FOUNDATIONS OF COMPUTATIONAL GEOMETRIC MECHANICS

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ABSTRACT. Geometric mechanics involves the application of geometric techniques to the study of Lagrangian and Hamiltonian mechanics. Recent work on discrete variational mechanics have resulted in systematic methods for the construction of symplectic-momentum integrators, but additional geometric machinery applicable to discrete spaces is necessary for the development of a coherent theory of computational geometric mechanics.

We review recent progress in developing some of the mathematical infrastructure that is necessary to more directly adopt the approach of geometric mechanics in the construction of computational algorithms. These mathematical tools include discrete notions of exterior calculus and connections on principal bundles.

The discrete exterior calculus is modeled on a primal simplicial complex, and a dual circumcentric cell complex. Discrete notions of differential forms, exterior derivatives, hodge stars, codifferentials, sharps, flats, wedge products, contraction, Lie derivative, and the Poincaré lemma are introduced, and their discrete properties are analyzed.

The discrete theory of connections on principal bundles is constructed by introducing the discrete analogue of the Atiyah sequence, with a connection corresponding to the choice of a splitting of the short exact sequence. Equivalent representations of a discrete connection are considered, and computational issues such as the order of approximation are addressed.

The applications include discrete Lagrangian reduction, geometric control theory, and a discrete notion of a Riemannian manifold, and the associated Levi-Civita connection and its curvature.

CONTENTS

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1. Introduction

Geometric mechanics [1, 25] has motivated the outgrowth of new and innovative numerical schemes that inherit many of the desirable conservation properties of the original continuous problem. It is the goal of computational geometric mechanics to more directly adopt the approach of geometric mechanics in the construction of computational algorithms, as well as the systematic analysis of their numerical conservation properties.

Variational integrators [26], which are based on discretizing the underlying Hamilton's variational principle, as opposed to the Euler–Lagrange equations, are automatically symplectic and momentum preserving. Forcing and dissipation can also be addressed by considering the discrete Lagrange–d'Alembert principle instead. In either the forced or conservative case, variational integrators exhibit excellent long time energy behavior that cannot be understood from their local error properties alone.

In understanding the global behavior of such variational integrators, and their underlying geometry, we are hindered by the absence of the discrete analogues of mathematical tools that geometric mechanics has come to rely on. These include the language of differential forms and exterior calculus, as well as geometric structure encoded in the form of connections on principal bundles.

In recent years, there has been increasing interest in the study of numerical schemes as dynamical systems in their own right, and if we are to repeat the success of geometric mechanics in elucidating the underlying geometry of such numerical methods, it is imperative that we develop more of the relevant mathematical infrastructure.

In particular, there has been interest in developing a theory of symmetry reduction at the discrete level, including work on the Discrete Euler–Poincaré [23, 24] equations, and the Discrete Routh equations [20]. The discrete analogues of exterior calculus and curvature that arose in the development of discrete reduction [21] motivated us to systematically develop a discrete theory of exterior calculus [12]. Similarly, attempts to develop a discrete analogue of the continuous theory of Lagrangian reduction [11] lead to the construction of discrete connections on principal bundles [22].

This paper will focus on recent advances that elucidate the appropriate discrete analogues of such machinery, and will also comment on the manner in which it relates to, and complements, prior work on discrete variational mechanics and asynchronous variational integrators. We will also address its applications to the geometric control theory of formations, discrete Levi-Civita connections, and mimetic vector difference operators.

There have been many attempts to discretize exterior calculus for computational purposes, but one crucial difficulty has been how to incorporate vector fields in an intrinsic manner. Most attempts can be categorized into two approaches, those derived from the ideas of Whitney [32], which involve thinking of differential forms as cochains on a simplicial triangulation, as in Bossavit [5, 6, 7] and Hiptmair [17, 18], and those that rely on logically rectangular meshes, as in Dezin [14], whereby the uniform connectivity of the mesh allows a tensor product based approach that builds on fundamentally one-dimensional theory.

