INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL

Int. J. Robust. Nonlinear Control 0000; **00**:1–33

Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc

Coordinate-free Formulation for Dynamics and Control of a Chain Pendulum on a Cart

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SUMMARY

A new multibody dynamics model, that is referred to as a chain pendulum on a cart, is presented. It is composed of an arbitrarily number of rigid links that are serially connected by spherical joints, and a cart that is moving in a horizontal plane. This yields rich dynamic structures on the nonlinear configuration manifold that is comprised of the copies of the unit-sphere and the two dimensional Euclidean space. In this paper, we provide Lagrangian mechanics, controllability analysis, and controller design for a chain pendulum on a cart. In particular, it is shown that any equilibrium can be locally asymptotically stabilized by linear control analysis, and a nonlinear control system is also presented to stabilize the inverted equilibrium for the special case of a single link. The unique contribution is that all of these dynamic models and control analysis are developed in a coordinate-free fashion such that the corresponding results can be uniformly applied to a chain pendulum on a cart with an arbitrary number of links. The proposed coordinate-free formulation completely avoids singularities and complexities that are associated with local coordinates of the nonlinear configuration manifold.

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KEY WORDS: Geometric control; Lagrangian mechanics; spherical pendulum

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Contract/grant sponsor: National Science Foundation; contract/grant number: CMMI-1243000, DMS-0726263, DMS-1001521, DMS-1010687, and CMMI-1029445.

1. INTRODUCTION

Dynamics of rigid bodies that are attached to a pivot, moving under the effects of gravity have been studied extensively with various assumptions [1, 2]. The planar pendulum with a single massless link is often studied in undergraduate-level engineering courses. It has been extended in several ways to obtain a spherical pendulum, a double pendulum, or triple pendulum [3–5]. The pendulum bob model also has been generalized to an axially-symmetric rigid body to obtain a Lagrange top [6], or to an arbitrary rigid body to yield a 3D pendulum [7]. The pendulum attached to a cart that moves on a horizontal plane is often used as a benchmark example for control of underactuated mechanical systems. Various control techniques have been applied to stabilize the inverted equilibrium by the control force acting on the cart. These include passivity-based approaches [8, 9], Lyapunov-based method [10], and controlled-Lagrangian systems [11]. Swing-up strategies [12, 13] and hybrid control systems also have been considered [14].

Despite extensive research activities over a long-time period, there has been little attention paid to dynamics and control of spherical pendula with a large number of links. This is mainly due to the technical challenges to derive the equations of motion in a concise form that is suitable for dynamics analysis and controller design. The configuration of a spherical pendulum is often expressed in terms of the set of angles representing the direction of each link, and it yields complicated equations of motion involving trigonometric functions. Such local coordinates also cause singularities that make it difficult to study dynamic properties globally. Alternatively, equality constraints that enforce the fixed length of links are incorporated, but explicit constraints introduce additional complexity in analysis and computation.

In this paper, we present a chain pendulum on a cart that is composed of a serial connection of an arbitrary number of rigid links and a cart. Each link is connected by a spherical joint, and one end is attached to a cart that is moving on a horizontal plane and actuated by a control force. The configuration manifold is given by the product of n-copies of the two-sphere S^2 and the two-dimensional Euclidean space $\mathbb{R}^2$, where n is the number of the links and the two-sphere S^2 denotes the manifold of the unit-vectors in $\mathbb{R}^3$.

We first derive the Euler–Lagrange equations for a chain pendulum on a cart based on Lagrangian mechanics on the two-sphere [15, 16]. In contrast to most of the literature that treats dynamic

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DOI: 10.1002/rnc

systems on S^2, the equations of motion for a chain pendulum on a cart are derived without using local coordinates and explicit constraints. This yields a remarkably compact expression, that is uniformly applicable for an arbitrary number of links, and it is globally valid for any configuration of the links.

Based on these, we study challenging control problems for the chain pendulum on a cart system. We give a complete description of the uncontrolled equilibria, and we prove that any of these equilibrium solutions can be made locally asymptotically stable using feedback. However, the generic feedback controller may have a small region of attraction as the stability analysis is based on the linearized dynamics. We also construct a stabilizing controller with a large region of attraction for the inverted equilibrium for a single link, that is a spherical pendulum on a cart. Otherwise, we claim that constructing stabilizing feedback controllers with large regions of attraction is very difficult, consistent with common intuition.

One of the contributions of this paper is providing an intrinsic and unified framework to study dynamics and control of chain pendulum on a cart systems. Dynamics and control of a spherical pendulum with an arbitrary number of links have not investigated before. The presented coordinate-free formulation based on geometric mechanics and control approaches enables us to study complicated pendulum dynamics globally in a concise fashion without singularities. The proposed framework can be applied to other complex multibody systems evolving on copies of the two-sphere.

This paper is organized as follows. Lagrangian dynamics of a chain pendulum on a cart system is described in Section 2. Their equilibrium structures and control strategies are presented in Sections 3 and 4, respectively. These are followed by numerical examples and conclusions in Sections 5 and 6, respectively. The proofs of some of the more technical results are deferred to Appendix A.

2. LAGRANGIAN DYNAMICS OF THE CHAIN PENDULUM ON A CART SYSTEM

2.1. *Chain Pendulum on a Cart*

Consider a chain pendulum on a cart system as illustrated in Figure 1. The inertial frame is defined by the unit vectors $e_1 = [1;\,0;\,0] \in \mathbb{R}^3$, $e_2 = [0;\,1;\,0] \in \mathbb{R}^3$, $e_3 = [0;\,0;\,1] \in \mathbb{R}^3$; we assume that e_3 is in the direction of gravity. Define $C = [e_1,\, e_2] \in \mathbb{R}^{3\times 2}$, which we will use to identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$.

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DOI: 10.1002/rnc

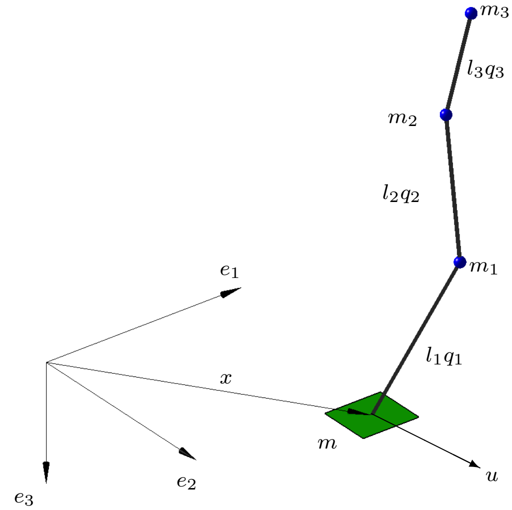

Figure 1. Chain pendulum on a cart ($n = 3$)

The cart translates on the horizontal e_1–e_2 plane, and it is subject to a horizontal control force $u \in \mathbb{R}^2$. The mass of the cart is denoted by m, and its position is described by $x \in \mathbb{R}^2$. A serial connection of n rigid links, connected by spherical joints, is attached to the cart. For simplicity, we assume that the mass of each link is concentrated at the outboard end of the link. The mass of the i-th link is denoted by m_i and the link length is denoted by l_i for $1 \leq i \leq n$, where the link attached to the cart is considered as the first link. The direction vector of the i-th link measured from m_{i-1} (or the cart when $i = 1$), to m_i in the inertial frame is given by $q_i \in \mathsf{S}^2$. Here, the two-sphere is defined as

$$\mathsf{S}^2 = \{q \in \mathbb{R}^3 \| \, \|q\| = 1\},$$

which corresponds to the two-dimensional unit-sphere embedded in $\mathbb{R}^3$. The configuration of a chain pendulum on a cart is described by the direction of each link and the location of the cart, and the corresponding configuration manifold is $(\mathsf{S}^2)^n \times \mathbb{R}^2$.

The kinematic equation for the direction vector of the i-th link q_i is given by

$$\dot{q}_i = \omega_i \times q_i = \hat{\omega}_i q_i, \tag{1}$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity of the i-th link satisfying $\omega_i \cdot q_i = 0$. The *hat* map $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ transforms a vector in $\mathbb{R}^3$ to a 3×3 skew-symmetric matrix, and is uniquely defined by the

DOI: 10.1002/rnc

property that $\hat{x}y = x \times y$ for any $x, y \in \mathbb{R}^3$:

$$\hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}.$$

The inverse of the hat map is denoted by the *vee* map $\vee : \mathfrak{so}(3) \to \mathbb{R}^3$.

Throughout this paper, the dot product of two vectors is denoted by $x \cdot y = x^T y$ for any $x, y \in \mathbb{R}^n$. The $n \times n$ identity matrix is denoted by I_n. The $n \times m$ matrix composed of zero elements is denoted by $0_{n\times m}$, and it is written as 0_n if $n = m$. A column-wise stack of matrices is written as $[A; B] = [A^T, B^T]^T \in \mathbb{R}^{(a+b)\times n}$ for $A \in \mathbb{R}^{a\times n}$ and $B \in \mathbb{R}^{b\times n}$. Several properties of the hat map are summarized in Appendix A.1.

