot 4 \cdot 3$ or 60 choices. Again, since we do not care about the order in which the coins are picked, we have counted each group of coins more than once. Since any permutation of the 3 coins leads to the same group of coins and therefore 3! permutations of the coins, we have counted each group exactly 3! times. Therefore, the number of groups of

3 coins chosen from a set of 5 coins is $\frac{5\cdot4\cdot3}{3!}$ or 10 .In fact, we can write down these 10 choices:

PND PNQ PNH PDQ PDH PQH NDQ NDH NQH DQH .

Example.

6. How many different sets of 2 coins can be chosen from a penny, nickel, dime, and quarter?

Solution. There are 4 choices for the first, 3 for the second, giving $4\cdot3 = 12$ ways of choosing. Since the order does not matter, and the 2 coins can be interchanged, we have to divide by 2! Therefore, the answer is $4\cdot3$ divided by 2! or 6. Another way to do this is just to write them all down:

PN PD PQ ND NQ DQ.

7. How many different sets of 4 balls can be chosen from a set of 5 balls containing one each in the colors white, black, red, yellow, and green?

Solution. There are 5 choices for the first, 4 for the second, 3 for the third, and 2 for the fourth, giving $5\cdot4\cdot3\cdot2$ or 120 ways to choose. Since there are 4! ways to permute , each set has been counted 4! times. Therefore, there are $\frac{5\cdot4\cdot3\cdot2}{4!}$ or 5 different sets. We can also figure this out by writing down all the possible sets. Each set must consist of all the colors except one:

WBRY WBRG WBYG WRYG BRYG .

Exercises:

8. How many different sets of 2 coins can be chosen from the 4 coins : penny, nickel, dime quarter?

9. How many groups of 3 balls can be chosen from a set of 4 balls containing one each in the colors red, green, blue, and white?

10. Jack has 10 baseball cards. He wants to give Jill 4 of them. How many different sets of cards could he give her?

3.3. A formula for combinations.

Now we will try to write down a general formula . For this, we shall do a few more calculations with larger numbers. Suppose that there are 10 objects, and we want to find the number of combinations of these is we pick 4 each time. If we follow the same reasoning as in the above examples, there are 10 choices for the first object, 10 - 1 or 9 for the second, 10-2 or 8 for the third, and 10-3 or 7 for the fourth.. We multiply these together, and divide by 4!. This gives us

$$\frac{10\cdot 9\cdot 8\cdot 7\cdot}{4!}$$

for the number of combinations of 10 objects taken in groups of 5. If we replace 10 by 12 and 5 by 4 we obtain

$$\frac{12\cdot 11\cdot 10\cdot 9}{4!}$$

for the number of combinations of 12 objects taken in groups of 4. For the general case we will write the letter n for the number of objects and the letter r for the number of objects in each group. (The number that r represents has to be a whole number which is no bigger than n.) Now the formula is obtained by multiplying together n times n-1 times n-2 and so forth until we have a product of r numbers, and then dividing by r! The last number that we multiply in the numerator is n+1-r. Therefore, in mathematical notation we can write the formula for the number of combinations of n objects taken in groups of r as

$$\frac{n(n-1)(n-2)...(n-r+1)}{r!}\ .$$

(Remember that the ... between (n-2) and (n-r+1) means multiply by all the whole numbers in between.)

Examples.

8 How many different sets of 3 balls can be chosen from a set of 8 balls of different colors?

Solution. Use the formula with n = 8 and r = 3. Then n-r+1 = n-2 Since n(n-1)(n-2) equals $8\cdot 7\cdot 6 = 336$, and r! = 3! = 6, the answer is 336/6 = 56.

9. How many 5 person casts can be chosen from a class of 18 students?

Solution. Here we take n = 18 and r = 5. Then

$$n(n-1)(n-2)(n-3)(n-4) = 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = 1{,}028{,}160.$$

Since $r! = 5! = 120$, the answer is 8,568.

10. In a party game, 5 children take off their shoes and mix them all together in a bag. Then one of the children picks 2 shoes from the bag. What is the probability that the 2 shoes are actually a pair?

Solution: First we shall count the number of combinations of 2 shoes taken from a bag of 10. The answer is $\frac{10 \cdot 9}{2!} = 45$ possible combinations. How many of these are pairs? Since there are 5 pairs, that is the number of combinations which give a pair. Then the probability is 5/45 or 1/9.

Exercises:

11. How many 4 person committees can be chosen from a group of 8 people?

12. How many sets of 3 coins can be chosen from the 6 coins: penny, nickel, dime, quarter, half dollar, silver dollar.

13. How many different 5 card hands are there containing only clubs. (There are 13 clubs in the deck.)

14. Eight balls, numbered one to eight, are in a jar. There are 2 balls each in the colors red, green, blue, and white. Suppose that 2 balls are chosen from the jar. What is the probability that the balls will be the same color?

