290 Qual Prep Problem Set 2 — Summer 2007

1. **Computing homology.**
   (a) Compute $H_*(S^n;\mathbb{Z})$ using induction and Mayer-Vietoris.
   (b) Prove the suspension isomorphism: There is a natural isomorphism $\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X)$ for all n and all X.
   (c) Compute the integral homology groups of $\mathbb{R}P^n, \mathbb{C}P^n$, and $\mathbb{H}P^n$ using the standard cell structure.
   (d) Compute the integral homology of the Klein bottle K.
   (e) Compute the integral homology of the torus $T = S^1 \times S^1$.
   (f) Compute the integral homology of $K \times T$.
   (g) Compute the integral homology of $K \times \mathbb{R}P^4$.

2. **Fun with degrees.** The degree of a map $f : S^n \to S^n$ is characterized by the condition

$$f_*(\alpha) = \deg(f) \cdot \alpha \quad \text{ for all } \alpha \in \widetilde{H}_n(S^n).$$

   (a) Prove that $\deg(f) = \deg(\Sigma f)$, where Σ denotes suspension.
   (b) Compute the mapping degree of a reflection $r : S^n \to S^n$ about some hyperplane through the origin of $\mathbb{R}^{n+1}$.
   (c) Compute $\deg(-\operatorname{id})$. Is the antipodal map homotopic to the identity map?
   (d) Show that every self-map f of an even dimensional sphere has either a fixed point or an antipodal point, i.e. there exists an $x \in S^n$ such that $f(x) = \pm x$.
   (e) Let's prove the Hairy Ball Theorem! A *vector field* on a sphere S^n is a map $v : S^n \to \mathbb{R}^{n+1}$. (Think of $S^n \subseteq \mathbb{R}^{n+1}$. Then such a map assigns to every point $x \in S^n$ a vector $v(x) \in \mathbb{R}^{n+1}$.) Show that a continuous nowhere-vanishing vector field on S^n exists if and only if n is odd.
   (f) Suppose $f : S^n \to S^n$ satisfies $f(x) = f(-x)$ for all x (i.e. f is an even map). Show that the degree of f is even. Moreover, show that if also n is even, then the degree of f is zero (this is actually easier, so you might want to try this first). (Hints are given in Hatcher, Section 2.2 Exercise 14.)
   (g) It can be shown that any map $\varphi : S^k \to S^k$ satisfying $\varphi(-x) = -\varphi(x)$ has odd degree. Use this to prove that if $n > m \geq 1$, then there is no continuous map $f : S^n \to S^m$ with $f(-x) = -f(x)$. Also prove that if $g : S^n \to \mathbb{R}^n$ is a continuous map and $n \geq 1$, then there is a point x with $g(x) = g(-x)$.

3. **Standard qual questions.**
   (a) Let X be a finite-dimensional CW-complex with only even-dimensional cells. Prove that $H_*(X;\mathbb{Z})$ is torsion-free.
   (b) Let H be a finitely generated abelian group and let $n > 0$. Show that there exists a finite CW-complex X with $\widetilde{H}_k(X) = H$ if $k = n$ and $\widetilde{H}_k(X) = 0$ otherwise. Now let $\{H_n\}_{n\in\mathbb{N}}$ be a sequence of finitely generated abelian groups. Prove that there is a CW-complex X with $\widetilde{H}_n(X) = H_n$ for all n.

(c) Let $X \vee Y$ be the one-point union of X and Y. Prove for each $k > 0$ there is a split short exact sequence

$$0 \to H_k(X \vee Y) \to H_k(X \times Y) \to H_k(X \times Y, X \vee Y) \to 0.$$

(d) Let T be the torus, which is obtained by identifying the edges of the unit square in the usual manner. Let $S^1 \vee S^1$ be the one-point union of circles which is the image of the boundary of the unit square. Compute $H_*(T, S^1 \vee S^1; \mathbb{Z})$ and the map

$$i_* : H_*(S^1 \vee S^1; \mathbb{Z}) \to H_*(T; \mathbb{Z}),$$

where $i : S^1 \vee S^1 \to T$ is the inclusion map.

(e) Let $Z = S^1 \vee S^1 \vee S^2$ be the one-point union of the two circles and a 2-sphere. Prove that $H_*(Z; \mathbb{Z})$ and $H_*(T; \mathbb{Z})$ are isomorphic, but that Z and T do not have the same homotopy type.

4. **The Euler characteristic, again.** For a graded abelian group $A = \bigoplus_i A_i$, define the Euler characteristic

$$\chi(A) = \sum_i (-1)^i \operatorname{rk} A_i.$$

We may consider a chain complex as a graded abelian group by forgetting about the differential. Show that for a chain complex (C, d), we have $\chi(C) = \chi(H(C, d))$, i.e. that the Euler characteristic of a chain complex is the same as the Euler characteristic of the graded group we get from taking the homology of the chain complex. Use this to prove that the two definitions of the Euler characteristic for a CW-complex coincide. That is, prove that for a finite CW-complex X, we have

$$\sum_i (-1)^i (\text{number of } i\text{-cells of } X) = \chi(X) = \sum_i (-1)^i \operatorname{rk} H_i(X; \mathbb{Z}).$$