2.2. *Lagrangian*

We find the expression for the Lagrangian of a chain pendulum on a cart. Using the definition of the matrix $C = [e_1, e_2]$, the location of the cart is given by $Cx \in \mathbb{R}^3$ with respect to the inertial frame. Let $x_i \in \mathbb{R}^3$ be the position of the i-th mass in the inertial frame. It can be written as

$$x_i = Cx + \sum_{a=1}^{i} l_a q_a, \tag{2}$$

and therefore, the velocity of the m_i is given by $\dot{x}_1 = C\dot{x} + \sum_{a=1}^{i} l_q q_a$. The total kinetic energy is composed of the kinetic energy of the cart and the kinetic energy of each mass:

$$T = \frac{1}{2}m\|\dot{x}\|^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \|C\dot{x} + \sum_{a=1}^{i} l_a \dot{q}_a\|^2. \tag{3}$$

Next, we find an alternative expression for the kinetic energy that is more suitable to derive the Euler–Lagrange equations. For simplicity, we first consider the part of the kinetic energy, namely T_x that is dependent on the motion of the cart. By expanding the last term of (3) and using the fact that $\|\dot{x}\| = \|C\dot{x}\|$, we obtain

$$T_x = \frac{1}{2}M_{00}\|\dot{x}\|^2 + \dot{x} \cdot \sum_{i=1}^{n} M_{0i}\dot{q}_i, \tag{4}$$

where the inertia matrices $M_{00} \in \mathbb{R}$, $M_{0i} \in \mathbb{R}^{2\times 3}$, and $M_{i0} \in \mathbb{R}^{3\times 2}$ are given by

$$M_{00} = m + \sum_{i=1}^{n} m_i, \quad M_{0i} = C^T \sum_{a=i}^{n} m_a l_i, \quad M_{i0} = M_{0i}^T \tag{5}$$

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for $i = 1, \dots, n$. The part of the kinetic energy, namely T_q, that is independent of $\dot{x}$ is given by

$$
\begin{aligned}
T_q &= \frac{1}{2}\sum_{i=1}^{n} m_i \|\sum_{a=1}^{i} l_a \dot{q}_a\|^2\\
&= \frac{1}{2}\sum_{i=1}^{n}\left(\sum_{a=i}^{n} m_a\right) l_i^2\|\dot{q}_i\|^2\\
&\quad + l_1\dot{q}_1\cdot(m_2 l_2\dot{q}_2 + m_3(l_2\dot{q}_2 + l_3\dot{q}_3)+\cdots)\\
&\quad + l_2\dot{q}_2\cdot(m_3 l_3\dot{q}_3 + m_4(l_3\dot{q}_3 + l_4\dot{q}_4)+\cdots)+\cdots,
\end{aligned}
$$

where T_q is decomposed into auto-terms and cross-terms between $\dot{q}_i$ and $\dot{q}_j$. This can be rewritten in a more compact form as

$$
T_q = \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\dot{q}_i\cdot\dot{q}_j, \tag{6}
$$

where the inertia constants $M_{ij} \in \mathbb{R}$ are given by

$$
M_{ij} = \left(\sum_{a=\max\{i,j\}}^{n} m_a\right) l_i l_j \tag{7}
$$

for $i, j = 1, \dots, n$.

From (4) and (6), the total kinetic energy is given by

$$
T = \frac{1}{2}M_{00}\|\dot{x}\|^2 + \dot{x}\cdot\sum_{i=1}^{n} M_{0i}\dot{q}_i + \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}\dot{q}_i\cdot\dot{q}_j, \tag{8}
$$

where the inertia matrices are given by (5), (7). The gravitational potential energy is composed of the gravitational potential energy of each mass, assuming that it is set to zero at the e_1–e_2 plane. From (2) and using $e_3 \cdot Cx = 0$ for any $x \in \mathbb{R}^2$, it can be written as

$$
V = -\sum_{i=1}^{n} m_i g x_i \cdot e_3 = -\sum_{i=1}^{n}\sum_{a=i}^{n} m_a g l_i e_3\cdot q_i. \tag{9}
$$

In short, the Lagrangian of a chain pendulum on a cart is given by $L = T - V$.

2.3. *Euler–Lagrange equations*

Next, we derive the Euler–Lagrange equations. In [15], a coordinate-free form of the Euler–Lagrange equations for the mechanical systems evolving on the two-spheres is derived from Hamilton's variational principle without using any explicit constraint on the length of q_i.

The central idea is to use the group action of the rotation group $SO(3)$ on S^2, and then express the variation of a curve on the two-sphere in terms of the exponential map. Let $q_i(t)$ be a curve on S^2

DOI: 10.1002/rnc

for $t \in [t_0, t_f]$. Its variation $q^\epsilon(t)$ is a family of curves on S^2 parameterized by $\epsilon \in (-c, c)$ for some positive constant c, satisfying $q^0(t) = q(t)$, $q^\epsilon(t_0) = q(t_0)$, and $q^\epsilon(t_f) = q(t_f)$ for any $\epsilon \in (-c, c)$ and $t \in [t_0, t_f]$ [17]. In this paper, we propose that variation of a curve on S^2 is expressed as

$$q_i^\epsilon(t) = \exp(\epsilon\hat{\xi}_i(t))q_i(t),$$

where $\xi_i(t)$ is a curve in $\mathbb{R}^3$ satisfying $q_i(t) \cdot \xi_i(t) = 0$ for all $t \in [t_0, t_f]$ and $\xi_i(t_0) = \xi_i(t_f) = 0$.

The above expression implies that for each t, $q_i^\epsilon(t)$ is obtained by rotating the unit-vector $q_i(t)$ about the axis $\xi_i(t)$ by the angle $|\epsilon|\|\xi_i(t)\|$. Therefore, it is straightforward to verify that it satisfies the three properties of the variation, and most importantly, it is guaranteed that the varied curve $q_i^\epsilon(t)$ has unit–length, i.e. $q^\epsilon(t) \in \mathsf{S}^2$ for any $t \in [t_0, t_f]$. The corresponding infinitesimal variation is given by

$$\delta q_i(t) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \exp(\epsilon\xi_i(t))q_i(t) = \xi_i(t) \times q_i(t). \tag{10}$$

Using these, the variation of the Lagrangian of a chain pendulum on a cart can be derived to obtain the Euler–Lagrange equations.

Proposition 1

Consider a chain pendulum on a cart, whose Lagrangian is given by (8) and (9). The Euler–Lagrange equations are given by

$$\begin{bmatrix} M_{00} & M_{01} & M_{02} & \cdots & M_{0n} \\ -\hat{q}_1^2 M_{10} & M_{11}I_3 & -M_{12}\hat{q}_1^2 & \cdots & -M_{1n}\hat{q}_1^2 \\ -\hat{q}_2^2 M_{20} & -M_{21}\hat{q}_2^2 & M_{22}I_3 & \cdots & -M_{2n}\hat{q}_2^2 \\ \vdots & \vdots & \vdots & & \vdots \\ -\hat{q}_n^2 M_{n0} & -M_{n1}\hat{q}_n^2 & -M_{n2}\hat{q}_n^2 & \cdots & M_{nn}I_3 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix}$$
$$= \begin{bmatrix} u \\ -\|\dot{q}_1\|^2 M_{11} q_1 - \sum_{a=1}^n m_a g l_1 \hat{q}_1^2 e_3 \\ -\|\dot{q}_2\|^2 M_{22} q_2 - \sum_{a=2}^n m_a g l_2 \hat{q}_2^2 e_3 \\ \vdots \\ -\|\dot{q}_n\|^2 M_{nn} q_n - m_n g l_n \hat{q}_n^2 e_3 \end{bmatrix}, \tag{11}$$

where the inertia matrices M_{ij} are defined by (5) and (7). Or equivalently, it can be written in terms of the angular velocities as

$$\begin{bmatrix} M_{00} & -M_{01}\hat{q}_1 & -M_{02}\hat{q}_2 & \cdots & -M_{0n}\hat{q}_n \\ \hat{q}_1 M_{10} & M_{11}I_3 & -M_{12}\hat{q}_1\hat{q}_2 & \cdots & -M_{1n}\hat{q}_1\hat{q}_n \\ \hat{q}_2 M_{20} & -M_{21}\hat{q}_2\hat{q}_1 & M_{22}I_3 & \cdots & -M_{2n}\hat{q}_2\hat{q}_n \\ \vdots & \vdots & \vdots & & \vdots \\ \hat{q}_n M_{n0} & -M_{n1}\hat{q}_n\hat{q}_1 & -M_{n2}\hat{q}_n\hat{q}_2 & \cdots & M_{nn}I_3 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \dot{\omega}_1 \\ \dot{\omega}_2 \\ \vdots \\ \dot{\omega}_n \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{j=1}^{n} M_{0j}\|\omega_j\|^2 q_j + u \\ \sum_{j=2}^{n} M_{1j}\|\omega_j\|^2 \hat{q}_1 q_j + \sum_{a=1}^{n} m_a g l_1 \hat{q}_1 e_3 \\ \sum_{j=1, j\neq 2}^{n} M_{2j}\|\omega_j\|^2 \hat{q}_2 q_j + \sum_{a=2}^{n} m_a g l_2 \hat{q}_2 e_3 \\ \vdots \\ \sum_{j=1}^{n-1} M_{nj}\|\omega_j\|^2 \hat{q}_n q_j + m_n g l_n \hat{q}_n e_3 \end{bmatrix}, \tag{12}$$

$$\dot{q}_i = \omega_i \times q_i. \tag{13}$$

Proof

See Appendix A.2 □

2.4. Remarks about the Chain Pendulum on a Cart System

In short, (11), or (12) and (13), provide the Euler–Lagrange equations for a chain pendulum on a cart with an arbitrary number of links. As they are developed in a coordinate-free fashion, singularities that arise when using local coordinates are completely avoided. The system of equations above for the chain pendulum on a cart system remains valid for arbitrary deformations of the chain with respect to the cart, assuming that all possible collisions of the chain pendulum with itself and of the chain pendulum with the cart are ignored.

In addition, it does not require any explicit constraints to preserve the unit-length of q_i, i.e., if the initial conditions are chosen such that $\|q_i(0)\| = 1$ and $q_i(0)\cdot\omega_i(0) = 0$, then it is guaranteed that $\|q_i(t)\| = 1$ and $q_i(t)\cdot\omega_i(t) = 0$ along the solutions of (11)–(13). This eliminates the additional computational burden to enforce that the numerical solutions evolves on the nonlinear configuration manifold, $(\mathsf{S}^2)^n \times \mathbb{R}^2$.

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DOI: 10.1002/rnc

Furthermore, the given Euler–Lagrange equations are remarkably concise, compared with the equations of motion derived in terms of the set of angles that represent the direction of each link. In such cases, they are often derived by using a symbolic computational tool due to their complexity. The Euler–Lagrange equations presented in (11)–(13) are well-organized and easy to interpret physically. For example, the left hand side of (12) represents the inertia terms, and the rigid hand side of (12) is composed of the effects of the centrifugal forces, gravity, and control force. By using these structures explicitly, we could provide further control analysis for a chain pendulum on a cart in the subsequent sections.

The proposed chain pendulum on a cart system encompasses various pendulum models as special cases. For example, the chain pendulum with the cart inertially fixed is a special case of the model above when $x \equiv 0$. If there is a single link, $n = 1$, the spherical pendulum on a cart system is obtained; if there are two serial links, $n = 2$, the double spherical pendulum on a cart system is obtained.

The planar chain pendulum on a cart is also a special case of the above model. Let $v \in \mathbb{R}^3$ be a fixed vector in the horizontal plane, i.e., $v \cdot e_3 = 0$. Suppose that the initial condition is chosen such that $\omega_i(0) \times v = 0$ and $q_i(0) \cdot v = 0$ for $i = 1, \ldots, n$ and the control input satisfies $v \cdot Cu = 0$. Then, from (12), we can show that the motion of links with respect to the cart remains confined to the vertical plane normal to v. The planar chain pendulum and cart has relatively simpler dynamics than the 3D chain pendulum and cart with the same number of links, since it has half the number of degrees of freedom.

