An introduction to the classification of manifolds, with the Poincaré conjecture as an example

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Abstract

These notes were prepared for a talk in the 'Food for Thought' graduate student seminar at UCSD. They are meant to give an intuitive introduction to the classification of manifolds, geared at a general audience of graduate students and prospective graduate students. Therefore, many, many details will be swept under the rug.

Manifolds arise naturally in areas of mathematics varying from topology to geometry to analysis, and are important in applications to many fields outside of math (e.g. physics). I'll begin by explaining what a manfiold is. Then I'll talk a bit about the classification of manifolds in small dimensions. After that, I'll state the Poincaré conjecture and explain how it is proved (for $n \geq 6$) using the h-cobordism theorem. This talk should be accessible to anyone who can see all the pretty pictures on the board.

I don't have time to put all the pretty pictures in this document, but I left space so that you can draw them in for yourself! Please email me if you find any errors: `maverett@math.ucsd.edu`.

1 What is a manifold, anyway?

The best example that everyone knows from daily life is the surface of the earth. If I'm standing on the surface of the earth, I look around and think that things look pretty much like a slightly-curved plane, but we all know that the earth is round. But round like an apple, not round like a pancake. So, locally we think that the surface of the earth looks almost like $\mathbb{R}^2$, but we know that globally something more is happening. We can imagine that the earth is made up of a lot of pretty-much-flat planes glued together to make something that is globally round. This is exactly what a manifold is: a space which locally looks like euclidean space $\mathbb{R}^n$.

What this means in math language is that if I pick any point in my space, I'd better be able to find some neighborhood of it which looks like $\mathbb{R}^n$. Just as the earth is a bunch of two-dimensional planes glued together into a sphere, we also require that the n be the same for each point in our space. Let's write down the formal definition, then explain what it means.

Definition 1.1. A *n-dimensional manifold* is a second-countable Hausdorff topological space together with a collection of open sets $\{U_\alpha\}$, such that $M = \bigcup_\alpha U_\alpha$, and homeomorphisms $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The maps φ_α are called *charts*.

Figure 1: Charts on a manifold.

Second-countable is just a technicality which ensures that our manifold is not too unwieldy. For those of you who know what a topology is, second-countability makes the number of open sets in the topology not too large. Hausdorff means that we can separate points. Throughout this talk I'll only work with manifolds without boundary. A *homeomorphism* is a continuous bijective map with a continuous inverse. This is just the correct notion of 'isomorphism' in the category of topological spaces.

2 Classification of Manifolds

Now that we know what a manifold is, we want to try to figure out what kinds of manifolds there are. We want to classify them up to homeomorphism. Intuitively, this means that we allow ourselves to stretch and shrink the manifolds, as long as we don't create holes or break any part of our manifold. For example, from a topological point of view, a sphere is a sphere, it doesn't matter how large the radius is. Another classic joke is that a topologist can't tell the difference between a coffee cup and a donut, since if we had a flexible enough donut, we could make a dent in it and enlarge that dent to be the container of the coffee cup, while smushing the rest of the donut down in to the handle of the coffee cup.

Figure 2: A coffee cup and a donut are the same!

Let's just start trying to classify manifolds and see what we can say. To make things easier, let's assume our manifolds are connected. This just means that we can walk from any point to any other point on our manifold. Some people might also notice that we're also assuming that our manifolds have no boundary.

2.1 Dimensions 0 and 1

Since $\mathbb{R}^0$ is just the point $\{0\}$, a 0-dimensional manifold is just a bunch of discrete points floating around, so a connected 0-dimensional manifold is nothing more than a point! That was easy.

What about dimension 1? Obviously one example is the real line $\mathbb{R}$. Another example is a circle S^1. You can see, just by trying to draw pictures, that this is pretty much all there should be. Up to homeomorphism, there's really nothing else that you can do to the real line and keep it a manifold, besides making a circle. Any other kind of gluing would create a point which looks like an × or a $\leftthreetimes$, so the resulting space can't be a manifold.[1]

2.2 Dimension 2

Manifolds of dimension 2 are surfaces and their classification is a beautiful classic result. It turns out that there are two families, with infinitely many members in each family, indexed by the natural numbers, including zero. The two families are *orientable* and *not orientable.* An orientable manifold is one on which you can walk all around and when you come back to the spot where you started, you are not turned inside out or upside down or anything silly like that. A classic example of a nonorientable manifold (with boundary) is the Möbius strip. It is not orientable since if an ant walks along the surface starting at a certain point, by the time he gets back to that point, his notion of left and right are switched.