15. Two families have 4 children each. In a game, each child writes his or her name on a piece of paper, and puts it in a bowl. Then 4 names are picked from the jar. How likely is it that all 4 are from the same family?

16. In Example 5, what is the answer if only 3 names are picked from the bowl? (Hint: you need to figure out how many different ways there are to pick sets of 3 names from 4.)

17. A jar contains 20 jelly beans. There are 5 in each of 4 different flavors mixed all together.. A girl picks 5 jelly beans. What is the probability that they are all the same flavor?

3.4 The numbers $\binom{n}{r}$.

There is another way to write down the formula for the number of combinations of n objects taken r at a time. It helps to look at an example. Suppose that $n = 10$ and $r = 4$. Since

$$6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 \quad \text{and} \quad 10! = 10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$$

we have

$$10\cdot9\cdot8\cdot7 = \frac{10!}{6!}.$$

Therefore, the number of combinations of 10 objects taken 4 at a time is

$$\frac{10\cdot9\cdot8\cdot7}{4!} = \frac{\frac{10!}{6!}}{4!} = \frac{10!}{6!\cdot4!}.$$

Finally, we might also notice that $6 = 10\text{-}4$, so that the answer might be written $\frac{10!}{(10-4)!\cdot4!}$: This works for any number n and any number r which is less than or equal to the number n. We define the number $\binom{n}{r}$ by the formula

$$\binom{n}{r} = \frac{n!}{(n-r)!\cdot r!}.$$

For example, if $n = 3$ and $r = 2$, then $\binom{3}{2} = \frac{3!}{1!2!} = 3$. (Notice that $\binom{n}{r}$ looks like a fraction, but it is not; there is no line between n and r. The number $\binom{n}{r}$ is sometimes called " n choose r")

We have shown:

The number of combinations of n objects taken in sets of size r is $\binom{n}{r}$.

Examples.

13. Calculate $\binom{7}{3}$.

Solution: $\binom{7}{3} = \frac{7!}{(7-4)!\cdot4!} = \frac{7!}{3!\cdot4!} = \frac{5040}{6\cdot24} = 35.$

14. How many different sets of 6 photographs can be made if the objects are to be chosen from a set of 8 photographs?

Solution: There are $\binom{8}{6} = \frac{8!}{2!\cdot6!} = 28$ different sets.

15. A class has 11 girls and 7 boys. The teacher will choose 5 girls and 4 boys to work

together. How many possiblities are there?

Solution: There are $\binom{11}{5} = 462$ ways to pick the 5 girls. Wow! There are $\binom{7}{4} = 35$ ways to pick the 4 boys. By the counting rule there are $462 \cdot 35 = 16{,}170$ ays to pick all 9 students. It will certainly take the teacher much longer to decide on the girls.

16. Five boys and 5 girls put their names into a bowl for a contest drawing. Four names will be chosen from the 10. How likely is it that only girls will be chosen?

Solution: There are $\binom{10}{4} = 210$ possibilities for choosing 4 names from the 10. This is the sample space. Now we have to find out how many outcomes lead to all 4 being girls. Since there are 5 girls, there are $\binom{5}{4} = 5$ ways of choosing 4 girls from the girls' names. The probability is $\frac{5}{210} = .02$. This is not very likely. If it happens, someone might suspect that it is not a fair contest.

Exercises.

18 Calculate $\binom{6}{3}$.

19. Calculate $\binom{17}{10}$.

20. Find the number of combinations of 5 colored balls, to be chosen from a bag of 10 balls of different colors.

21. In Example 16. above, find the probability of all girls being chosen if there are 6 boys and 6 girls.

22. Five pairs of different colored socks are all mixed up in a drawer. If a person picks 2 socks without looking, what is the probability that he will get a matched pair?

23. Six students are chosen from a group of 7 boys and 8 girls. If they are chosen randomly, what is the probability that 3 girls and 3 boys will be chosen?

4. Combining probabilities.

4.1. What happens when outcomes are not equally likely?

So far, we have only talked about probabilities when the outcomes are equally likely. In that case, the probability of an event is the number of outcomes in the event, divided by the total number of outcomes. A probability of 1 means the outcome is certain, while a probability of 0 means that it is impossible. If the probability of an event is, say, 5/16, it means that if we do an experiment, such as tossing a coin, 16 times, we expect that the event will occur about 5 times.

We can use this idea to define the probability of anyoutcome. Suppose, for instance, that the coin we toss is not fair. We toss the coin 100 times, but get only 35 heads. If this is the typical proportion of heads for 100 tosses, then we would say that the probability of the coin coming up heads is 35/100 or .35.