3. DYNAMIC PROPERTIES OF THE UNCONTROLLED CHAIN PENDULUM ON A CART SYSTEM

In this section, we find the equilibrium of a chain pendulum on a cart assuming that there is no control input, and a coordinate-free form of the linearized equations is presented to approximate its dynamics near a selected equilibrium.

DOI: 10.1002/rnc

3.1. Equilibrium Configurations

The equilibrium configurations in $(\mathsf{S}^2)^n \times \mathbb{R}^2$ are given by the cart in a fixed location and all links in the chain pendulum aligned vertically, i.e.,

$$q_i \times e_3 = 0, \quad \text{for } i = 1, \ldots, n.$$

Consequently, if we restrict ourselves to the equilibria where the cart is located at the origin, i.e., $x = 0$, then there are 2^n possible equilibrium configurations where $q_i = \pm e_3$ for $i = 1, \ldots, n$. We introduce binary variables s_i to describe a specific equilibrium. Let

$$s_i = \begin{cases} 1 & \text{if } q_i = e_3 \text{ (along gravity)}, \\ -1 & \text{if } q_i = -e_3 \text{ (opposite to gravity)} \end{cases} \tag{14}$$

for $i = 1, \ldots, n$. Then, an ordered n-tuple $s = (s_1, s_2, \ldots, s_n)$, specifies a particular equilibrium, and the direction of the i-th link at the equilibrium is given by $q_i = s_i e_i$.

Among the 2^n choices, a few characteristic equilibria are selected as follows. The equilibrium for which all links are aligned with the gravity direction, $q_i = e_3$ for $i = 1, \ldots, n$, is referred to as the *hanging equilibrium*; the equilibrium for which all links are aligned opposite to the gravity direction $q_i = -e_3$ for $i = 1, \ldots, n$, is referred to as the *inverted equilibrium*. Two other interesting equilibria correspond to the cases where all adjacent links point in opposite directions, that is $q_i \cdot q_{i+1} = -1$. These two equilibria, one with the first link pointing in the direction of e_3 and the other with the first link pointing in the direction of $-e_3$, are referred to as *folded equilibria* since the chain pendulum is completely folded in each case.

3.2. Linearized Equations of Motion

Next, we derive linearized equations of motion for the local dynamics near each equilibrium. Consider a specific equilibrium where the direction of the i-th link is given by $q_i = s_i e_3$. From (10), the variation of q_i can be written as

$$\delta q_i = \xi_i \times s_i e_3, \tag{15}$$

where $\xi_i \in \mathbb{R}^3$ satisfying $\xi_i \cdot s_i e_3 = 0$, or equivalently $\xi_i \cdot e_3 = 0$. The variation of ω_i is given by $\delta\omega_i \in \mathbb{R}^3$ with $\delta\omega_i \cdot e_3 = 0$. Therefore, the third components of ξ_i and $\delta\omega_i$ for any equilibrium configuration are zero. This reflects the fact that the variation of a curve q_i on S^2 evolves on the

DOI: 10.1002/rnc

two-dimensional tangent space of S^2 at q_i, that corresponds to the plane normal to q_i. Consequently, the linearized equations of motion are expressed in terms of the first two elements of ξ_i, i.e., the state vector of the linearized equation is composed of $C^T\xi_i \in \mathbb{R}^2$. As a result, the dimension of the corresponding unconstrained linearized equation is $2n+2$. The linearized equations can be derived by substituting (15) into (11) and ignoring the higher-order terms, and they are summarized as follows.

Proposition 2

Consider an equilibrium of a chain pendulum on a cart, specified by $s = (s_1, \dots, s_n)$ and $x = 0$. The linearization of the Euler–Lagrange equations (11) can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}\mathbf{x} = \mathbf{B}u, \tag{16}$$

or equivalently

$$\begin{bmatrix} \mathbf{M}_{xx} & \mathbf{M}_{xq} \\ \mathbf{M}_{qx} & \mathbf{M}_{qq} \end{bmatrix} \begin{bmatrix} \delta\ddot{x} \\ \ddot{\mathbf{x}}_q \end{bmatrix} + \begin{bmatrix} 0_2 & 0_{2\times 2n} \\ 0_{2n\times 2} & \mathbf{G}_{qq} \end{bmatrix} \begin{bmatrix} \delta x \\ \mathbf{x}_q \end{bmatrix} = \begin{bmatrix} I_2 \\ 0_{2n\times 2} \end{bmatrix} u,$$

where the corresponding sub-matrices are defined as

$$\begin{aligned}
\mathbf{x}_q &= [C^T\xi_1;\, \dots\, ;\, C^T\xi_n] \in \mathbb{R}^{2n},\\
\mathbf{M}_{xx} &= M_{00} I_2 \in \mathbb{R}^{2\times 2},\\
\mathbf{M}_{xq} &= \begin{bmatrix} -s_1 M_{01}\hat{e}_3 C & -s_2 M_{02}\hat{e}_3 C & \cdots & -s_n M_{0n}\hat{e}_3 C \end{bmatrix} \in \mathbb{R}^{2\times 2n},\\
\mathbf{M}_{qx} &= \mathbf{M}_{xq}^T \in \mathbb{R}^{2n\times 2},\\
\mathbf{M}_{qq} &= \begin{bmatrix} M_{11} I_2 & s_{12} M_{12} I_2 & \cdots & s_{1n} M_{1n} I_2 \\ s_{21} M_{21} I_2 & M_{22} I_2 & \cdots & s_{2n} M_{2n} I_2 \\ \vdots & \vdots & & \vdots \\ s_{n1} M_{n1} I_2 & s_{n2} M_{n2} I_2 & \cdots & M_{nn} I_2 \end{bmatrix} \in \mathbb{R}^{2n\times 2n},\\
\mathbf{G}_{qq} &= \mathrm{diag}[s_1 \sum_{a=1}^{n} m_a g l_1 I_2,\, \dots, s_n m_n g l_n I_2] \in \mathbb{R}^{2n\times 2n},
\end{aligned}$$

where $s_{ij} = s_i s_j$ for $i, j = 1, \dots, n$.

Proof

See Appendix A.3. □

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Stability of each equilibrium can be determined from the eigenstructure of the linearized dynamics. The eigenvalues of (16) are the roots of $\det[\lambda^2\mathbf{M}+\mathbf{G}]=0$. Note that there are zero eigenvalues since the first two columns and rows of $\mathbf{G}$ are zero. These correspond to the dynamics of the cart given by $\delta\ddot{x}=u$.

For the hanging equilibrium, given by $s_i=1$ for all $i=1,\ldots,n$, the matrix $\mathbf{G}_{qq}$ becomes positive-definite, and the eigenvalues become imaginary. Then, the stability of the hanging equilibrium for the complete nonlinear dynamics is inconclusive from the linearized equation (16) as $\mathrm{Re}[\lambda]=0$. But, it can be shown that the hanging equilibrium is stable in the sense of Liapunov modulo the cart position. For each of the other 2^n-1 equilibrium configurations, there exist a pair of positive and negative eigenvalues, which implies that it is an unstable saddle.

4. CONTROL ANALYSIS

Various control problems can be posed for the chain pendulum on a cart system. For example, feedback control might be used to achieve asymptotic stabilization of a natural equilibrium solution. However, control system design appears to be more challenging as the number of links of the chain pendulum increases. In fact, our conjecture is that, in an experimental trial, no human would be able to asymptotically stabilize the inverted equilibrium of a chain pendulum on a cart that has more than a few links, no matter how long each of the links are.

In this section, we present several control results, but we also recognize the limitations of these results. In spite of the above conjecture, we can show that any natural equilibrium of a chain pendulum and cart system, with an arbitrary number of links, can be made locally asymptotically stable by feedback. This result follows by showing that the linearized dynamics of each natural equilibrium is completely controllable. This positive theoretical result provides encouragement to seek stabilizing nonlinear feedbacks. However, we are able to present concrete results that provide a large domain of attraction only for the special case of a single link, that is $n=1$.

4.1. Local Controllability and Stabilization

Based on the linearized equations (16), we show that local controllability can be determined by studying a reduced system.

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Proposition 3

Consider an equilibrium of a chain pendulum on a cart, specified by $s = (s_1, \ldots, s_n)$ and $x = 0$. Suppose that the inertia matrix $\mathbf{M}$ of the linearized equation (16) is positive-definite. Then, the equilibrium is completely controllable if and only if the following reduced system is completely controllable

$$\mathbf{M}_{qq}\ddot{\mathbf{x}}_q + \mathbf{G}_{qq} = \mathbf{M}_{xq}\mathbf{u} \tag{17}$$

for a control input $\mathbf{u} \in \mathbb{R}^2$, or equivalently

$$\mathrm{rank}[\lambda^2\mathbf{M}_{qq} + \mathbf{G}_{qq},\ \mathbf{M}_{qx}] = 2n \tag{18}$$

for any $\lambda \in \mathbb{C}$.

Proof

See Appendix A.4. □

This proposition states that the controllability of (16) is equivalent to the controllability of a reduced system given by (17), which represents the dynamics of the chain pendulum, without the cart, where the input matrix is given by $\mathbf{M}_{qx}$ that corresponds to the inertia coupling between the chain dynamics and the cart dynamics.

If the linear equation (17) is controllable, then it is guaranteed that the nonlinear equations for the chain pendulum and cart are locally controllable in some neighborhood of that equilibrium and the equilibrium can be made asymptotically stable using state feedback. In other words, any of the natural equilibrium solutions of the chain pendulum and cart system can be made asymptotically stable by state feedback. We present a particular control system as follows.

Proposition 4

Consider an equilibrium of a chain pendulum on a cart, specified by $s = (s_1, \ldots, s_n)$ and $x = 0$. Assume that the corresponding linear equation is controllable (as characterized by Proposition 3). The control feedback force u is:

$$u = -K_x x - K_{\dot{x}}\dot{x} - \sum_{i=1}^{n}\{K_{q_i}C^T(s_i e_3 \times q_i) + K_{\omega_i}C^T\omega_i\}, \tag{19}$$

for controller gain matrices $K_x, K_{\dot{x}}, K_{q_i}, K_{\omega_i} \in \mathbb{R}^{2\times 2}$ for $i = 1, \ldots, n$. Then, there exist values of the controller gains such that the equilibrium of the controlled system is locally asymptotically stable.