Figure 3: A Möbius Strip.

Orientable manifolds are classified by their *genus.* This is essentially the number of 'holes' they have. The orientable surface of genus 0 is the sphere S^2 (the surface of the earth!), the orientable surface of genus 1 is the torus T (a donut!), and higher genus surfaces are connected sums of tori (donuts glued together!).

$$M_g = \underbrace{T \# T \# \cdots \# T}_{g \text{ copies}}$$

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[1] By definition we are only allowed to glue along open sets. To make an ×, we would have to glue along a closed set, and to make a $\leftthreetimes$ we would have to glue along a half-open set.

Connected sum of two surfaces just means that you cut a disk out of each surface and glue the surfaces back together along the boundary circles of the holes you've created.

Figure 4: A connected sum of tori.

Now we've seen how to make orientable surfaces by gluing together tori. We also had the example of the Möbius strip, which looks almost like a surface except for the fact that it has boundary. If we want to make a non-orientable surface, we might try starting with the Möbius strip and try to get rid of the boundary. Notice that the boundary of the Möbius strip is a circle, so we can glue a disk to it, since the boundary of a disk is also a circle. If we do this, what we get is certainly a 2-dimensional manifold, and it's not on our list yet, since it's not orientable (it has a Möbius band inside it!). This guy is called $\mathbb{R}P^2$.

Another way we might try to make the Möbius strip into a surface is to glue another Möbius strip to it, identifying their boundaries. If you do this, you end up with the Klein bottle. It turns out that these are all you need to get all surfaces.

Theorem 2.3 (Classification of Surfaces)**.** *Any closed 2-manifold is homeomorphic to one of the following:*

1. S^2*, or*
2. *A connect sum of* g *copies of the torus* T*, or*
3. *A connect sum of* g *copies of the torus* T *plus one copy of* $\mathbb{R}P^2$*, or*
4. *A connect sum of* g *copies of the torus* T *plus one copy of the Klein bottle* K*.*

Non-orientable surfaces do not embed in $\mathbb{R}^3$, so it's difficult to visualize them, but I want to show you a neat way of visualizing spaces that might not embed in $\mathbb{R}^3$. We can get an idea of what they look like by to drawing them cut apart and remembering where the cuts are. For example, we can make a torus by taking a piece of paper and gluing the long sides together to get a cylinder, and then gluing the edges of the cylinder to each other to make a donut. We can represent this by drawing a square and coloring the sides we glue, paying attention to the direction that we glue them to each other. We can make the orientable surface of genus g by identifying the edges of a $4g$-gon in a similar way. The picture below shows how to get an orientable surface of genus 2 from an octagon.

This method of drawing also helps us with non-orientable surfaces. As we saw above, the basic building blocks of non-orientable surfaces are the real projective plane $\mathbb{R}P^2$ and the Klein bottle K, which can be drawn as a square with the edges identified as in the picture below.

Figure 5: A torus, a surface of genus 2, $\mathbb{R}P^2$, and K drawn as identification spaces.

2.4 Dimension 3

Dimension 3 turns out to be incredibly hard although by now a lot is known. Since answering questions about topology is in general hard, we often introduce machines that convert topological questions into algebraic questions, and then use the machinery of algebra to help answer the topological questions. One such machine is the fundamental group.

The *fundamental group* of a topological space is homotopy classes (rel boundary) of maps from the unit interval into the space, which map the endpoints of the interval to a chosen basepoint in the space. That is, as a set,

$$\pi_1(X, x_0) = \{\gamma : I \to X \mid \gamma(0) = x_0 = \gamma(1)\}/\sim$$

where the equivalence relation $\sim$ is given by *homotopy*. Two paths are homotopic if one can be continuously deformed through time to the other. The requirement that the paths start and end at the point x_0 just means that they are loops at some fixed point $x_0 \in X$. The group structure on $\pi_1(X, x_0)$ is given by *concatenation* of paths. Given two loops α and β, we get a loop $\alpha * \beta$ by first running through α for $0 \leq t \leq \frac{1}{2}$ and then through β for $\frac{1}{2} \leq t \leq 1$. This is still a loop at x_0 since $\alpha(0) = x_0 = \beta(1)$. The unit in the group is the constant loop at x_0. Although we needed the basepoint to define the fundamental group, it turns out that it is independent of the choice of basepoint, as long as X is path connected.