Other examples where events are not equally likely come from problems we have looked at already. For instance, if a bowl of fruit contains 3 oranges, 2 apples, and 2 bananas, what is the probability of the outcome of choosing an apple? (Here we will assume that a piece of fruit is picked without looking.) Since there are 7 choices, of which 2 are apples, the probability of picking an apple is 2/7.

Examples.

1. A die is tossed 200 times. The number 5 comes up only 25 times. What probability should be given to the event that a 5 comes up?

Solution: The best guess for the probability is 25/200 or 1/8.

2.. A sample space contains 10 outcomes, all of which are equally likely. What is the probability of any one of the events?

Solution: Any one event should happen 1 time in every 10 experiments, so the probability is 1/10.

3. There are 7 boys and 11 girls in a class. If a student is chosen at random, what is the

probability that the student is a girl?

Solution: If all 18 students are equally likely to be picked, then a girl will be picked 11 times out of 18, on average. Therefore the probability is 11/18.

4: A coin is tossed many times. For every 5 heads there are 6 tails. What is the probability of heads coming up on a given toss?

Solution: According to the information, there are an average of 5 heads for every 11 tosses, since the other 6 tosses come up tails. Therefore, the probability of heads is 5/11 .

Exercises:

1 A coin is tossed 500 times. Tails come up 200 times. What is the probability of getting tails on a toss of that coin? (Or rather, what is the best guess for the probability?)

2 What is the probability of picking an ace from a standard deck of 52 cards? (Hint: How many aces are in the deck? Assume all the cards are equally likely?)

3. There are 10 boys and 8 girls in a group at camp. If a camper is chosen at random, what is the probability that the camper selected will be a boy?

4. .A coin is tossed often. For every 7 heads there are only 4 tails. What is the probability of getting heads on a given toss?

4.2. **The probability of one event or another happening.**

If we know the probability of two different events, what can we say about the probability that at least one of the events happens? It depends on how the events are related to each other. We will look at an example. The probability of getting a 5 when a die is tossed is 1/6 . The probability that the number which turns up is 3 or 4 is 2/6 or 1/3. What is the probability that the number which turns up is 5, 3 or 4 . To find out, we count the number of outcomes in this event, and then divide by the total number of outcomes in the sample space:

Outcomes: {5} {3,4} Total outcomes in event: 3

Sample space: {1,2,3,4,5,6} Total outcomes in sample space : 6

Therefore the probability is 3/6, or 1/2 It is not a coincidence that this is 1/6 + 2/6, the sum of the other 2 probabilities.. We are just adding fractions with the same denominator.

The rule for the probability of at least one of two events to occur:

Suppose two events cannot occur at the same time. Then the probability that at least one of the events occurs is the sum of the probabilities that each of the events occurs.

Examples.

5. A coin (but not a fair one!) is tossed twice. The probability of heads coming up both times is 9/25. The probability of heads coming up exactly once is 12/25. What is the probability that heads come up at least once?

Solution: In 25 tosses there are about 9 which result in 2 heads and another 12 which result in exactly one head. Therefore, there are 9+12 = 21 tosses out of 25 that result in at least one head. . The probability of having at least one head in a two tosses of this (unfair) coin is 21/25, which is 9/25 + 12/25.

6. Suppose again that a fair die is thrown once. The probability that the number which comes up is less than 4 is 3/6 or 1/2, since 1, 2, and 3 are included in this. The probability that the number which comes up is even is also 3/6, since 2, 4, and 6 are the possiblities. What is the probability that the number that comes up is either less than 4 or an even number ?

Solution: We first count the outcomes in this event:

Outcomes: {1,2,3} {2,4,6} Total: {1,2,3,4,6} .

In this case, the number of outcomes, which is 5, is less than the sum of the outcomes in the first two events. This is because having a number less than 4 come up and having an even number come up , can occur at the same time: when the number 2 comes up. The probability that the third event occurs is 5/6, which is less than 3/6 + 3/6 , which is the sum of the probabilities of the first two events.

8. A pair of (unfair) dice are thrown. The probability that the sum of the numbers is 2 is .1, while the probability that the sum of the number is 12 is .05. What is the probability that the sum will be less than 3 or greater than 11?

Solution. The two events cannot both occur, so that the probability of at least one occurring is the sum: .1 + .05, or 1.05.

9. A jelly bean is picked from a jar containing 3 black jelly beans, 4 white jelly beans, and 7 red jellybeans. What is the probability that either a black one or a white one will be picked?

Solution: The probability of picking a black jelly beans is 3/14 (since there are 14 jelly beans in the jar.) The probability of picking a white one is 4/14. Therefore the probability of picking a white one or a black one is the sum : 7/14 or 1/2. Of course, we could have done this just by counting the number of outcomes in the event of picking a black or a white jelly bean.

