290 Qual Prep Problem Set 1 **Summer 2007**

1. **Euler characteristic.** Recall that the Euler characteristic can be defined as

$$\chi(X) := \sum_i (-1)^i (\text{number of } i\text{-cells of } X).$$

It can be shown that this is a topological invariant of X.

(a) Let $p : (E, e) \to (B, b)$ be a covering and assume that B is path connected and that $\chi(B)$ exists. If B is a CW-complex and $p^{-1}(b)$ has cardinality n, prove that $\chi(E) = n \cdot \chi(B)$.

(b) Compute the Euler characteristic of the following spaces by finding CW-structures on them.

i. The n-sphere S^n. (Note that the $n = 2$ case is Euler's famous formula for polyhedra.)
ii. The n-disk D^n.
iii. The orientable surface of genus g, Σ_g.
iv. The non-orientable surface of genus g, N_g.
v. The torus T^2.
vi. The Klein bottle K.
vii. Real projective space $\mathbb{R}P^n$.
viii. Complex projective space $\mathbb{C}P^n$.

(c) Which compact connected surfaces can cover the Klein bottle? Can a genus 2 surface be the covering space of a torus?

(d) Let X be a space and let $U, V \subseteq X$ be open subsets such that $U \cup V = X$. Establish a relation between $\chi(U), \chi(V), \chi(U \cap V)$, and $\chi(X)$, assuming that they are all defined, and prove it, using the homology definition of the Euler characteristic

$$\chi(X) = \sum_i (-1)^i \operatorname{rk} H_i(X; \mathbb{Z}) = \sum_i (-1)^i \dim_{\mathbb{Q}} H_i(X; \mathbb{Q}).$$

2. **Standard qual questions.**

(a) Let $i : S^1 \to D^2$ denote the inclusion. Prove that there is no continuous map $r : D^2 \to S^1$ such that $ri = \mathrm{id}_{S^1}$. Use this to prove that every continuous map $f : D^2 \to D^2$ has a fixed point, i.e. a point x such that $f(x) = x$.

(b) Prove the fundamental theorem of algebra: Suppose that

$$f(x) = c_0 + c_1 x + \ldots c_{n-1} x^{n-1} + x^n$$

is a polynomial with coefficients $c_i \in \mathbb{C}$ and degree $n > 0$. Then there is a complex number a such that $f(a) = 0$.

3. **The fundamental group of CW-complexes.** Assume X is a finite CW-complex. Let $X^{(n)}$ denote the n-skeleton of X.

(a) Assume $\dim(X) = 2$ and $X^{(0)} = \{x_0\}$ and that there are attaching maps $\varphi_1, \ldots, \varphi_n : S^1 \to X$ that send $1 \in S^1$ to x_0. Let $a_1, \ldots, a_m$ be the 1-cells of X. Interpret the statement

$$\pi_1(X, x_0) = \langle a_1, \ldots, a_m \mid [\varphi_i] = 1 \text{ for } i = 1, \ldots n \rangle$$

in a meaningful way, and prove it.

(b) Compute $\pi_1(S^n)$ for $n > 1$.

(c) Assume X is path connected. Prove that $\pi_1(X) = \pi_1(X^{(2)})$. *Hint: Use induction on the number of cells of dimension* > 2.

(d) Show that for every finitely presented group G, there exists a finite CW-complex X such that $G \cong \pi_1(X)$.

4. **Computing the fundamental group.**

   (a) Let $X \subseteq \mathbb{R}^3$ be the union of n distinct lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

   (b) Let $M_k := S^1 \cup_{\varphi_k} D^2$, where $\varphi_k : S^1 \to S^1$ sends $z \mapsto z^k$. Compute $\pi_1(M_k)$.

   (c) Let M be the Möbius band, obtained as a quotient of $I \times I$ by identifying $(t, 0) \sim (1-t, 1)$. Compute $\pi_1(M)$.

   (d) Let X be a CW-complex with m n-cells. Compute $\pi_1(X^{(n)}/X^{(n-1)})$.

   (e) Compute $\pi_1(\mathbb{R}P^n)$ and $\pi_1(\mathbb{C}P^n)$ for all n.

   (f) Take a solid cube and identify its faces in pairs, as shown below. Compute the fundamental group of this space.

5. **Covering Spaces.**

   (a) Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

   (b) Find all the connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

   (c) Let X be the space obtained from $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover of X, and describe the action of $\pi_1(X)$ on the universal cover.

   (d) Describe the universal cover of the cube from the previous problem. Deduce that the space is a 3-manifold.