290 Qual Prep Practice Qual — Summer 2007

This is a practice qualifying exam. I suggest that you complete your studying for the qual before even looking at this exam. About a week before the qual, take this exam under the circumstances in which you will take the actual qual (i.e. 3 hour time limit, no distractions, no books or notes). At the end of the 3 hours, stop. The next day, use the solutions to check your answers. Discuss with your friends. Use the remaining time before the qual to review those concepts which tripped you up on this exam.

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You have three hours. Books and notes are not allowed. Try to outline solutions to all problems before getting bogged down in details on any one in particular.

1. Construct a space X with
$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k = 0,3,4 \\ \mathbb{Z}/3 & k = 1 \\ \mathbb{Z}/8 & k = 2 \\ 0 & \text{else} \end{cases}$$
and construct a space Y with
$$\pi_1(Y) = \langle a, b \mid abab^{-1} \rangle.$$
Compute $H_k(Y;\mathbb{Z})$ for all k for your space Y and also compute $H_k(X \times Y;\mathbb{Z})$ for all k.

2. Define the Euler characteristic of a space by
$$\chi(X) = \sum_i (-1)^i \operatorname{rk} H_i(X;\mathbb{Z}).$$
Let X and Y be spaces such that the Euler characteristics $\chi(X)$ and $\chi(Y)$ are defined. Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

3. Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

4. Is there a closed, oriented manifold with the same cohomology groups as $\mathbb{C}P^2$ but with a different cup product structure? If so, provide an example (with proof that it has the desired properties!). Otherwise, prove that none such exists. What do you think about $\mathbb{C}P^3$?

5. Suppose α and β are curves in $\mathbb{D} = \{\vec{v} \in \mathbb{R}^2 \mid ||\vec{v}|| \leq 1\}$ joining distinct pairs of antipodal points in the boundary of the disk. Prove by a topological argument that α and β must intersect. *Hint:* What space do you get when you identify antipodal points on the boundary of $\mathbb{D}$? What is its cohomology?

6. Prove that the torus $T = S^1 \times S^1$ can be covered by three, but not two, contractible open sets.

7. Let M be a connected, closed, oriented n-manifold and suppose there is a degree 1 map $f : S^n \to M$. Show that M has the integral cohomology of a sphere.

Good luck!