Exercises.

Exercises:

5. A fair die is tossed once. The probability that a number less than 3 comes up is 2/6 or 1/3. The probability that a number larger than 4 comes up is also 2/6 or 1/3. What is the probability that a number less than 3 or a number greater than 4 comes up?

6. A fair die is tossed once. The probability that an odd number will come up is 3/6 or 1/2. The probability that a number greater than 4 will come up is 2/6 or 1/3. What is the probability that either an odd number or a number greater than 4 will come up?

7. A coin which is not fair is tossed twice. The probability of tails coming up twice is 4/9. The probability of tails coming up exactly once is also 4/9. What is the probability that tails come up at least once?

8. A die which is not fair is tossed once. The probability of a 5 coming up is 1/7. The probability of either 1 or 3 coming up is 3/7. What is the probability of 1,3,or 5 coming up?

9. Two dice are tossed. The probability that the sum is 2 is 1/36,while the probability that the sum is 11 is 2/36. What is the probability that the sum is either 2 or 11?

10. There are 10 fifth graders, 12 sixth graders and 9 seventh graders in a group from which a student will be chosen. How likely is that the student chosen will be either a sixth grader or a seventh grader?

11. Three unfair dice are tossed. Suppose the probability that the sum is 3 is 3/216 and the probability that the sum is 18 is 4/216. What is the probability that the sum is either 3 or 18?

4.4. How likely is it that an event will not occur?

The addition rule can be used to answer the following question: If you know the probability that an event will occur, what is the probability that the event will not occur? Since an event cannot both happen and not happen, the probability that either the event happens or it does not happen is the sum of the probability that it happens and the probability that it does not happen. The sum must be 1, since it is certain that any event either happens or does not happen. That means that the probability that an event does not happen is 1 minus the proability that it does happen. We can state this as a rule.

If the probability that an event occurs is the number p , then the probability that the event does not occur is the number 1-p.

Examples.

10. The probability that a certain coin will land heads is .49. What is the probability that it will land tails?

Solution. The probability of tails is 1 - .49 or .51.

11. A candy is picked from a jar containing lemon, orange, and cherry candies. If the probability of picking a lemon candy is 3/11, and the probability of picking an orange candy is 5/11, what is the probability of picking a cherry candy?

Solution. The probability of picking a lemon or orange candy is 3/13 + 4/13 or 7/13 by the rule for either of two events to occur. Therefore the probability of picking a cherry candy is 1 - 7/13 or 6/13.

12. Two girls each pick a number from 0 to 9 at random. What is the probability that the two numbers are different?

Solution. We will first find the probability that the two numbers are the same. First, since there are 10 numbers for each choice, there are $10 \cdot 10$ or 100 possible ways the girls could choose the numbers. Which lead to the same number? Exactly 10 of these, one for each number. Therefore the probability that they choose the same number is 10/100 or 1/10. Then the probaility that they choose different numbers is 1- 1/10 or 9/10.

13. Suppose now that three girls will each choose a number between 0 and 9. What is the probability that at least two of the numbers chosen will be the same?

Solution. This time it is easier to start by finding the probability that all the numbers will be different. There are 10 ways for the first girl to choose a number. If all the numbers are to be different, there are only 9 choices for the second girl. Finally there are only 8 choices for the third girl. Therefore there are $10 \cdot 9 \cdot 8$ or 720 ways to choose all the numbers to be different. There are $10 \cdot 10 \cdot 10$ or 1000 ways to choose the numbers. The probability is 720/1000 or 18/25 that the numbers are all different. The probability that at least two are the same is therefore 1 - 18/25 or 7/25.

Exercises.

11. The probability that a coin that you are given will land heads is .53 .
What is the probability that it will land tails?

12. The probability that a certain coin will land tails is 12/25. What is the probability that it will land heads?

13. A jar of jelly beans contains red, green, yellow and black jelly beans. If there are 5 red jelly beans out of a total of 51 in all, what is the probability that the one chosen will not be

red?

14. Two boys pick a number from 1 to 9 at random. What is the probability that the two numbers chosen will be different?

15. A fair die is tossed twice. What is the probability that the two numbers which come up will be different?

16. Three boys each choose a number from 1 to 9. What is the probability that at least two of the numbers will be the same?