Our discrete theory of exterior calculus [12] will be based on the simplicial cochain approach. A fundamental concept that is necessary to incorporate discrete vector fields into a theory of discrete differential forms is the notion of primal simplicial complexes and dual cell complexes. In particular, it will become clear that dual complexes based on circumcentric duality is critical in the construction of discrete vector differential operators that exhibit good numerical properties.

As mentioned, the work on discrete connections on principal bundles [22] was motivated by attempts to develop a theory of Lagrangian reduction for discrete variational mechanics. In particular, discrete connections and their induced continuous counterparts are necessary to coordinatize the reduced spaces in an intrinsic fashion. This coordinatization is a natural consequence of the five lemma in homological algebra when we think of connections as splittings of the Atiyah sequence.

These ideas can be extended to the case of semi-discretized Riemannian manifolds, and for such spaces, we can construct a discrete Levi-Civita connection. By adopting ideas from discrete exterior calculus, we propose a method of identifying regions of high curvature for abstract simplicial meshes, thereby providing a quality metric that allows the identification of regions for which subdivision and refinement is necessary in the interest of numerical accuracy.

This paper will, whenever possible, provide motivation and intuition for the definitions we adopt in the construction of our discrete theory of exterior calculus and connections. In addition, the correspondence between the continuous and discrete theories will be addressed.

2. Discrete Theory of Exterior Calculus

The purpose of this section is to construct a discrete theory of exterior calculus that involve discrete differential forms, discrete vector fields, and discrete operators that act on them. Whenever possible, these discrete operators will be defined as combinatorial operations on the cochain representation of discrete differential forms. This is desirable for ease of implementation, and computational efficiency.

Once discrete forms and vector fields are defined a calculus can be developed by defining the discrete exterior derivative ($\mathbf{d}$), codifferential (δ) and Hodge star ($*$) for operating on forms, discrete wedge product ($\wedge$) for combining forms, discrete flat ($\flat$) and sharp ($\sharp$) operators for going between vector fields and one forms and discrete contraction operator ($\mathbf{i}_X$) for combining forms and vector fields. The contraction operator, and the Lie derivative ($\pounds_X$) will be constructed using their dynamic definition.

A discrete divergence in any dimension and curl in $\mathbb{R}^3$ can also be defined. A discrete Laplace-deRham operator (Δ) can be defined using the usual definition of $\mathbf{d}\delta + \delta\mathbf{d}$. When applied to functions this is the same as the discrete Laplace-Beltrami operator (∇^2) which is the defined as $\operatorname{div} \circ \operatorname{curl}$. We construct these operators in this section.

2.1. Primal Simplicial Complex and Dual Cell Complex.

2.1. Primal Simplicial Complex and Dual Cell Complex. In constructing the discretization of a continuous problem in the context of our formulation of discrete exterior calculus, we first discretize the manifold of interest as a simplicial complex. While this is typically in the form of a simplicial complex that is embedded into Euclidean space, it is only necessary to have an abstract simplicial complex, along with metric on adjacent vertices. This abstract setting will be addressed further toward the end of this section.

We will now recall some basic definitions of simplices and simplicial complexes, which are standard from simplicial algebraic topology. A more extensive treatment can be found in Munkres [28].

Definition 2.1. *A p-**simplex** is the convex span of $p+1$ geometrically independent points,*

$$\sigma^p = [v_0, v_1, \dots, v_p] = \left\{ \sum_{i=0}^{p} \alpha^i v_i \mid \alpha^i \geq 0, \sum_{i=0}^{n} \alpha^i = 1 \right\}$$

The points $v_0, \dots, v_p$ are called the ***vertices*** *of the simplex, and the number p is called the* ***dimension*** *of the simplex. Any simplex spanned by a (proper) subset of $\{v_0, \dots, v_p\}$ is called a* ***(proper) face*** *of σ^p. If σ^q is a proper face of σ^p then we write $\sigma^q \prec \sigma^p$.*