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Proof

See Appendix A.5. □

In some sense, these results are surprising, since they indicate that any natural equilibrium of the chain pendulum on a cart system is stabilizable in a generic sense and this is true for any number of links. This is inconsistent with our conjecture that such systems should be exceedingly difficult to stabilize—certainly beyond the ability of a human controller if the chain pendulum has more than several links. Of course, the local stabilization result is only a theoretical result and does not guarantee that stabilization can be practically achieved in an experiment. The region of attraction may be very small and/or the resulting closed loop may be highly sensitive to disturbances and uncertainty.

4.2. Stabilization of the inverted equilibrium for a single link chain pendulum

A nonlinear feedback controller is presented that asymptotically stabilizes the inverted equilibrium when $n = 1$. This model corresponds to an inverted spherical pendulum on a cart. For $n = 1$, the equations of motion (13) can be rewritten as

$$\begin{bmatrix} (m+m_1)I_2 & -m_1 l_1 C^T \hat{q}_1 \\ m_1 l_1 \hat{q}_1 C & m_1 l_1^2 I_3 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \dot{\omega}_1 \end{bmatrix} = \begin{bmatrix} m_1 l_1 \|\omega_1\|^2 C^T q_1 + u \\ m_1 g l_1 \hat{q}_1 e_3 \end{bmatrix}. \tag{20}$$

The nonlinear feedback is constructed using standard approaches: partial feedback linearization and use of a control Liapunov function. The procedure is similar to one taken for a planar pendulum on a cart [18], but the controller presented here is for the more complicated problem of stabilization of a spherical pendulum; in addition, the current development uses a coordinate-free formulation.

We first make the cart dynamics linear by introducing the feedback transformation

$$u = ((m+m_1)I_2 + m_1 C^T \hat{q}_1^2 C)^{-1}\mu - m_1 g C^T \hat{q}_1^2 e_3 - m_1 l_1 \|\omega_1\|^2 C^T q_1, \tag{21}$$

where $\mu \in \mathbb{R}^2$ is the transformed control input. It can be shown that the matrix in the first term of the right hand side is invertible if $m > 0$. Substituting it into (20) and rearranging, we obtain the transformed equations of motion, with linear cart dynamics,

$$\ddot{x} = \mu, \tag{22}$$

$$\dot{\omega}_1 = \frac{g}{l_1}\hat{q}_1 e_3 - \frac{1}{l_1}\hat{q}_1 C \mu. \tag{23}$$

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We now construct a control Liapunov function. It is convenient to select the output variable $y \in \mathbb{R}^2$ as:

$$y = x + k_q l_1 C^T q_1. \tag{24}$$

This output can be interpreted as the projection onto the horizontal plane of the position of a fictitious point on the link whose distance from the pivot is $k_q l_1$ for a positive constant k_q. A Liapunov function is chosen in terms of y as

$$\mathcal{V} = \frac{1}{2}k_y\|y\|^2 + \frac{1}{2}\|\dot{y}\|^2 + k_w\mathcal{W}, \tag{25}$$

for $k_y, k_w > 0$, where the additional term $\mathcal{W}$ is defined to make $\mathcal{V}$ positive-definite on $\mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2)$ in a neighborhood of the inverted equilibrium:

$$\mathcal{W} = -\frac{k_q}{2}l_1^2\|\omega_1\|^2 + k_q g l_1(1 + q_1^T e_3) + \frac{1}{2}\|\dot{x}\|^2.$$

For suitable values of the constants, this Liapunov function can be shown to be positive-definite in a neighborhood of the inverted equilibrium. It also serves as a control Liapunov function in the sense that the transformed control can be selected in feedback form to make the time derivative of the Liapunov function non-positive. The stabilization result and details of the proof are now presented.

Proposition 5

Consider the spherical pendulum on a cart system given by (20) when $n = 1$. The feedback control force on the cart is chosen as (21), where the transformed control input μ is given in feedback form by

$$\mu = ((1+k_w)I + k_q C^T\hat{q}_1^2 C)^{-1} \times (-k_y y - k_{\dot{y}}\dot{y} + k_q g C^T\hat{q}_1^2 e_3 + k_q l\|\omega_1\|^2 C^T q_1), \tag{26}$$

for positive constants $k_y, k_{\dot{y}}, k_q, k_w$ with $k_q > 1 + k_w$, where the output $y \in \mathbb{R}^2$ is given by (24). Then, the inverted equilibrium with $(-e_3, 0, 0, 0) \in \mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2)$ is asymptotically stable with a computable domain of attraction.

Proof

See Appendix A.6. □

The obtained nonlinear feedback control is obtained by substituting (26) into (21). This feedback is quite complicated, which reflect the complex coupled dynamics of the spherical pendulum and

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the cart, the fact that these dynamics evolve on a manifold $\mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2)$ with nontrivial curvature, the under-actuation of the system, and that the selected output function has relative degree two.

As stated in the proposition, there is a computable domain of attraction of the inverted equilibrium using the Liapunov function introduced previously and given in Appendix A.6 as

$$\begin{aligned}\mathcal{V}(q_1, x, \dot{q}_1, \dot{x}) = &\frac{1}{2}k_y\|y\|^2 + k_w k_q g l_1(1 + q_1^T e_3) + \frac{1+k_w}{2}\|\dot{x}\|^2\\ &- k_q l_1 \dot{x}^T C^T \hat{q}_1\omega_1 + \frac{1}{2}k_q l_1^2(k_q\|C^T\hat{q}_1\omega_1\|^2 - k_w\|\omega_1\|^2).\end{aligned}$$

Here we provide a two step construction procedure that leads to a domain of attraction. This construction first requires solving a nonlinear programming problem on $\mathbb{R}^3$:

$$\begin{aligned}&\textit{maximize}\ \frac{1}{2}k_y k_q l_1\|C^T q_1\|^2 + k_w g(1 + q_1^T e_3),\\ &\textit{subject to}\ \|q_1\|^2 = 1,\ q_1^T e_3 < -\sqrt{\frac{1+k_w}{k_q}}.\end{aligned}$$

This nonlinear programming problem is guaranteed to have a solution since it involves maximization of a continuous function on a compact set. Denote one solution by $\tilde{q}_1$ so that $\tilde{q}_1 \in \mathsf{S}^2$.

The next step is to select a constant $0 < K \leq 1$ such that the scaled sublevel set satisfies the inclusion property in $\mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2)$:

$$\{(q_1, x, \dot{q}_1, \dot{x})|\, \mathcal{V}(q_1, x, \dot{q}_1, \dot{x}) < K\mathcal{V}(\tilde{q}_1, 0, 0, 0)\} \subset \left\{(q_1, x, \dot{q}_1, \dot{x})|q_1^T e_3 < -\sqrt{\frac{1+k_w}{k_q}}\right\}.$$

Such a K is guaranteed to exist since it is shown in Appendix A.6 that $\mathcal{V}(q_1, x, \dot{q}_1, \dot{x})$ is positive-definite in a neighborhood of the inverted equilibrium defined by the larger set above.

Consequently, a domain of attraction is given by the compact scaled sublevel set

$$\left\{(q_1, x, \dot{q}_1, \dot{x}) \in \mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2)|\, \mathcal{V}(q_1, x, \dot{q}_1, \dot{x}) < K\mathcal{V}(\tilde{q}_1, 0, 0, 0)\right\}.$$

That is, any solution that starts in this compact scaled sublevel set remains in it and asymptotically approaches the inverted equilibrium. As usual, this provides a conservative estimate of the domain of attraction of the inverted equilibrium.

4.3. Comments on stabilization of the chain pendulum and cart system for $n > 1$

It may be possible to generalize the above stabilization results for a chain pendulum with a single link to cases where the chain pendulum has multiple links, that is $n > 1$. The key is to select a control

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Liapunov function based on an output variable that represents the projection of a location on the last link of the chain pendulum. A control Liapunov function is given by (25), where the additional term $\mathcal{W}$ is selected using constructive nonlinear control approaches [19]. This approach would seem to be conceptually natural, but important difficulties arise due to the fact that the relative degree of the output function and the degree of under-actuation increase as the number of links increase. Thus the control transformation and the control Liapunov function are highly complex even for relatively few links. All of these factors imply that the complexity of a feedback controller, constructed in this way, increases rapidly as the number of links increases. This would seem to give rise to both practical difficulties in constructing the feedback as well as loss of closed loop robustness.

It is well known for the simple planar pendulum that the required control bandwidth increases as the link length decreases. The same effect holds for the chain pendulum on a cart system and control difficulties increase as the link lengths decrease. These difficulties are independent of the challenges identified above.

We do not claim that the concrete stabilization results presented above are complete. However, major challenges would need to be overcome to go beyond those results. In this sense, the chain pendulum and cart system provides a benchmark example that is at the frontier of what contemporary nonlinear control techniques are able to handle.

5. NUMERICAL EXAMPLES

We consider several numerical examples where we illustrate the results presented in this paper. Examples are given for both uncontrolled and controlled dynamics of a chain pendulum and cart system.

We consider a chain pendulum model with five identical links, i.e. $n = 5$. This system has twelve degrees of freedom with two force control inputs on the cart. The properties of the cart and the links are chosen as

$$m = 0.5\,\mathrm{kg}, \quad m_i = 0.1\,\mathrm{kg},\ l_i = 0.1\,\mathrm{m}\ \text{ for } i = 1, \ldots, 5.$$

Throughout this section, the following units are used: $\mathrm{kg}, \mathrm{m}, \mathrm{sec}$ and rad, unless otherwise specified.

First, simulation results for the uncontrolled case are presented. The initial condition is a small perturbation from one of the completely folded equilibria, given by $s = (-1, 1, -1, 1, -1)$. More

DOI: 10.1002/rnc

explicitly, the initial conditions are chosen as

$$\begin{aligned} x(0) &= [0.2; 0.1], \quad \dot{x}(0) = [0; -0.1], \\ q_i(0) &= s_i e_3 = (-1)^i e_3 \text{ for } i = 1, \dots, 4, \\ q_5(0) &= [\sin 1^\circ; 0; -\cos 1^\circ], \\ \omega_i(0) &= 0_{3\times 1} \text{ for } i = 1, \dots, 5, \end{aligned} \tag{27}$$

where the fifth link is perturbed by 1° from the folded equilibrium, and the cart is slightly displaced from the origin. The corresponding simulation results are shown in Figure 2. The given initial condition guarantees that the relative motion of the links with respect to the cart always lies in the e_1–e_3 plane, which is depicted in Figure 2(c). It shows the complicated trajectories traced by the last mass. Figure 2(d) demonstrates the complex energy transfer between the gravitational potential energy, the kinetic energy of the cart, and the kinetic energy of the links.