17. Three dice are tossed. What is the probability that at least two of the die will come up with the same number?

4.5. What happens when two events can both occur?

Let's calculate the probability of either of two events occurring if both can occur at the same time. Suppose that a student will be chosen from a group of 7 fifth graders and 9 sixth graders. Eight of the students are girls, of whom 3 are fifth graders. How likely is that the student chosen will be either a fifth grader or a girl (or a fifth grade girl)? There are 7 fifth graders and 8 girls. If we add 7 and 8 we are counting the fifth grade girls twice. The answer is 7+8 minus 3,the number of fifth grade girls. The number of fifth graders and girls is 7+8-3=12. The probability that a fifth grader or a girl will be chosen is 12/16 or 3/4. This is the same as the probability that a fifth grader will be chosen (7/16) plus the probability that a girl will be chosen (8/16) minus the probability that a fifth grade girl will be chosen (3/16).

The general rule is:

The probability that either one of two events occurs is the sum of the probability that each one occurs minus the probability that they both occur at the same time.

Examples.

14. The probability that an ace is picked from a deck of cards is 4/52. The probability that a heart is picked from a deck is 13/52. The probability that the ace of hearts is picked is 1/52.

What is the probability that either an ace or a heart will be picked?

Solution. By using the rule, the probability that either an ace or a heart is picked is the probability that an ace is picked (4/52) plus the probability that a heart is picked (13/52) minus the probability that both an ace and a heart are picked (1/52). The answer is 4/52 + 13/52 - 1/52 or 16/52, which is 4/13.

15. A number is picked between 1 and 98. What is the probability that it will either be an even number or that it will be between 1 and 30?

Solution. There are 98 numbers in all, of which half, or 49, are even. Therefore, the probability of picking an even number is 49/98 or 1/2. The probability that it is between 1 and 30 is 30/98 (since there are 30 such numbers.) There are 15 even mumbers between 1 and 30, so that the probability of picking one is 15/98. By the rule, the answer is 49/98 + 30/98 - 15/98 or 24/98, which is 12/49.

Exercises.

1. In a deck of 52 cards, the probability is 26/52 that a red card will be picked. The probability that a four will be picked is 4/52. The probability that a red four will be picked is 2/52. What is the probability that a four or a red card will be picked?

2. Suppose that the deck of cards also contains two jokers. What is the probability that a heart or an ace will be picked if one card is chosen?

3. A number is picked between 1 and 50. What is the probability that it will be either an odd number or a number bigger than 25?

4. A pair of dice ar thrown. The probability that the sum is a number bigger than 4 is 30/36. The probability that the sum is odd is 18/36. The porbability that the sum is 3 is 2/36. What is the probability that the sum is either an odd number or a number less than 4?

4.5. Computational project.

A famous problem in probability is the Birthday Problem. A group of 25 people are in a room. What is the probability that at least two of the people have the same birthday? (We will assume that there are always 365 days in a year and that it is equally likely that a person is born on any day.) We could ask the question for a group of any size. As a mathematical problem this is the same as the problem where each person chooses a number at random from 1 to 365. Then you ask how likely it is that two people will pick the same number.

This problem is very similar to the one where each person in a group picks a number from 1 to 9. This problem is the same as if everyone were to pick a number from 1 to 365. It is easier to calculate the probability that no two people pick the same number. Suppose there are 5 people. The first person has 365 choices, the second has 364 (since he cannot pick the same number as the first), the third has 363 choices (since 2 are already taken), and so forth. Therefore, there are $365 \cdot 364 \cdot 363 \cdot 362 \cdot 361$ ways that the 5 people can have different birthdays (or pick different numbers). On the other hand, there are $365 \cdot 365 \cdot 365 \cdot 365 \cdot 365$ ways that the 5 people can pick numbers between 1 and 365. Therefore, the probability that 5 people have different birthdays is

$$\frac{365 \cdot 364 \cdot 363 \cdot 362 \cdot 361}{365 \cdot 365 \cdot 365 \cdot 365 \cdot 365} .$$

If we calculate the numerator and denominator separately, we get extremely big numbers (and we will get even bigger ones when we have more than 5 people.) A better way to do this calculation is to write it as

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} ,$$

which is approximately .98 . By the rule for the probability of an event not occuring, the probability that at least 2 people have the same birthday is only .02 (which is 1 - .98).

It might seem as if you would need a very large group of people before it became more likely than not that two people would have the same birthday. Surprisingly, in a group of 23 people, the probability that at least 2 will have the same birthday is more than 1/2 . With a hand calculator, or a computer program, calculate the probability that two people will have the same birthday for groups of people up to 25. What happens for 50 people?

5. How likely is it that two events will both occur?

The answer to this question depends on how the two events are related to each other (or if they are related at all!) There can be examples of two events which can never both occur at the same time (such as having a coin turn up both heads and tails at the same time) and others which must occur at the same time (such as having a coin turn up both heads and not tails). The interesting cases are in between, and we will look at many examples of these.