Example 2.1. *Consider 3 noncolinear points v_0, v_1 and v_2 in $\mathbb{R}^3$. Then these three points individually are examples of 0-simplices which are assumed to have no orientation. Examples of 1-simplices are the oriented line segments $[v_0, v_1]$, $[v_1, v_2]$ and $[v_0, v_2]$. By writing the vertices in that order we have given orientations to these 1-simplices, i.e. $[v_0, v_1]$ is oriented from v_0 to v_1. The triangle $[v_0, v_1, v_2]$ is a 2-simplex oriented in counter clockwise direction. Note that the orientation of $[v_0, v_2]$ does not agree with that of the triangle.*

Definition 2.2. *A* ***simplicial complex*** *K in $\mathbb{R}^N$ is a collection of simplices in $\mathbb{R}^N$ such that,*

(1) *Every face of a simplex of K is in K.*
(2) *The intersection of any two simplices of K is a face of each of them.*

Definition 2.3. *A* ***simplicial triangulation*** *of a polytope $|K|$ is a simplicial complex K such that the union of the simplices of K recover the polytope $|K|$.*

Definition 2.4. *If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex in its own right, and it is called a* ***subcomplex*** *of K. One subcomplex of K is the collection of all simplices of K of dimension at most p, which is called the* ***p-skeleton*** *of K and is denoted $K^{(p)}$.*

We will also use the notion of a circumcentric dual or Voronoi mesh of the given primal mesh. We will point to the importance of this choice later on in Section 2.5 and 2.7.1. We call the Voronoi dual a circumcentric dual since the dual of a simplex is its circumcenter (equidistant from all vertices of the simplex).

Definition 2.5. *The* ***circumcenter*** *of a p-simplex σ^p is given by the center of the p-circumsphere, where the p-circumsphere is the unique p-sphere that has all $p+1$ vertices of σ^p on its surface. Equivalently, the circumcenter is the unique point in the p-dimensional affine space that contains the p-simplex that is equidistant from all the $p+1$ nodes of the simplex. We will denote the circumcenter of a simplex σ^p by $c(\sigma^p)$.*

The circumcenter of a simplex σ^p can be obtained by taking the intersection of the normals to the $(p-1)$-dimensional faces of the simplex, where the normals are emanating from the circumcenter of the face. This allows us to recursively compute the circumcenter.

If we are given the nodes which describe the primal mesh, we can construct a simplicial triangulation by using the Delaunay triangulation, since this ensures that the circumcenter of a simplex is always a point within the simplex. Otherwise we assume that a nice mesh has been given to us, i.e it is such that the circumcenters lie within the simplices. While this is not be essential for our theory it makes some proofs simpler. For some computations the Delaunay triangulation is desirable in that it reduces the maximum aspect ratio of the mesh, which is a factor in determining the rate at which the corresponding numerical scheme converges. But in practice there are many problems for which Delaunay triangulations are a bad idea. See for example Schewchuk [29]. We will address such computational issues in a separate work.

Definition 2.6. *The* ***circumcentric subdivision*** *of a simplicial complex is given by the collection of all simplices of the form,*

$$[c(\sigma_0), \dots, c(\sigma_p)],$$

where $\sigma_0 \prec \sigma_1 \prec \dots \prec \sigma_p$, or equivalently, that σ_i is a proper face of σ_j for all $i < j$.

We construct a circumcentric dual to a p-simplex using the circumcentric duality operator.

Definition 2.7. *The* ***circumcentric duality operator*** *is given by*

$$\star(\sigma^p) = \sum_{\sigma^p \prec \sigma^{p+1} \prec \dots \prec \sigma^n} \epsilon_{\sigma^p, \dots, \sigma^n} \left[c(\sigma^p), c(\sigma^{p+1}), \dots, c(\sigma^n)\right]$$

where the $\epsilon_{\sigma^p,\ldots,\sigma^n}$ coefficient ensures that the orientation of $\left[c(\sigma^p), c(\sigma^{p+1}), \ldots, c(\sigma^n)\right]$ is consistent with the orientation of the primal simplex, and the ambient volume form.

Orienting σ^p is equivalent to choosing a ordered basis, which we shall denote by $dx^1 \wedge \ldots \wedge dx^p$. Similarly, $\left[c(\sigma^p), c(\sigma^{p+1}), \ldots, c(\sigma^n)\right]$ has an orientation denoted by $dx^{p+1} \wedge \ldots \wedge dx^n$. If the orientation corresponding to $dx^1 \wedge \ldots \wedge dx^n$ is consistent with the volume form on the manifold, then $\epsilon_{\sigma^p,\ldots,\sigma^n} = 1$, otherwise it takes the value -1.

