# What Is Algebraic Topology?

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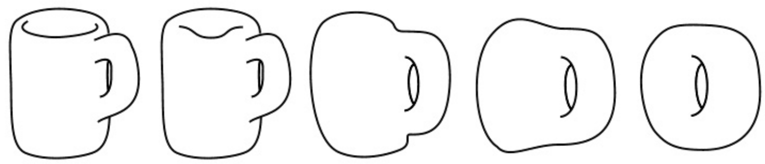

Algebraic topology is a branch of mathematics that uses algebraic objects, such as numbers and equations, to study geometric objects called "spaces." Examples of spaces include the surface of the earth, the set of solutions to an equation in several variables, or even something as large and complicated as the entire universe. The main goal is to understand properties that are intrinsic to the shape of a space—properties that occur regardless of size and are not disturbed by any amount of stretching or bending (but not breaking, tearing, or puncturing). Compare a standard coffee cup to a doughnut, and you'll notice that they both have exactly one hole. If the coffee cup were made of some malleable material, rather than fired clay, you can imagine gently pulling the bottom of the part of the cup that holds the coffee up, and then pushing this around the handle until the cup resembled the doughnut. In the eyes of the topologist, the coffee cup and the doughnut are essentially the same—we say that they are *homeomorphic*. Of course, in everyday life, the rigid shape of the coffee cup is important, since you can't very well drink from a donut, but a topologist seeks to understand the coarser qualities of shapes, as a first step towards understanding their geometry. One of the ultimate goals of topology is to classify spaces up to homeomorphism. In the three-dimensional case, Gregori Perelman's recent proof of Thurston's Geometrization conjecture and the famous Poincaré conjecture completes the classification for an important class of spaces.

To study spaces, the algebraic topologist builds an extensive tool box of "mathematical machines" that associate to a space an algebraic object, such as a sequence of numbers, a polynomial, or a collection of abelian groups. Consider a soccer ball, for example. A typical soccer ball is a sphere stitched out of leather pentagons and hexagons. If you add up the number of pentagons and hexagons, subtract the number of straight seams, and then add the number of points where the seams intersect, you get the number two. If you choose a different set of shapes to cover the ball with and do the same counting, you again get two. This always happens, no matter what shape you choose to cover the soccer ball with, no matter whether you have an average sized soccer ball or one the size of the moon. This number is called the Euler characteristic, and it describes an intrinsic property of the sphere that is not affected by stretching, squishing, or otherwise distorting (so long as you don't tear or puncture). Because it

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does not change under such distortions, the Euler characteristic is an *invariant* of the sphere, and there is a mathematical machine which will output this number if you put in the sphere.

Algebraic topology is full of these beautiful mathematical machines that convert geometric information into algebraic information. Indeed, it is the job of the homotopy theorist to construct and operate these machines. Though homotopy theory is an extremely abstract and technical field, its tools have proven enormously effective in solving concrete problems in geometry and its methods influence algebraic geometers and number theorists alike. Algebraic topology builds bridges between seemingly disparate fields, drawing connections across the realm of mathematics, and as such, its machines are employed to solve problems ranging from algebraic to analytic.

The most fundamental of these mathematical machines is cohomology, which, though born in algebraic topology, has become an incredibly powerful tool central to the whole of modern mathematics. Cohomology is a general term for various collections of abelian groups associated to geometric objects. Cohomology theories tend to be difficult to describe, but once they are constructed, they become extremely useful, especially since they are relatively easy to compute. The primary example is singular cohomology, which is responsible for the two we associated to the soccer ball above. Singular cohomology puts this number in a broader context and gives us a tool to calculate similar numbers for all kinds of spaces, rather than just spheres. In an appropriate sense, singular cohomology keeps track of the number of holes a space has in various dimensions. A more sophisticated example of a cohomology theory is $K$-theory, which contains information about vector bundles over a space. Fields medalist Alexander Grothendieck invented $K$-theory to formulate the famous Grothendieck-Riemann-Roch theorem and since then $K$-theory has been employed to prove extremely interesting and strong theorems relating analysis, geometry, and algebra. One monumental application of $K$-theory is the Atiyah-Singer index theorem, which relates purely topological data to analytic data of solutions to a differential equation. Another application of $K$-theory is Adams's proof that the only division algebras over $\mathbb{R}$ are the real numbers, the complex numbers, the quaternions, and the octonians. The only known proofs of this purely algebraic result use algebraic topology in an essential way.