5.1 When events independent of each other.

Sometimes one event does not depend on another. For instance, if coins are tossed, the result of the second toss does not depend on the result of the first toss. That is, the probability of getting tails on the second toss is 1/2 whether or not we know what the first toss was. On the other hand, it often happens that one event does depend on another. Suppose that you have two jelly beans, one white and one black. If one jelly bean is picked and you see its color, then you know the color of the remaining jelly bean.

We will say that two events are independent if the probability that the second event will occur is the same whether or not we know if the first event has occurred. The results of tosses of dice are independent, but the outcomes of picking jelly beans are not.

If two events are completely independent of each other then it is easy to calculate the probability that they will both occur.. An example is given by the tosses of a coin. If a fair coin is tossed twice, there are 4 possible outcomes: HH HT TH TT. The probability that a head is tossed first and then a tail is 1/4. This probability is the product of the probabilities that the first toss is a head and the probability that the second toss is a tail.

The rule for the probability that several events occur:

If two events are independent, then the probability that they both occur is the product of the probabilities that each occurs. If more than two events are independent, the probability that

they all occur is still the product of the probabilities that each occurs.

Examples:

1. A coin is tossed 3 times. What is the probability that the first toss is heads, and the second and third are tails?

Solution. The three events (getting heads on the first toss, tails on the second, and tails on the third) are independent and each has probability 1/2. Therefore the probability is 1/8. Of course, we can also figure this out by counting all possible outcomes.

2. An unfair die is thrown twice. The probability that a 1 comes up is 7/36, while the probability that a 6 comes up is 5/36. What is the probability that the first toss is a 1 and the second a 6?

Solution. The events are independent, so we multiply probabilities. The answer is 7/36 time 5/36 or 35/1296.

3. A bag contains 10 black marbles and 15 white marbles. One marble is picked and then put back into the bag. Then a second marble is picked. What is the probability that the first marble chosen is black and the second white?

Solution. Since the first marble is replaced before the second one is picked, the color of the second marble does not depend in any way on the color of the first. Therefore the events are independent. The probability of picking a black marble is 10/25 or 2/5. The probability of picking a white marble is 15/25 or 3/5. By the rule, the probability of picking a black one and then a white one is 2/5 times 3/5 or 6/25.

Exercises.

1. A girl tosses a fair coin 4 times. How likely is it that it comes up tails first and then heads three times?

2. A fair die is tossed twice. What is the probability that a 6 will come up first and then a five?

3. A coin is tossed and then a die is thrown. What is the probability that the coin will come

up tails and a 3 will come up on the die?

4. A boy tosses an unfair die three times. The probability that a 1 comes up is 7/36 while the probability that a 6 comes up is 5/36. What is the probability that the first toss is a 6 and the next two are 1's?

5. A bowl of fruit contains 5 apples and 8 pears. A fruit is picked without looking, and is then put back. Afterwards another fruit is picked. How likely is it that the first fruit was a pear and the second fruit was an apple?

6. A card is picked from a deck, and then replaced. A second card is then picked. What is the probability that the first card was an ace and the second card was a 2? (Hint: There are 52 cards in a deck of which there are 4 of each type.)

5.2 Events which depend on each other: conditional probability.

Your estimate of how likely it is that something will happen depends on how much information you have. Additional information can change your guess. Let's look at a situation similar to the one we discussed before. Suppose there are 3 marble in a bag: one white, one red, and one black. One marbles is picked (and is not put back) and then another is picked. If we know that the first was red, then we can say that the probability that the second is white is 1/2 (since white and black are the only remaining colors). If we have no information about the color of the first one, then the probability that the second is white is only 1/3.

The probability that an event, which we'll call A, occurs, given that event, called B, has occurred, is called the conditional probability of A given B. (Sometimes the word "conditional" is left out.) In the case of the marbles, the conditional probability that the second marble is white, given that the first marble is red, is 1/2.

Examples.

4 Two cards are picked from a deck. What is the conditional probability that the second card picked is an ace, given that the first card picked is an ace?

Solution. There are 52 cards in the deck, of which 4 are aces. After one ace is picked there are 51 cards remaining of which 3 are aces. The answer is 3/51.

5. Suppose instead that the first card picked is a king. What is the probability that the second one will be an ace?

Solution. After the king is picked, there are 4 aces remaining in the 51 cards. The answer is 4/51.

6. Two dice are thrown, one red and one blue. What is the probability that the red die comes up 1 given that the sum of the two is 3?

Solution. There are only two ways that the sum can be 3: if red comes up 1 and blue comes up 2 or if red comes up 2 and blue comes up 1. These two possibilities are eqally likely, so the answer is 1/2.

7. A prince comes from a family of 2 children. What is the probability that the other child is his sister?

Solution. The possibilities are that both children are boys or that the other child is a girl. Therefore, the probability that the other child is his sister is 1/2.

8. A coin is tossed three times. What is the conditional probability that the first toss was heads, given that two of the tosses were tails?