We immediately see from the construction of the circumcentric duality operator that the dual elements can be realized as a submesh of the first circumcentric subdivision, since it consists of elements of the form, $[c(\sigma_0), \ldots, c(\sigma_p)]$, which are by definition part of the first circumcentric subdivision.

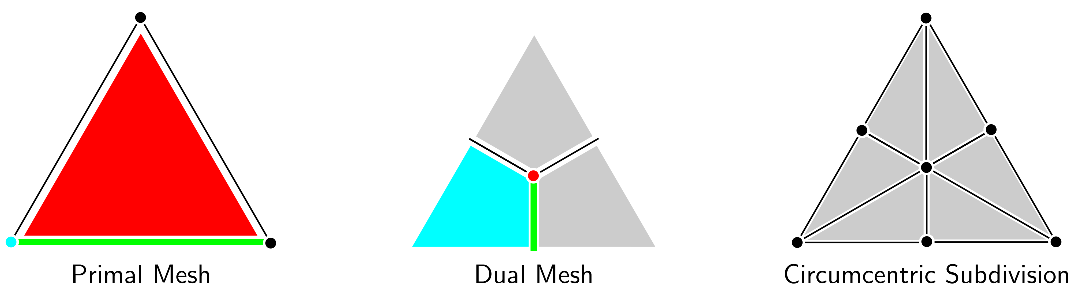
Example 2.2. *The circumcentric duality operator maps a $0-$simplex into the convex hull generated by the circumcenters of n-simplices that contain the 0-simplex,*

$$\star(\sigma^0) = \left\{ \sum \alpha_{\sigma^n} c\,(\sigma^n) \mid \alpha_{\sigma^n} \geq 0, \sum \alpha_{\sigma^n} = 1, \sigma^0 \prec \sigma^n \right\},$$

and the circumcentric duality operator maps a $n-$simplex into the circumcenter of the $n-$simplex,

$$\star(\sigma^n) = c(\sigma^n).$$

This is more clearly illustrated in the following diagram, where the primal and dual elements are color coded to represent the dual relationship between the element in the primal and dual mesh.



The choice of a circumcentric dual is significant since it allows us to recover geometrically important objects such as normals to $(n-1)$-dimensional faces, which are obtained by taking their circumcentric dual, whereas if we were to use a barycentric dual, the dual to a $(n-1)$-dimensional face would not be normal to it.

Notice that given an oriented simplex σ^p represented by $[v_0, \ldots, v_p]$, the orientation is equivalently represented by $(v_1 - v_0) \wedge (v_2 - v_1) \wedge \ldots \wedge (v_p - v_{p-1})$, which we denote by,

$$[v_0, \ldots, v_p] \sim (v_1 - v_0) \wedge (v_2 - v_1) \wedge \ldots \wedge (v_p - v_{p-1}),$$

which is an equivalence at the level of orientation. It would be nice to express our criterion for determining the orientation of the dual cell in terms of the $p+1$ vertex representation.

To determine the orientation of the $(n-p)$-simplex given by $[c(\sigma^p), c(\sigma^{p+1}), \ldots, c(\sigma^n)]$, or equivalently, $dx^{p+1} \wedge \ldots \wedge dx^n$, we consider the n-simplex given by $[c(\sigma^0), \ldots, c(\sigma^n)]$, where $\sigma^0 \prec \ldots \prec \sigma^p$. This is related to the expression $dx^1 \wedge \ldots \wedge dx^n$ up to a sign determined by the relative orientation of $[c(\sigma^0), \ldots, c(\sigma^p)]$ and σ^p. Thus we have,

