290 Qual Prep Problem Set 0 — Summer 2007

Since this is our first meeting, we'll work on these problems together in class today.

1. **A warmup joke problem.** Let G be the homophony group, the group generated by the 26 letters of the English alphabet with relations $w_1 = w_2$ if these are two English words which are pronounced the same (e.g. great = grate). Prove that the group is trivial! (This joke was invented by Zagier and others, according to Justin.)

2. **Homotopies.**
   (a) A space is *contractible* if it is homotopy equivalent to a point. Show that a space is contractible if and only if its identity is homotopic to a constant map.
   (b) Show that any non-surjective map $f : X \to S^n$ is homotopic to a constant map.
   (c) Show that $\mathbb{R}^{n+1} - \{0\}$ is homotopy equivalent to S^n.
   (d) Show that when n is odd, the antipodal map $S^n \to S^n$, given by negation of unit vectors $x \to -x$, is homotopic to the identity map on S^n.

3. **Calculating the fundamental group.** Consider the pictures below. They represent solid polygons with their boundaries identified according to the arrows shown. Describe the fundamental group in each case. Is the hexagonal one homeomorphic to a torus?

4. **Necessary algebra.**
   (a) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. Show that the following conditions are equivalent:
      i. There exists a homomorphism $r : B \to A$ with $ri = \mathrm{id}_A$.
      ii. There exists a homomorphism $s : C \to B$ with $ps = \mathrm{id}_C$.
      iii. $B \cong A \oplus C$, and the sequence is $0 \to A \to A \oplus C \to C \to 0$ with the obvious inclusion and projection.
   (b) Suppose the commutative diagram of abelian groups below has exact rows.

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   Show that if f_1, f_2, f_4, and f_5 are isomorphisms, then so is f_3. Can you weaken this statement?

(c) Show that a long exact sequence may be written as a collection of short exact sequences. For each exact sequence of abelian groups below, say as much as possible about the unknown group G and homomorphism α.

i. $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$

ii. $0 \to \mathbb{Z} \to G \to \mathbb{Z}/2 \to 0$

iii. $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$

iv. $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

v. $0 \to \mathbb{Z}/3 \to G \to \mathbb{Z}/2 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$