Solution. The sample space is {HTT THT TTH}. Therefore the probability is 1/3.

Exercises.

7. Two cards are picked from a deck. What is the conditional probability that the second is a queen, given that the first is a queen?

8. Three cards are picked from a deck. What is the probability that the third is a queen, given that the first is a jack and the second a king?

9. Two dice are thrown: one red and one blue. What is the probability that the blue comes up 1, given that the sum is 4.

10. A family has 2 children. What is the probability that both are boys, given that the younger child is a boy?

11. A coin is tossed twice. What is the probability that the first toss is heads, given that heads comes up at least once?

12. A coin is tossed three times What is the probability that the second toss is tails, given that tails come up at least twice?

5.3. **The probability that two events will both occur.**

Suppose that two events are not independent of each other. How likely is it that they will both occur? For instance, if two cards are picked from a deck, how likely is it that the first will be a queen and the second a jack? There are 4 ways to pick the queen and then 4 ways to pick the jack. By the basic rule of counting, there are 16 ways to pick a queen and then a jack. To figure out the number of ways two cards can be picked, we note that there are 52 ways to pick the first card and 51 ways to pick the second card. This makes a total of 52 times 51 or 2652 ways of picking two cars. Therefore the probability is 16/2652. This fraction is equal to $\frac{4 \cdot 4}{52 \cdot 51}$. That is the probability that the first card is a queen times the conditional probability that the second card is a jack, given that the first card is a queen.

This rule works for any two events:

The probability that events A and B both occur is the product of the probability that A occurs and the conditional probability that B occurs, given A.

We can write this rule as an equation. We will write p(A) for the number which is the probability that event A occurs, and write p(AB) for the number that is the probability that events A and B both occur. Now write $p(B \mid A)$ for the conditional probability that B occurs, given that A has occurred. Now we can write the rule as the equation

$$p(AB) = p(A) \cdot p(B \mid A).$$

Another way to write this rule is to divide both sides of the equation by the number p(A). This gives the equation for the rule in reverse order:

$$p(B \mid A) = \frac{p(AB)}{p(A)}.$$

Examples.

9. Two cards are picked from a deck. What is the probability that they are both aces?

Solution. The probability that the first is an ace is 4/52 or 1/13. The probability that the second is an ace, given that the first was, is 3/51 or 1/17. By the rule, the answer is 1/13 times 1/17 or 1/663. We could also figure this out by calculating how many ways there are to pick a set of two cards from a deck.

10. A die is tossed twice. What is the conditional probability that the first toss came up 4, given that the sum is 7?

Solution. This time we'll use the reverse rule. We first figure out the probability that the first toss is 4 and the sum is 7. There is only one way this can happen: (4,7) out of 36 possible tosses. The ways the sum can be 7 can be listed: (1,6) (2,5), (3,4), (4,3), (5,2), (6,1). The probability that the sum is 7 is 1/6. The probability that the first toss is a 4 and the sum is a 7 is 1/36. The probability that the sum is a 7 is 1/6. Now the equation for the reverse rule says that the conditional probability is $\frac{1/36}{1/6}$ or 1/6 . We could figure this out just as well by listing the 6 ways that the sum can be 7 and seeing that only one of them has 4 as the first toss.

11. Two jelly beans are picked at random from a jar containing 5 red and 7 green jelly beans. What is the probability that the first picked is green and the second is red?

Solution. We will find the probability that a green jelly bean is picked first, and then the conditional probability that a red one is picked second, given that a green one was picked first. Since there are a total of 12 jelly beans, the probability of picking a green one is 7/12. After the green one is picked there are 11 remaining jelly beans of which 5 are red. Therefore the conditional probability of picking a red one second, given that the first was green, is 5/11. By the rule, the probability that the first picked was green and the second red is 7/12 times 5/11 or 35/132. Unlike Example 9, this problem cannot be done by looking at the ways of picking groups of 2 jelly beans. That is because the order in which they are picked matters.

12. Things get more complicated when 3 jelly beans are chosen and when there are more

colors. Suppose that a jar contains 5 red, 7 green, 4 white, and 12 black jelly beans. Three jelly beans will be picked. What is the probability that the first picked is white, the second red, and the third green?

Solution. First we find the probability that first is white and the second is red. There are 26 jelly beans in all, of which 4 are white. The probability that the first picked is white is 4/26 or 2/13. Now we find the probability that the second is red, given that the first is white. There are 25 remaining of which 5 are red, so the conditional probability is 5/25 or 1/5. By the rule, the probability that the first is white and the second is red is 2/13 times 1/5. Next we find the probability that the third is green, given that first is white and the second is red. There are 24 remaining jelly beans of which 7 are green. The conditional probability is 7/24. By using the rule again, we see that the probability that the first is white, the second red, and the third green is 7/24 times the probability that the first is white and the second red. The answer is 2/13 times 1/5 times 7/24 or 7/780.