Figure 6: Homotopic paths and loops.

Let's do an example to get an idea of what the fundamental group detects about our space. Consider $X = \mathbb{C} - \{0\}$ and pick $x_0 = 1$. Then loops which do not wrap around the origin can all be pulled back to the constant loop at the basepoint, and hence represent the identity element in $\pi_1(\mathbb{C} - \{0\})$. If a loop goes around the origin, say once, then there are no deformations that we can do to pull it back to the constant loop at the basepoint, since the puncture at 0 is in the way. So this would represent a nontrivial element in $\pi_1(\mathbb{C} - \{0\})$. In fact, you can see that loops that wrap around twice cannot be deformed to loops that wrap around once, and so on, so that we get a equivalence classes corresponding to the number of times that the loops wrap around the point 0. Hence $\pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z}$, with the isomorphism given by

$$[\gamma] \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{z}\, dz.$$

The fundamental group is detecting this hole in our space.

Figure 7: Loops in $\mathbb{C} - \{0\}$.

Let's see if the fundamental group is useful for distinguishing some surfaces. Let's look at the sphere and the torus. Thinking of S^2 as $\mathbb{R}^2$ with a point at infinity, take the basepoint to be $s_0 := 0 \in \mathbb{R}^2$. The stereographic projection gives us a homeomorphism $S^2 - \infty \cong \mathbb{R}^2$, and obviously any loop which does not go through infinity can be shrunk down to a point, since it's just a loop in $\mathbb{R}^2$. If a loop passes through infinity, we have enough room to deform it a bit, since S^2 is 2-dimensional, so that it doesn't pass through infinity, and then we can shrink it down to a point. So every loop can be shrunk to a point, and hence $\pi_1(S^2, s_0) = 0$. For the torus, we have two generating classes of loops that cannot be shrunk down to a point, and so $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$. Yay, the fundamental group can tell the difference between these two surfaces! It turns out that the fundamental group can completely classify surfaces.

Figure 8: Computing $\pi_1(S^2)$ and $\pi_1(T)$.

The fundamental group is an example of a *topological invariant*, which is a machine that eats topological spaces and gives back some other kind of object (e.g. a group or a sequence of groups or vector spaces). If you feed in two spaces that are homotopy equivalent, it gives you back the same group. Homotopy equivalence is a notion of 'sameness' that is weaker than homeomorphism.

Around the turn of the 20^{th} century, Poincaré developed, among other things, the idea of the fundamental group and also of another topological invariant called homology, which gives you a sequence of abelian groups when you feed in a topological space. Poincaré first conjectured (1900) that homology could detect the spheres completely. Then, in 1904, he wrote a paper constructing a counterexample, now known as the Poincaré homology sphere.[2] Poincaré proved that his space was not even homotopy equivalent to the 3-sphere by showing that it has a nontrivial fundamental group. So Poincaré's space can't be homotopy equivalent to S^3, since the fundamental group of S^3 is trivial, for the same reason that the fundamental group of S^2 is trivial. Poincaré then asked if the only simply connected closed 3-manifold is the 3-sphere.

Conjecture 2.5 (Poincaré, 1904)**.** *Every simply connected closed* 3*-manifold is homeomorphic to* S^3*.*

This 100 year old problem is only recently thought to have been solved! It is currently one of the Clay Math millennium prize problems, with a reward of $1,000,000 promised for the solution. In 2003, Grigori Perelman circulated a few preprints in which he follows through a method outlined by Richard Hamilton to prove the Thurston geometrization conjecture. This conjecture classifies 3-manifolds by saying that every one of them has a certain one of eight geometric structures. Perelman's proof uses Ricci flow, which is a hot topic of research here at UCSD. We have Ben Chow and Lei Ni who are very active in this field. By now (early 2006), most people believe that Perelman's proof does not contain any major flaws. Thus, the Poincaré conjecture may be one of the first Clay Math problems to be solved!

