290 Qual Prep Problem Set 3 — Summer 2007

1. **Cup products.**
   (a) Suppose a space X has an open cover $X = \bigcup_{i=1}^{n} U_i$ such that each inclusion $U_i \hookrightarrow X$ is nullhomotopic. Show that all n-fold cup products vanish in $H^*(X;\mathbb{Z})$.
   (b) Show that all cup products are trivial on ΣX for any space X.
   (c) Show that if M is a closed manifold with the homotopy type of a suspension. Show that M has the integral cohomology of a sphere.
   (d) Compute the cohomology rings $H^*(\mathbb{C}P^n;\mathbb{Z})$ and $H^*(\mathbb{R}P^n;\mathbb{Z}/2)$. What are $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ and $H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$?
   (e) Compute the cohomology rings $H^*(T;\mathbb{Z}/2)$ and $H^*(K;\mathbb{Z}/2)$, where T is the torus and K is the Klein bottle.

2. **Degrees of maps between manifolds.** For a map $f : M \to N$ between n-dimensional orientable manifolds, we can define the degree of f by the equation

$$f_*([M]) = \deg(f) \cdot [N]$$

   where $[M] \in H_n(M;\mathbb{Z})$ and $[N] \in H_n(N;\mathbb{Z})$ are their fundamental classes.
   (a) Show that a self-homotopy equivalence $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ must have degree $+1$. Note that this implies that $\mathbb{C}P^k$ does not admit an orientation-reversing homeomorphism if k is even.
   (b) Suppose $k > 0$ and $l > 0$. Show that any map $f : S^{k+l} \to S^k \times S^l$ must have degree zero.
   (c) Let M_i denote the closed orientable surface of genus i. Show that there is a map $f : M_g \to M_h$ with $\deg(f) = 1$ if and only if $g \geq h$.

3. **Standard qual questions.**
   (a) Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.
   (b) Suppose M is a $2k$-dimensional closed orientable manifold. Suppose that $H_{k-1}(M;\mathbb{Z})$ is torsion-free. Show that $H_k(M;\mathbb{Z})$ is torsion-free.
   (c) Using the fundamental class in $\mathbb{Z}/2$ homology, there is an obvious definition of the mod 2 degree for a continuous map between closed manifolds (not necessarily oriented). Show that there is no map of nonzero mod 2-degree between the Klein bottle and the torus, in either direction.
   (d) Give an example of a space X such that $H^i(X;\mathbb{Z}) = \mathbb{Z}$ for all i, and such that the cohomology ring $H^*(X;\mathbb{Z})$ is finitely generated.
   (e) Let M be a compact manifold with boundary. Show that M does not retract onto ∂M.
   (f) Suppose M is a compact orientable contractible manifold with boundary. Show that ∂M has the integral homology of a sphere.
   (g) Let M be an odd dimensional closed manifold. Show that the Euler characteristic $\chi(M)$ vanishes. (Hint: First define the "mod 2 Euler characteristic.")

(h) Prove that $\mathbb{R}P^2$ is not the boundary of a compact 3-manifold. (Hint: Suppose there is such an M. Glue two copies of M together along their common boundary and then use part (g).)

4. **Orientability.**

(a) Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.

(b) Suppose X is a topological group which is also a manfiold. Show that X is orientable.

(c) Suppose M is a compact $\mathbb{Z}/3$-orientable manifold. Show that M is orientable.

(d) Show that $M \times N$ is orientable if and only if M and N are orientable.

(e) Show that any covering space of an orientable manifold is orientable. You may assume that the covering space is a manifold (though you should probably know how to prove this for the qual). Note that this is another proof that no genus 2 surface can cover the torus, as in 1.(c) from the first problem set, since by the Euler characteristic calculation, such a surface would be non-orientable.