Second, the feedback control presented at Proposition 4 is applied to stabilize the hanging equilibrium of the above case. The initial conditions are same as in (27), and the controller gains are chosen from a linear quadratic regulator with the weighting matrices $Q = \mathrm{diag}[8I_2, I_{2n}, 8I_2, I_{2n}]$ and $R = I_2$. We define the following variables that measure the direction errors and the angular velocity errors of links:

$$e_q = \sum_{i=1}^{n} \|q_i - s_i e_i\|, \quad e_\omega = \sum_{i=1}^{n} \|\omega_i\|,$$

which represent the accumulated errors for the direction and the angular velocity of the links. Figure 3 illustrates that the chain pendulum on a cart asymptotically approaches the hanging equilibrium.

Next, we consider another case of the control system presented in Proposition 4 to stabilize a partially folded equilibrium given by $s = (-1, -1, -1, 1, 1)$, i.e., the first three links are opposite to gravity, and the remaining last two links are aligned with gravity. Initial conditions are given as follows:

$$\begin{aligned} q_1(0) &= -[\sin 6^\circ; 0; \cos 6^\circ], \\ q_2(0) = q_3(0) &= -[0; \sin 4^\circ; 0; \cos 4^\circ], \\ q_4(0) &= [\sin 5^\circ \cos 4^\circ; -\sin 5^\circ \sin 4^\circ; \cos 5^\circ], \\ q_5(0) &= [-\sin 35^\circ; 0; \cos 35^\circ]. \end{aligned}$$

The initial conditions for $x(0), \dot{x}(0)$ and $\omega_i(0)$ are identical to (27). Figure 4 illustrates that the controlled trajectories asymptotically converges to the folded equilibrium $s = (-1, -1, -1, 1, 1)$. But, in this case, the chosen initial condition is sufficiently close to the selected equilibrium. It is observed that if the initial conditions are perturbed further, the resulting trajectory does not converge.

We now show the closed loop responses of a spherical pendulum on a cart ($n = 1$) using the nonlinear feedback given by (21) and (26) that is selected to stabilize the inverted equilibrium. The cart mass, link mass and link length are identical to the above cases, and the control parameters are given by $k_y = 4$, $k_{\dot{y}} = 2.8$, $k_w = 1$, and $k_q = 14$. The initial conditions are chosen as $x(0) = [0.2; 0.1]$, $\dot{x}(0) = [0; 0.1]$, $q_1(0) = [0.64; -0.64; -0.42]$, and $\omega_1(0) = 0$. The initial angle between the link and the upward vertical direction is $65°$. The corresponding closed loop simulation responses are illustrated in Figure 5.

6. CONCLUSIONS

We present a chain pendulum on a cart model that is composed of an arbitrary number of serially-connected links and a planar cart that is actuated by a control input. This encompasses various pendulum models, such as the double planar pendulum and the spherical pendulum on a cart. Euler–Lagrange equations are developed on the configuration manifold $(\mathsf{S}^2)^n \times \mathbb{R}^2$, and controllability is studied. A feedback control system that stabilizes any equilibrium of a chain pendulum on a cart is presented, and a nonlinear control system that stabilizes the inverted equilibrium is derived for the special case when the number of the links is $n = 1$.

The modeling, analysis of dynamics, control analysis and design, and computations are carried out directly on the nonlinear configuration manifold in terms of a geometric coordinate-free framework. This approach provides a remarkably compact form of the equations of motion, that enables us to analyze properties of their uncontrolled dynamics and at least some control properties for an arbitrary number of the links. Singularities and complexities, that are associated with local coordinates or explicit constraints to enforce that the state evolves on the manifold, are completely avoided. This important fact has not been appreciated by many researchers in dynamics and control who continue to formulate many dynamics and control problems in local coordinates, which could otherwise be analyzed with greater ease in a coordinate-free framework.

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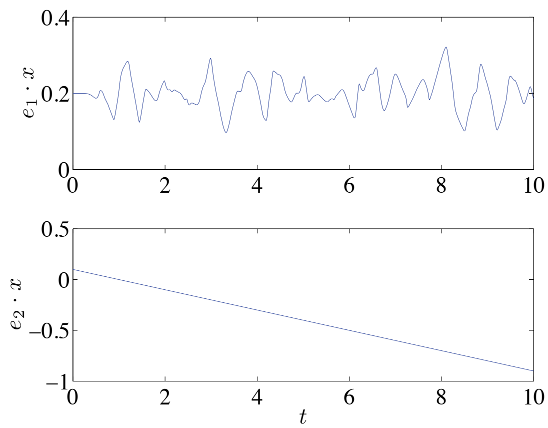
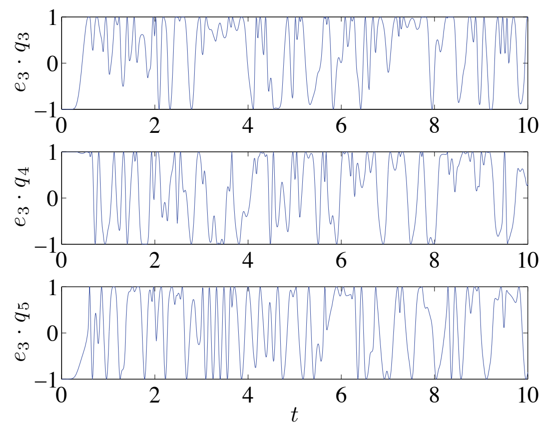
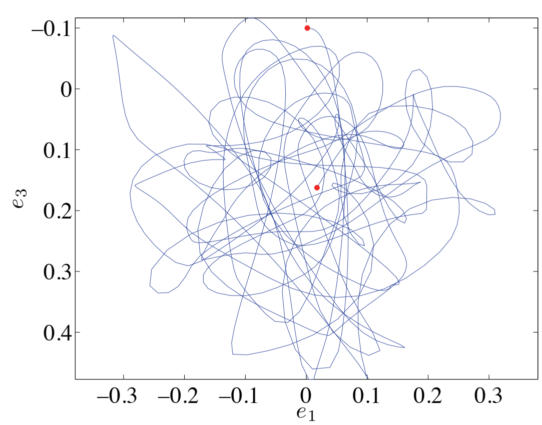


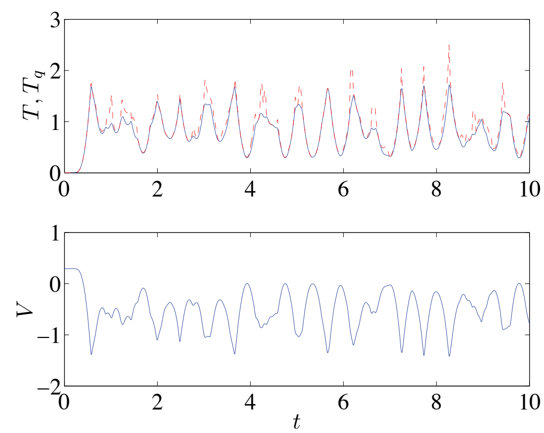
(a) Cart position x



(b) Third components of q_3, q_4, q_5



(c) Location of m_5 with respect to the cart ($\sum_{i=1}^{5} l_i q_i$) in the e_1–e_3 plane



(d) Kinetic energy T (T_q in (6): red,dotted) and potential energy V

Figure 2. Uncontrolled response: perturbation from a folded equilibrium

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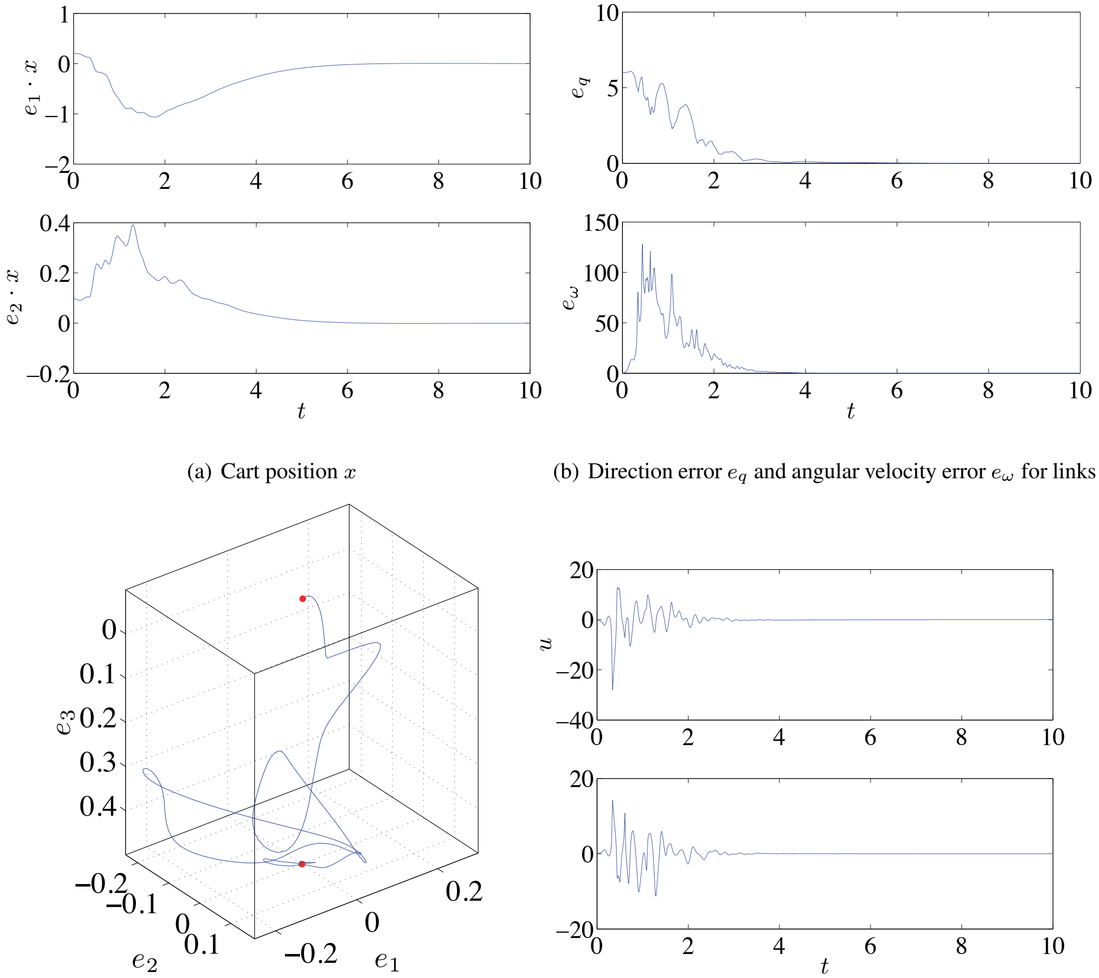