2.6 Dimension 4

Dimension 4 turns out to be surprisingly mysterious, and there is a lot of active research in this area. In dimensions 4 and higher, there is a theorem which states that classifying manifolds is at least as hard as classifying groups.

Theorem 2.7. *Given any finitely presented group* G *and an integer* $n \geq 4$*, we can find an* n*-dimensional manifold with* $\pi_1(M) = G$*.*

In general, given a finite presentation of a group, we can't decide what group it is, and so the same problem holds for manifolds: given a manifold, we can often write down its fundamental group in terms of generators and relations, but determining what group that is is not possible in general. One way to get around this problem is to first try to classify manifolds with trivial fundamental group. In the simply connected case, we at least have some hope of classifying higher dimensional manifolds.

3 The generalized Poincaré conjecture

Now I want to switch gears a bit so that I can talk about the h-cobordism theorem, which can be used to prove the generalized Poincaré conjecture for $n \geq 6$. The genralized Poincaré

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[2] There is actually a great description of this guy as a dodecahedron with opposite faces identified with a $\frac{1}{5}^{th}$ twist.

conjecture says that closed manifolds homotopy equivalent to S^n are in fact homeomorphic to S^n. Again, if you don't know what homotopy equivalent means, just think of it as a form of 'sameness' for topological spaces, like homeomorphism, but much weaker. For example, manifolds of different dimension can be homotopy equivalent, but homeomorphisms cannot change the dimension of a manifold. For example, a point is homotopy equivalent to $\mathbb{R}^n$, but they're obviously not homeomorphic.

Theorem 3.1 (Generalized Poincaré Conjecture)**.** *Every closed manifold which is homotopy equivalent to* S^n *is homeomorphic to* S^n*.*

In 1961, Smale proved the generalized Poincaré conjecture for $n \geq 5$, and was awarded the Fields medal. More recently (1982), Freedman proved the $n = 4$ case, for which he was also awarded the Fields medal. Freedman used to be on the faculty here at UCSD, but now he's building quantum computers for Microsoft.

I want to talk a little about Smale's methods, so I need to briefly introduce a few concepts. A manifold is *smooth* if all of its transition functions are smooth. Smooth manifolds are a natural place to do calculus, since we can say what smooth functions are by saying that a function is smooth if it is smooth when composed with the coordinate charts. We can also define a stronger form of 'sameness' for smooth manifolds. A *diffeomorphism* is a homeomorphism which is differentiable in coordinate charts. From now on, all manifolds are smooth, unless otherwise noted.

A *cobordism* between two n-manifolds M and N is an $(n+1)$-manifold W (with boundary) whose boundary is the disjoint union of M and N.[3] An important situation is when the cobordism W is diffeomorphic to the cylinder.

$$(W; M, N) \cong M \times (I; \{0\}, \{1\})$$

Figure 9: A cobordism which is diffeomorphic to a cylinder.

In this case, M is diffeomorphic to N by restricting the diffeomorphism $W \cong M \times I$ to the boundary. An *h-cobordism* is a cobordism where the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences. The following amazing theorem is a generalization of the crux of Smale's proof.

Theorem 3.2 (h-cobordism, Smale, 1962)**.** *Any h-cobordism between simply connected manifolds of dimension greater than* 5 *is diffeomorphic to the cylinder.*

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[3] I am sweeping details about orientation under the rug.

The proof of the h-cobordism theorem is via a method called *surgery*, which was developed by Smale, and later put to amazing use by Milnor and Kerviare [2]. Surgery basically involves cutting out and gluing in bits of W in order to alter until it is diffeomorphic to a cylinder with boundary $M \coprod N$. This is by now a very well-developed and sophisticated theory in which algebra and geometry play very well together. For more about surgery, see Ranicki [4] or Lück [3].