13. A boy has two coins in his pocket, one of which is a fair coin and the other a coin with heads on both sides! He takes one of the coins and flips it twice. It comes up heads both times. What is the probability that it is the fair coin.

Solution. This is a tough one! We want to find a conditional probability, so we use the reverse rule. We first calculate the probability that, without knowing which coin it is, heads come up in the two tosses. For the two coins together there are 4 sides of which 3 are heads. The probability of getting heads on any toss is 3/4. Therefore, the probability of getting heads on both tosses is 3/4 time 3/4 or 9/16. Now we calculate the probability that the coin is fair, given no information about any tosses. Since there are two coins, of which one is fair, that probability is 1/2. Using the reverse rule, we get that the probability that the coin is fair, given that it came up heads in two tosses is 1/2 divided by 9/16 or 9/32 .

Exercises.

13. A jar contains 12 jolly beans, 6 red and 6 green. Two jelly beans are picked. What is the

probability that they are both red?

14. Mary has a bag of marbles consisting of 7 white ones, 10 black ones, and 18 multicolored ones. She picks two without looking. How likely is it that the first one picked is black and the second one white?

15. Mary puts the two marbles back into the bag and then picks three more, one at a time. What is the probability that the first is white, the second black and the third multicolored?

16. Three cards are picked at random from a deck. What is the probability that all three cards are aces?

17. Do Example 10 directly instead of using the reverse rule.

18. A bag contains 5 red marbles and 10 green ones. A boy picks two at random. What is the probability that the first one picked is red and the second one is green?

19. There are 15 candies in a bag: 5 lemon, 2 orange, and 8 grape. Two candies are picked without looking. What is the probability that the first one picked is orange and the second lemon?

20. Suppose a bag contains the same 15 candies as in Exercise 19. Three candies are picked from the bag. What is the probability that the first picked is orange, the second lemon, and the third grape?

21. A girl has two coins in her pocket. One of the coins is a fair coin and the other is tails on both sides? She flips it three times, and it comes up tails each time. What is the probability that it is the fair coin?

5.4. **Quality testing: a computational project.**

This is an example of how conditional probability can be used in industry. Suppose that you are in charge of checking the quality of light bulbs. being made in a factory. By careful testing, you find that approximately one out of 10 bulbs made is defective. The bulbs which are not defective will light each time they are tested, while the defective bulbs light only about half the time. You can test a light bulb by trying to light it a certain number of times. If it fails to light any time, then it is discarded. The question is how many times a bulb should be tested.

The answer depends on how important it is that bulbs not be defective. If it is all right to have one out of 10 bulbs defective, then there is no reason to do any testing. (Testing is expensive; also, too much testing wears out the bulbs.) Let's check the result of testing each bulb twice. We want to find the conditional probability that a bulb is not defective, given that it lights both times when tested. By the reverse rule, we find the probability that it is not defective and lights both times, and divide by the probability that a random bulb picked will light twice. The first is easy: 9 out of 10 bulbs are not defective, and they are light each time. The probability that a bulb is not defective and lights both times is 9/10. Now we need to find the probability that a random bulb lights both times. By the rule in Chapter 4, this is the sum of two numbers: the probability that it is not defective and lights both times and the probability that it is defective and lights both times. (A bulb cannot be both defective and not defective.) The probability that it is defective and lights both times is 1/10 times 1/4 or 1/40. (To see this, think about 40 bulbs. Of these, 4 are defective, since 4/40 = 1/10. Out of 4 defective bulbs, only one will light twice, since the probability of lighting twice is 1/2 times 1/2 or 1/4.) Therefore, the probability that a bulb will light twice is 9/10 + 1/40 or 37/40. The conditional probability is 9/10 divided by 37/40 or 36/37. (If you are having difficulty with diving one fraction by another, convert to decimals and use a calculator: 9/10 divided by 37/40 is approximately .9 divided by .925 or .97. By testing each bulb twice and discarding those which fail either time, we increase the number of nondefective bulbs to 97 out of 100, and decrease the number of defective ones to 3 out of 100.

Now <u>you</u> do some calculations. How many defective bulbs will remain if each is tested 3 times. What is the answer if each is tested 4times or 5 times or 6 times. What is the least number of tests which will leave only one out of 1000 defective?

Study questions

1. Think of an example of two events which cannot both occur.

2. When are two events independent? Make of examples of pairs of events which are independent and pairs which are not.

3. What is the rule for the probability of two events both occurring if they are independent?

4. What is conditional probability? Give an example.

5. How do you find the probability that two events will both occur?