(a) Cart position x

(b) Direction error e_q and angular velocity error e_ω for links

(c) Location of m_5 with respect to the cart ($\sum_{i=1}^{5} l_i q_i$)

(d) Control force u

Figure 3. Controlled response: asymptotic stabilization of the hanging equilibrium $s = \{1, 1, 1, 1, 1\}$

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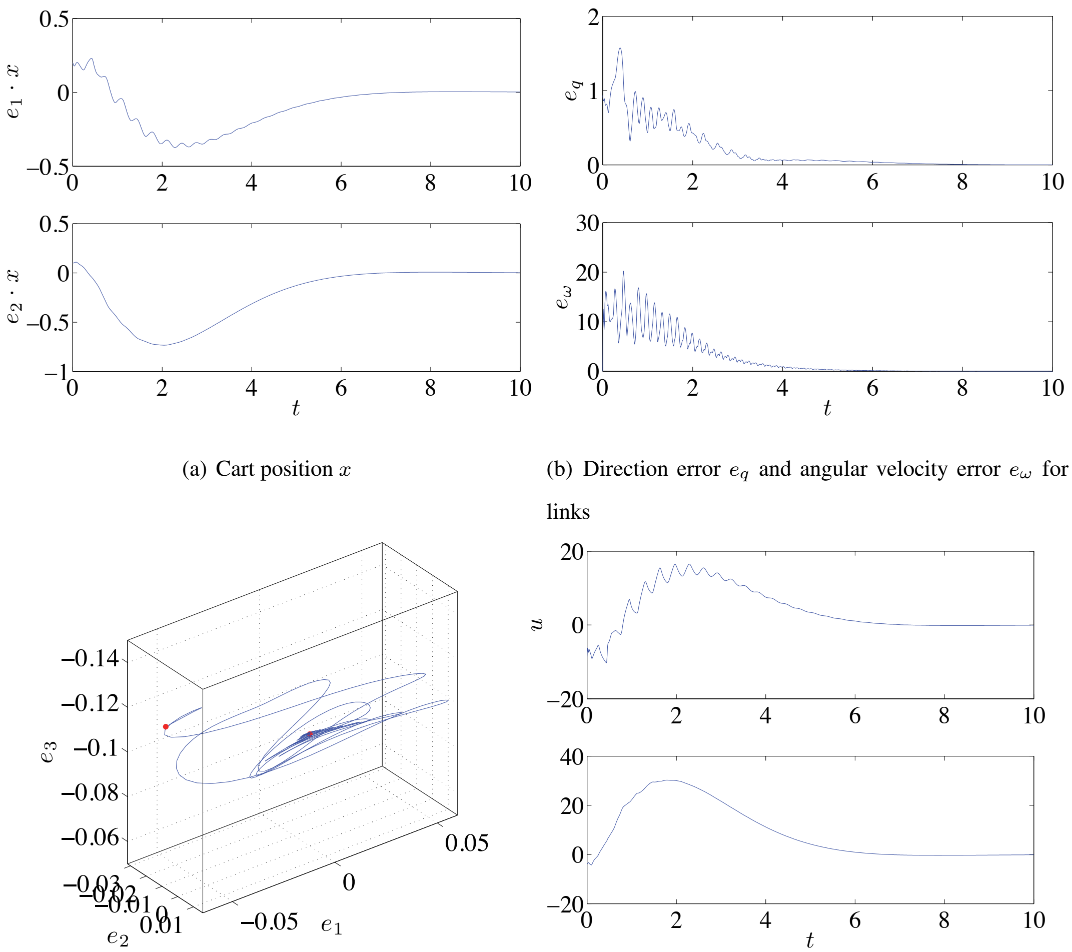

(a) Cart position x

(b) Direction error e_q and angular velocity error e_ω for links

(c) Location of m_5 with respect to the cart ($\sum_{i=1}^{5} l_i q_i$)

(d) Control force u

Figure 4. Controlled response: asymptotic stabilization of a partially-folded equilibrium $s = \{-1, -1, -1, 1, 1\}$

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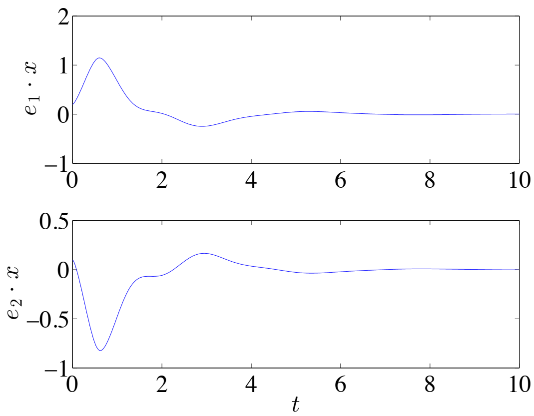
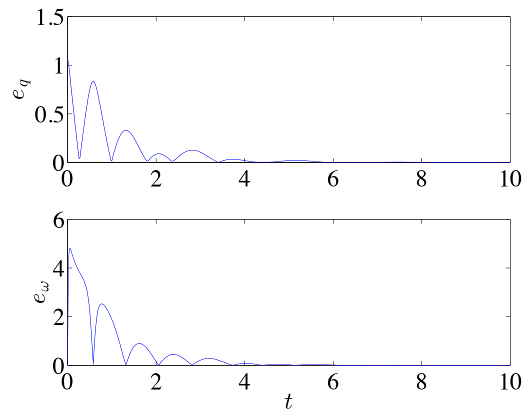
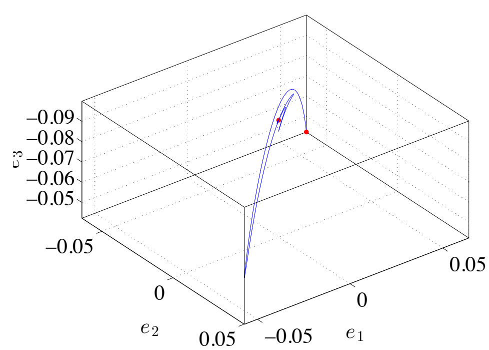


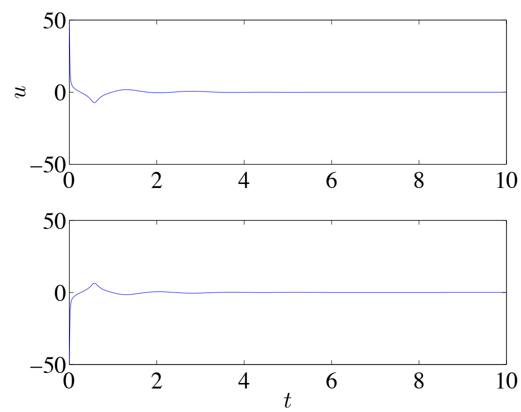
(a) Cart position x



(b) Direction error e_q and angular velocity error e_ω for link



(c) Location of m_1 with respect to the cart



(d) Control force u

Figure 5. Nonlinear control: asymptotic stabilization of the inverted equilibrium when $n = 1$

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A. APPENDIX

A.1. Properties of the hat map

Several properties of the hat map are given as follows:

$$\hat{x}y = x \times y = -y \times x = -\hat{y}x, \tag{28}$$

$$x \cdot \hat{y}z = y \cdot \hat{z}x = z \cdot \hat{x}y, \tag{29}$$

$$\hat{x}\hat{y}z = (x \cdot z)y - (x \cdot y)z, \tag{30}$$

$$C^T C = -C^T \hat{e}_3^2 C = I_2, \tag{31}$$

$$\hat{e}_3 = \hat{e}_3 CC^T = CC^T \hat{e}_3, \tag{32}$$

for any $x, y, z \in \mathbb{R}^3$.

A.2. Proof of Proposition 1

The overall procedure to derive Euler–Lagrange equation is as follows. For the given Lagrangian, the action integral is defined as $\mathfrak{G} = \int_{t_0}^{t_f} L dt$. From the expression for the variation on S^2 defined in (10), the variation of the action integral is derived, which, by the Lagrange–d'Alembert principle, is equal to the negative of the virtual work done by the control input.

First, the variation of the Lagrangian is derived as follows. From (8) and (9), it is straightforward to derive the variations of the Lagrangian with respect to x and $\dot{x}$ as

$$\begin{aligned}
&\mathbf{D}_x L \cdot \delta x = 0,\\
&\mathbf{D}_{\dot{x}} L \cdot \delta \dot{x} = (M_{00}\dot{x} + \sum_{i=1}^{n} M_{0i}\dot{q}_i) \cdot \delta \dot{x},
\end{aligned}$$

where $\mathbf{D}_x L$ represents the derivative of L with respect to x. From (10), the variation of q_i is $\delta q_i = \xi_i \times q_i$ for $\xi_i \in \mathbb{R}^3$ with $\xi_i \cdot q_i = 0$. The variation of the Lagrangian with respect to q_i is given by

$$\mathbf{D}_{q_i} L \cdot \delta q_i = \sum_{a=i}^{n} m_a g l_i e_3 \cdot (\xi_i \times q_i) = -\sum_{a=i}^{n} m_a g l_i \hat{e}_3 q_i \cdot \xi_i,$$

where (29) has been used. The variation of $\dot{q}_i$ is given by

$$\delta \dot{q}_i = \dot{\xi}_i \times q_i + \xi_i \times \dot{q}_i.$$

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From this and using (29) repeatedly, we obtain

$$\begin{aligned}\mathbf{D}_{\dot{q}_i} L \cdot \delta\dot{q}_i &= (M_{i0}\dot{x} + \sum_{j=1}^{n} M_{ij}\dot{q}_j) \cdot (\dot{\xi}_i \times q + \xi_i \times \dot{q}_i)\\ &= \hat{q}_i(M_{i0}\dot{x} + \sum_{j=1}^{n} M_{ij}\dot{q}_j) \cdot \dot{\xi}_i + \hat{\dot{q}}_i(M_{i0}\dot{x} + \sum_{j=1}^{n} M_{ij}\dot{q}_j) \cdot \xi_i.\end{aligned}$$

