Handout 2 - The Jones Polynomial

A knot invariant. The Jones polynomial is an example of a knot invariant. That means:

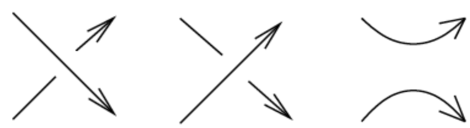
- If two knots are the *same*, then their Jones polynomials are *equal*.
- If two knots have *different* Jones polynomials, then they are *not the same!*
- The Jones polynomial is not perfect, though. It can happen that two *different* knots have the *same* Jones polynomial.

Definition. For an oriented link L, let V_L denote its Jones polynomial. The Jones polynomial is defined by the two equations below.

$$V_0 = 1$$

$$\frac{1}{t}V_{L_+} - tV_{L_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{L_0}$$

where V_0 is the Jones polynomial of the unknot and L_+, L_-, and L_0 are three links whose diagrams are identical except for at one specific crossing, where they differ by:



L_+ $\qquad$ L_- $\qquad$ L_0

The equation above is called the *skein relation*. It looks pretty scary, but all it says is that if two knots differ only by one crossing, then we know how their polynomials are related to each other.

Example. To get a feel for what this means, let's look at a couple of examples of links that differ at a crossing. We'll use these on the back of this page to compute the Jones polynomial of the trefoil. We can relate two copies of the unknot with the unlink with two components as below.

$$L_+ = 0 \qquad L_- = 0 \qquad L_0 = 00$$

We can relate the Hopf link, the unknot, and the unlink with two components as below.

$$L_+ = H \qquad L_- = 00 \qquad L_0 = 0$$

We can relate the trefoil, the unknot, and the Hopf link as below.

$$L_+ = T \qquad L_- = 0 \qquad L_0 = H$$

Example. We can compute the Jones polynomial of the trefoil by building up from the Hopf link, which we can compute from building up from the unlink with two components, which we can compute from building up from the unknot. We'll denote the Jones polynomials of the trefoil, the Hopf link, and the unlink with two components by V_T, V_H, and V_{00}, respectively.

To compute the Jones polynomial of the unlink with two components, notice that we can take two copies of the unknot as L_+ and L_-, and then the unlink with two components is L_0. So in this case, the skein relation becomes

$$\frac{1}{t}V_0 - tV_0 = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{00}.$$

Substituting $V_0 = 1$ into the above equation gives

$$\frac{1}{t} - t = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{00}$$

and solving for V_{00} gives

$$V_{00} = \frac{\frac{1}{t} - t}{\sqrt{t} - \frac{1}{\sqrt{t}}} = \frac{\frac{1}{t} - t}{\sqrt{t} - \frac{1}{\sqrt{t}}} \cdot \frac{\sqrt{t} + \frac{1}{\sqrt{t}}}{\sqrt{t} + \frac{1}{\sqrt{t}}} = \frac{(\frac{1}{t} - t)(\sqrt{t} + \frac{1}{\sqrt{t}})}{t - \frac{1}{t}} = -\sqrt{t} - \frac{1}{\sqrt{t}}.$$

To compute the Jones polynomial of the Hopf link, notice that we can take $L_+ = H$, and then L_- is the unlink with two components and L_0 is the unknot. In this case, the skein relation becomes

$$\frac{1}{t}V_H - tV_{00} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_0.$$

Substituting in $V_0 = 1$ and what we just computed for V_{00} gives

$$\frac{1}{t}V_H - t\left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right) = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

and solving for V_H gives

$$V_H = t\left(t\left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right) + \sqrt{t} - \frac{1}{\sqrt{t}}\right) = -\sqrt{t} - t^2\sqrt{t}$$

Again, we saw that if we look at one crossing on the right-handed trefoil and focus on one crossing of type L_+, then L_- is the unknot and L_0 is the Hopf link. Then the skein relation gives

$$\frac{1}{t}V_T - tV_0 = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_H$$

and substituting in $V_0 = 1$ and what we just computed for V_H gives

$$\frac{1}{t}V_T - t = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)(-\sqrt{t} - t^2\sqrt{t}).$$

Solving for V_T gives the Jones polynomial of the right-handed trefoil.

$$V_T(t) = t + t^3 - t^4$$

Yay!

The non-hand drawn images were taken from Vaughan Jones's recent paper on the Jones polynomial.

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