Proof of the Poincaré conjecture for $n \geq 6$. Suppose M is a closed simply connected manifold homotopy equivalent to S^n, say via a map $f : M \to S^n$, where $n \geq 6$. Disjointly embed two n-disks D_0^n, D_1^n in M and define W to be what's leftover after deleting the interiors of these disks.

$$W := M - (\mathring{D}_0^n \cup \mathring{D}_1^n)$$

It turns out that W is a simply connected h-cobordism between ∂D_0^n and ∂D_1^n. To see this, let $U_0 = M - \mathring{D}_0^n$ and $U_1 = M - \mathring{D}_1^n$ so that $W = U_0 \cap U_1$, $M = U_0 \cup U_1$, and $U_0 \simeq U_1 \simeq \star$. Now use Mayer-Vietoris to show that $H_*(W) \cong H_*(S^{n-1})$ and that the inclusions $i_k : \partial D_k^n \to W$ induce isomorphisms on homology. Since everything in sight is obviously simply connected, Whitehead's theorem implies that i_k is a homotopy equivalence for $k = 0, 1$, and hence that W is a simply connected h-cobordism as desired. So by the h-cobordism theorem[4] we can find a diffeomorphism

$$F : \partial D_0^n \times ([0,1]; \{0\}, \{1\}) \to (W; \partial D_0^n, \partial D_1^n)$$

which is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$ and induces some diffeomorphism $g : \partial D_0^n \to \partial D_1^n$.

Figure 10: The diffeomorphism from a cylinder to W.

Now we need a little trick from topology, called the *Alexander trick*. It says that any homeomorphism $\varphi : S^{n-1} \to S^{n-1}$ can be extended to a homeomorphism $D^n \to D^n$ as follows. Using polar coordinates on D^n, define $\widetilde{\varphi}(t \cdot x) := t \cdot \varphi(x)$, for $t \in [0,1]$. This map is obviously a local diffemorphism away from 0, and it turns out that it is only a local homeomorphism at 0.

Applying the Alexander trick to g, we obtain a homeomorphism $\widetilde{g} : D_0^n \to D_1^n$. Consider the closed cylinder D obtained by gluing two copies of D_0^n to $\partial D_0^n \times [0,1]$ along the boundary circles ∂D_0^n via the identity:

$$D := D_0^n \times \{0\} \cup_{\mathrm{id}} \partial D_0^n \times [0,1] \cup_{\mathrm{id}} D_0^n \times \{1\}$$

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[4] This is where we use the assumption on the dimension of S^n.

Recall that we have a decomposition $M = W \cup D_0^n \cup D_1^n$ and define a homeomorphism

$$H : D \to M$$

by the identity on the first copy of D_0^n, by F on $\partial D_0^n \times [0,1]$, and by $\widetilde{g}$ on the other copy of D_0^n.

Figure 11: The homeomorphism H

This completes the proof, since D is obviously homeomorphic to S^n. $\square$

Notice that in the proof, we used the Alexander trick to extend a self-homeomorphism of S^{n-1} to a self-homeomorphism of D^n. We could try to smooth out the cone point in the Alexander trick so that we extend self-diffeomorphisms of S^{n-1} to self-diffeomorphisms of D^n. It turns out that we can't always do this! The point is that there are so-called *exotic spheres*, which are spheres with non-standard smooth structures. The standard smooth structure on the sphere S^{n-1} is the one induced by its standard embedding as a submanifold of $\mathbb{R}^n$. We can smooth out the cone point if and only if the manifold M we started with is diffeomorphic to S^n with the standard smooth structure.

The existence of exotic spheres is another amazing theorem in topology. Since the Poincaré conjecture is true, homeomorphism classes of spheres are exactly homotopy classes. We say Σ^n is a *homotopy sphere* if it is an n-dimensional manifold which is homotopy equivalent to S^n. An *exotic sphere* is a homotopy sphere which is not diffeomorphic to S^n. In 1963 Milnor and Kervaire [2] computed the number of diffeomorphism classes of spheres for each dimension. The results are a truly amazing sequence of numbers. The following table lists the number of exotic spheres in each dimension.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Exotic S^n	1	1	1	∞?	1	1	28	2	8	6	992	1	3	2	16256

Table 1: Exotic structures on S^n

To wrap up, what we've seen is that in classification of manifolds: dimension 1 is easy; dimension 2 is classical; dimension 3 is hard, but known, and geometry is very important; dimension 4 is crazy; in dimensions 5 and higher we have a beautiful machine which gives us great results.

References

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