Using these, the variation of the action integral can be written as

$$\begin{aligned}\delta\mathfrak{G} = \int_{t_0}^{t_f} &\{M_{00}\dot{x} + \sum_{i=1}^{n} M_{0i}\dot{q}_i\} \cdot \delta\dot{x}\\ &+ \sum_{i=1}^{n}\{\hat{q}_i(M_{i0}\dot{x} + \sum_{j=1}^{n} M_{ij}\dot{q}_j)\} \cdot \dot{\xi}_i\\ &+ \sum_{i=1}^{n}\{\hat{\dot{q}}_i(M_{i0}\dot{x} + \sum_{j=1}^{n} M_{ij}\dot{q}_j) - \sum_{a=i}^{n} m_a g l_i \hat{e}_3 q_i\} \cdot \xi_i \, dt.\end{aligned}$$

Integrating by parts and using the fact that variations at the end points vanish, this reduces to

$$\begin{aligned}\delta\mathfrak{G} = \int_{t_0}^{t_f} &-\{M_{00}\ddot{x} + \sum_{i=1}^{n} M_{0i}\ddot{q}_i\} \cdot \delta x\\ &+ \sum_{i=1}^{n}\{-\hat{q}_i(M_{i0}\ddot{x} + \sum_{j=1}^{n} M_{ij}\ddot{q}_j) - \sum_{a=i}^{n} m_a g l_i \hat{e}_3 q_i\} \cdot \xi_i \, dt.\end{aligned}$$

According to the Lagrange–d'Alembert principle, the sum of the variation of the action integral, and the integral of the virtual work done by the control force on the cart, namely $\int_{t_0}^{t_f} u\delta x \, dt$, is zero. This implies that the expression within the first pair of braces in the above equation is equal to $-u$, and the expression within the second pair of braces is parallel to q_i for any $1 \leq i \leq n$, as ξ_i is perpendicular to q_i. Therefore, we obtain

$$M_{00}\ddot{x} + \sum_{i=1}^{n} M_{0i}\ddot{q}_i = u, \tag{33}$$

$$-\hat{q}_i^2(M_{i0}\ddot{x} + \sum_{j=1}^{n} M_{ij}\ddot{q}_j) + \sum_{a=i}^{n} m_a g l_i \hat{q}_i^2 e_3 = 0. \tag{34}$$

Equation (34) is rewritten to obtain an explicit expression for $\ddot{q}_i$. As $q_i \cdot \dot{q}_i = 0$, we have $\dot{q}_i \cdot \dot{q}_i + q_i \cdot \ddot{q}_i = 0$. Using this and (30), we have

$$-\hat{q}_i^2\ddot{q}_i = -(q_i \cdot \ddot{q}_i)q_i + (q_i \cdot q_i)\ddot{q}_i = (\dot{q}_i \cdot \dot{q}_i)q_i + \ddot{q}_i.$$

Substituting this into (34),

$$M_{ii}\ddot{q}_i - \hat{q}_i^2(M_{i0}\ddot{x} + \sum_{\substack{j=1\\ j\neq i}}^{n} M_{ij}\ddot{q}_j) = -M_{ii}\|\dot{q}_i\|^2 q_i - \sum_{a=i}^{n} m_a g l_i \hat{q}_i^2 e_3. \tag{35}$$

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It is straightforward to see that (33) and (35) are equivalent to (11).

This can be slightly rewritten in terms of the angular velocities. Since $\dot{q}_i = \omega_i \times q_i$ for the angular velocity ω_i satisfying $q_i \cdot \omega_i = 0$, we have

$$\begin{aligned}\ddot{q}_i &= \dot{\omega}_i \times q_i + \omega_i \times (\omega_i \times q_i)\\ &= \dot{\omega}_i \times q_i - \|\omega_i\|^2 q_i = -\hat{q}_i\dot{\omega}_i - \|\omega_i\|^2 q_i.\end{aligned}$$

Substituting this into (33) and (35), and using the fact that $\dot{\omega}_i \cdot q_i = 0$, we obtain

$$M_{00}\ddot{x} - \sum_{i=1}^{n} M_{0i}(\hat{q}_i\dot{\omega}_i + \|\omega_i\|^2 q_i) = u, \tag{36}$$

$$M_{ii}\dot{\omega}_i + \hat{q}_i(M_{i0}\ddot{x} - \sum_{\substack{j=1\\ j\neq i}}^{n} M_{ij}(\hat{q}_j\dot{\omega}_j + \|\omega_j\|^2 q_j)) = \sum_{a=i}^{n} m_a g l_i \hat{q}_i e_3, \tag{37}$$

which are equivalent to (12).

A.3. Proof of Proposition 2

At the given equilibrium, the direction of the i-th link is given by $q_i = s_i e_3$. From (10), its variation can be written as

$$\delta q_i = \xi_i \times s_i e_3,$$

where $\xi_i \in \mathbb{R}^3$ satisfies $\xi_i \cdot e_3 = 0$. Therefore, we have $\delta\dot{q}_i = \dot{\xi}_i \times s_i e_3$. Alternatively, from (1), $\delta\dot{q}_i$ can be written as

$$\delta\dot{q}_i = \dot{\xi}_i \times s_i e_3 = \delta\omega_i \times s_i e_3 + 0 \times (\xi_i \times s_i e_3) = \delta\omega_i \times s_i e_3.$$

Since both sides of the above equation is perpendicular to e_3 and $s_i = \pm 1$, this is equivalent to $e_3 \times (\dot{\xi}_i \times e_3) = e_3 \times (\delta\omega_i \times e_3)$, which yields

$$\dot{\xi} - (e_3 \cdot \dot{\xi})e_3 = \delta\omega_i - (e_3 \cdot \delta\omega_i)e_3,$$

from (30). Since $\xi_i \cdot e_3 = 0$, we have $\dot{\xi} \cdot e_3 = 0$. As $e_3 \cdot \delta\omega_i = 0$ from the constraint, this reduces to

$$\dot{\xi}_i = \delta\omega_i, \tag{38}$$

which is the linearized equation for the kinematic equation (1).

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Substituting these into (12) and ignoring higher-order terms, we obtain the linearized equations of motion. As an example, a few selected terms in (12) can be linearized as follows:

$$\begin{aligned}
-M_{0i}\hat{q}_i\dot{\omega}_i &\approx -s_i M_{0i}\hat{e}_3\ddot{\xi}_i = -s_i M_{0i}\hat{e}_3 CC^T\ddot{\xi}_i,\\
-M_{ij}\hat{q}_i\hat{q}_j\dot{\omega}_j &\approx -s_i s_j M_{ij}\hat{e}_3^2\ddot{\xi}_j = [s_i s_j M_{ij} C^T\ddot{\xi}_j;\, 0],\\
m_a g l_i \hat{q}_i e_3 &\approx -s_i m_a g l_i \hat{e}_3(\xi_i \times e_3) = [-m_a g l_i C^T\xi_i;\, 0],
\end{aligned}$$

where we have used the fact that $\hat{e}_3CC^T = \hat{e}_3$ and $\hat{e}_3^2 = \mathrm{diag}[-1,-1,0]$. More generally, we obtain

$$\begin{bmatrix}
M_{00}I_{2\times 2} & -M_{01}\hat{e}_3C & -M_{02}\hat{e}_3C & \cdots & -M_{0n}\hat{e}_3C\\
C^T\hat{e}_3M_{10} & M_{11}I_2 & M_{12}I_2 & \cdots & M_{1n}I_2\\
C^T\hat{e}_3M_{20} & M_{21}I_2 & M_{22}I_2 & \cdots & M_{2n}I_2\\
\vdots & \vdots & \vdots & & \vdots\\
C^T\hat{e}_3M_{n0} & M_{n1}I_2 & M_{n2}I_2 & \cdots & M_{nn}I_2
\end{bmatrix}$$

$$\times\begin{bmatrix}\delta\ddot{x}\\ C^T\ddot{\xi}_1\\ C^T\ddot{\xi}_2\\ \vdots\\ C^T\ddot{\xi}_n\end{bmatrix} = \begin{bmatrix} u\\ -\sum_{a=1}^n m_a g l_1 C^T\xi_1\\ -\sum_{a=2}^n m_a g l_2 C^T\xi_2\\ \vdots\\ -m_n g l_n C^T\xi_n\end{bmatrix}, \tag{39}$$

which is equivalent to (16).

A.4. Proof of Proposition 3

Suppose that $\mathbf{M}$ is invertible. It has been shown that the linearized system (16) is controllable, if and only if

$$\mathrm{rank}[\lambda^2\mathbf{M}+\mathbf{G},\ \mathbf{B}] = 2n+2 \tag{40}$$

for any generalized eigenvalue λ satisfying $\det[\lambda^2\mathbf{M}+\mathbf{G}]=0$ (see [20, 21]). This is a generalization of the Popov-Belevitch-Hautus (PBH) eigenvalue test to second-order systems. While it is not explicitly stated in the above references [20, 21], it is straightforward to find an equivalent condition in terms of eigenvectors, to obtain a PBH eigenvector test for second-order systems.

We claim that *(40) holds if and only if there is no generalized left eigenvector that is orthogonal to* $\mathbf{B}$*, i.e. for any non-zero eigenvector* $\mathbf{v}_i\in\mathbb{R}^{2n+2}$ *satisfying* $\mathbf{v}_i^T(\lambda_i^2\mathbf{M}+\mathbf{G})=0$*, we have* $\mathbf{v}_i^T\mathbf{B}\neq$

DOI: 10.1002/rnc

$0_{1\times 2}$. The proof is as follows:

(*Sufficiency*) Suppose that there is a generalized eigenvector $\mathbf{v}_i$ that is orthogonal to $\mathbf{B}$. Left-multiplying (40) by $\mathbf{v}_i$ yields

$$\mathbf{v}_i^T[\lambda^2\mathbf{M}+\mathbf{G},\ \mathbf{B}] = [\mathbf{v}_i^T(\lambda^2\mathbf{M}+\mathbf{G}),\ \mathbf{v}_i^T\mathbf{B}].$$

But, we have $\mathbf{v}_i^T(\lambda^2\mathbf{M}+\mathbf{G}) = 0_{1\times 2n+2}$ when $\lambda=\lambda_i$ from the definition of the generalized left eigenvector, and $\mathbf{v}_i^T\mathbf{B} = 0_{1\times 2}$ from the assumption. Therefore, the rows of the matrix $[\lambda^2\mathbf{M}+\mathbf{G},\ \mathbf{B}]$ are linearly dependent, and its rank is less than $2n+2$. This shows sufficiency.

(*Necessity*) Suppose that the matrix given in (40) is rank-deficient for some λ_i. Then, there exists a vector $\mathbf{v}_i$ satisfying

$$\mathbf{v}_i^T[\lambda_i^2\mathbf{M}+\mathbf{G},\ \mathbf{B}] = [0_{1\times 2n+4}],$$

which implies that $\mathbf{v}_i$ is a generalized left eigenvector that is orthogonal to $\mathbf{B}$.

Using the presented PBH eigenvector test, we show that (18) implies (40). More specifically, we show that if (40) is false, then (18) is false. Suppose that there exists a generalized eigenvector $\mathbf{v}_i = [\mathbf{v}_x;\mathbf{v}_q]$ that is orthogonal to $\mathbf{B}$. Then, we have $\mathbf{v}_i^T\mathbf{B} = \mathbf{v}_x^T I_2 + \mathbf{v}_q^T 0_{2n\times 2} = \mathbf{v}_x^T = 0_{1\times 2}$ from the definition of $\mathbf{B}$ in (16). As $\mathbf{v}_i$ is the left eigenvector, we also have

$$\begin{aligned}\mathbf{v}_i^T[\lambda_i^2\mathbf{M}+\mathbf{G}] &= [0_{2\times 1}^T,\ \mathbf{v}_q^T]\begin{bmatrix}\lambda_i^2\mathbf{M}_x & \lambda_i^2\mathbf{M}_{xq}\\ \lambda_i^2\mathbf{M}_{qx} & \lambda_i^2\mathbf{M}_{qq}+\mathbf{G}_{qq}\end{bmatrix}\\ &= \begin{bmatrix}\lambda_i^2\mathbf{v}_q^T\mathbf{M}_{qx} & \mathbf{v}_q^T(\lambda_i^2\mathbf{M}_{qq}+\mathbf{G}_{qq})\end{bmatrix}\\ &= \begin{bmatrix}0_{1\times 2} & 0_{1\times 2n}\end{bmatrix}. \end{aligned} \tag{41}$$

When $\lambda_i = 0$, this yields $\mathbf{v}_q^T\mathbf{G}_{qq} = 0_{1\times 2n} \Rightarrow \mathbf{v}_q = 0_{2n\times 1}$ as $\mathbf{G}_{qq}$ is invertible. This is not possible as it contradicts the fact that $\mathbf{v}_i = [0_{2\times 1};\mathbf{v}_q] \neq 0_{2n+2\times 1}$. Therefore, $\lambda_i \neq 0$. Then, (41) implies

$$\mathbf{v}_q^T(\lambda_i^2\mathbf{M}_{qq}+\mathbf{G}_{qq}) = 0_{1\times 2n}, \quad \mathbf{v}_q^T\mathbf{M}_{qx} = 0_{1\times 2}, \tag{42}$$

which states that there exists a non-zero generalized left eigenvector of (17) that is orthogonal to $\mathbf{M}_{qx}$. Therefore (18) is false.

As a last step, we show (40) implies (18). If (18) is false, there exists a non-zero eigenvector $\mathbf{v}_q$ satisfying (42). Let $\mathbf{v}_i = [0_{2\times 1};\mathbf{v}_q]$. Then, it is orthogonal to $\mathbf{B}$ as $\mathbf{v}_i^T\mathbf{B} = 0_{1\times 2}$. And $\mathbf{v}_i$ is a generalized left eigenvector of $(\mathbf{M},\mathbf{G})$ as it satisfies (41). In short, (18) is equivalent to (40).

A.5. Proof of Proposition 4

We show local asymptotical stability by using the linearized equation of motion. From (10), the third term of the control input is linearized as

$$K_{q_i} C^T (s_i e_3 \times q_i) \approx K_{q_i} C^T (s_i e_3 \times (\xi_i \times s_i e_3) = -K_{q_i} C^T \hat{e}_3^2 \xi_i = K_{q_i} C^T \xi_i,$$

where we used the fact that $C^T \hat{e}_3^2 = -C^T$. Therefore, the linearized control input is given by

$$u = -K_x \delta x - K_{\dot{x}} \delta \dot{x} - \sum_{i=1}^{n} \{K_{q_i} C^T \xi_i + K_{\omega_i} C^T \delta\omega_i\}.$$

From (38) and the definition of the state vector $\mathbf{x} = [\delta x; C^T\xi_1; \ldots; C^T\xi_n]$ in (16), u can be written as

$$u = -\mathbf{K}_{\mathbf{x}} \mathbf{x} - \mathbf{K}_{\dot{\mathbf{x}}} \dot{\mathbf{x}},$$

where $\mathbf{K}_{\mathbf{x}} = [K_x, K_{q_i}, \ldots, K_{q_n}] \in \mathbb{R}^{2\times 2n+2}$ and $\mathbf{K}_{\dot{\mathbf{x}}} = [K_{\dot{x}}, K_{\omega_1}, \ldots, K_{\omega_n}] \in \mathbb{R}^{2\times 2n+2}$. Since (16) is controllable, we can choose the controller gains $\mathbf{K}_{\mathbf{x}}, \mathbf{K}_{\dot{\mathbf{x}}}$ such that the equilibrium is asymptotically stable for the linearized equation (16). According to Theorem 4.7 in [22], the equilibrium becomes locally asymptotically stable for the nonlinear dynamics (12).

A.6. Proof of Proposition 5

Define the following domain near the inverted equilibrium:

$$D = \{(q_1, x, \omega_1, \dot{x}) \in \mathsf{T}(\mathsf{S}^2 \times \mathbb{R}^2) \,|\, q_1^T e_3 < -\sqrt{\frac{1+k_w}{k_q}}\}. \tag{43}$$

We first show that $\mathcal{V}$ is positive-definite in D. The time-derivative of y is given by

$$\dot{y} = \dot{x} - k_q l_1 C^T \hat{q}_1 \omega_1. \tag{44}$$

Substituting it into (25) and rearranging, we obtain

$$\begin{aligned}\mathcal{V} = \frac{1}{2} k_y \|y\|^2 + k_w k_q g l_1 (1 + q_1^T e_3) + \frac{1+k_w}{2} \|\dot{x}\|^2 \\ - k_q l_1 \dot{x}^T C^T \hat{q}_1 \omega_1 + \frac{1}{2} k_q l_1^2 (k_q \|C^T \hat{q}_1 \omega_1\|^2 - k_w \|\omega_1\|^2).\end{aligned}$$

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Let α be the angle between $\dot{q}_1 = \omega_1 \times q_1$ and $-e_3$. We have $\|C^T \hat{q}_1 \omega_1\| = \|C^T \dot{q}_1\| = |\sin\alpha|\|\omega\|$ as C^T represents the projection onto the plane normal to e_3. Substituting this,

$$\begin{aligned}\mathcal{V} &\geq \frac{1}{2} k_y \|y\|^2 + k_w k_q g l_1 (1 + q_1^T e_3) \\ &+ \frac{1}{2} \begin{bmatrix} \|\dot{x}\| \\ \|\omega_1\| \end{bmatrix}^T \begin{bmatrix} 1 + k_w & -k_q l_1 |\sin\alpha| \\ -k_q l_1 |\sin\alpha| & k_q l_1^2 (k_q \sin^2\alpha - k_w) \end{bmatrix} \begin{bmatrix} \|\dot{x}\| \\ \|\omega_1\| \end{bmatrix}.\end{aligned} \tag{45}$$

The matrix in the last term of the above expression is positive-definite if $\sin^2\alpha > \frac{1+k_w}{k_q}$, or the angle between $\dot{q}_1$ and $-e_3$ is sufficiently close to 90°. Since q_1 is normal to $\dot{q}_1$, the angle between q_1 and $-e_3$ is greater than $90^\circ - \alpha$, which yields $(q_1 \cdot e_3)^2 < \cos^2(90^\circ - \alpha) = \sin^2\alpha$. Therefore, $\mathcal{V}$ is positive-definite in the domain D given by (43).

We now compute $\dot{\mathcal{V}}$. From (23), (44), the second derivative of y is given by

$$\ddot{y} = (I + k_q C^T \hat{q}_1^2 C)\mu - k_q g C^T \hat{q}_1^2 e_3 - k_q l_1 \|\omega_1\|^2 C^T q_1. \tag{46}$$

From these, $\dot{\mathcal{W}}$ can be written as

$$\begin{aligned}\dot{\mathcal{W}} &= -k_q l_1^2 \omega_1^T \dot{\omega}_1 + k_q g l e_3^T \hat{\omega} q + \dot{x}^T \mu \\ &= k_q \omega_1^T (-g l_1 \hat{q}_1 e_3 + l_1 \hat{q}_1 C \mu) + k_q g l_1 e_3^T \hat{\omega}_1 q_1 + \dot{x}^T \mu \\ &= \mu^T (\dot{x} - k_q l C^T \hat{q} \omega) = \mu^T \dot{y}.\end{aligned}$$

Therefore, the time-derivative of the Liapunov function is given by

$$\dot{\mathcal{V}} = \dot{y}^T (k_y y + \ddot{y} + k_w \mu).$$

Using (26), the transformed control input μ is chosen to satisfy

$$k_y y + \ddot{y} + k_w \mu = -k_{\dot{y}} \dot{y},$$

so that

$$\dot{\mathcal{V}} = -k_{\dot{y}} \|\dot{y}\|^2.$$

Thus, the inverted equilibrium is stable.

To show asymptotic stability, we use LaSalle's theorem. Let $\dot{y} \equiv 0$ and $\ddot{y} \equiv 0$. This implies that y is fixed and $k_y y + k_w \mu = 0$. Assume (towards contradiction) that $y \neq 0$. Then, $\mu \neq 0$ and the cart translates with a fixed acceleration from (22). This contradicts the fact that y is fixed, since

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$\|C^T q_1\| \leq 1$ in (24). Therefore, $y \equiv 0$, $\mu \equiv 0$, $\ddot{x} \equiv 0$, and $\dot{\omega}_1 = \frac{g}{l_1}\hat{q}_1 e_3$. The resulting equation of motion for ω_1 implies that the link follows the solution of a spherical pendulum with a fixed pivot, i.e. it is either fixed or oscillating. But, it cannot oscillate as $\dot{y} \equiv 0$ and $\dot{x}$ is fixed in (44). Therefore $q_1 \equiv \pm e_3$ and $x \equiv 0$ from (24). In summary, in the neighborhood of $q = -e_3$ given by (43), the inverted equilibrium with $x = 0$ is the only solution in the set where $\dot{y} \equiv 0$. Thus, by LaSalle's principle, the inverted equilibrium is asymptotically stable.

A conservative estimate of the region of attraction is given by a bounded sublevel set of the Liapunov function that is contained in the domain D. From (43), the initial condition should satisfy $q_1(0)^T e_3 < 0$ in order to ensure asymptotic convergence to the inverted equilibrium.

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DOI: 10.1002/rnc