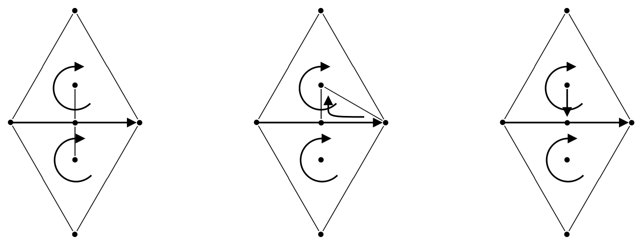
$$dx^1 \wedge \ldots \wedge dx^n \sim \operatorname{sgn}([c(\sigma^0), \ldots, c(\sigma^p)], \sigma^p)[c(\sigma^0), \ldots, c(\sigma^n)]$$

Then, we need to check that $dx^1 \wedge \ldots \wedge dx^n$ is consistent with the volume form on the manifold, which is represented by the orientation of σ^n. Thus, we have that the correct orientation for the $[c(\sigma^p), c(\sigma^{p+1}), \ldots, c(\sigma^n)]$ term is given by,

$$\operatorname{sgn}([c(\sigma^0), \ldots, c(\sigma^p)], \sigma^p) \cdot \operatorname{sgn}([c(\sigma^0), \ldots, c(\sigma^n)], \sigma^n).$$

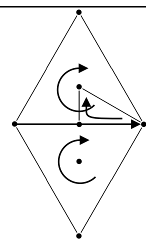
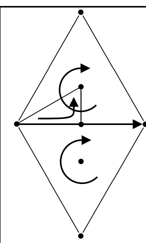
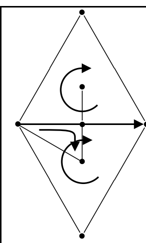
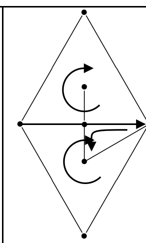
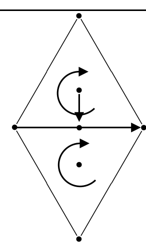
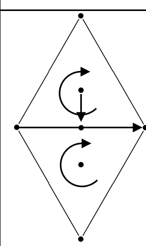
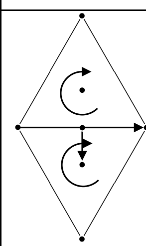
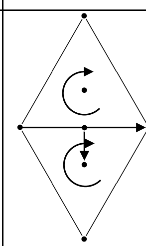
These two representations of the choice of orientation for the dual cells are equivalent, but the combinatorial definition above might be preferable for the purposes of implementation.

Example 2.3. *We would like to determine the orientation of the dual of a* 1*-simplex, in* 2*-dimensions, given the orientation of the two neighboring* 2*-simplices.*



Consider this first choice of the 2*-simplex of the form,* $[c(\sigma^0), c(\sigma^1), c(\sigma^2)]$*. Notice that the orientation is consistent with the given orientation of the* 2*-simplex, but it is not consistent with the orientation of the primal* 1*-simplex, so the orientation should be reversed, to give,*

We summarize the results for the other possible choices of 2*-simplices of the form,* $[c(\sigma^0), c(\sigma^1), c(\sigma^2)]$ *in the following table.*

$[c(\sigma^0), c(\sigma^1), c(\sigma^2)]$				
$\mathrm{sgn}([c(\sigma^0), c(\sigma^1)], \sigma^1)$	$-$	$+$	$+$	$-$
$\mathrm{sgn}([c(\sigma^0), c(\sigma^1), c(\sigma^2)], \sigma^2)$	$+$	$-$	$+$	$-$
$\mathrm{sgn}([c(\sigma^0), c(\sigma^1)], \sigma^1)$ $\cdot\, \mathrm{sgn}([c(\sigma^0), c(\sigma^1), c(\sigma^2)], \sigma^2)$ $\cdot [c(\sigma^1), c(\sigma^2)]$				

While the circumcentric duality operator is a map from the primal simplicial complex to the dual cell complex, we can formally extend the circumcentric duality operator to a map from the dual cell complex to the primal simplicial complex. However, we need to be slightly careful about the orientation of primal simplex we recover from applying the circumcentric duality operator twice.

We have that, $\star\star(\sigma^p) = \pm\sigma^p$, where the sign is chosen to ensure the appropriate choice of orientation. If, as before, σ^p has an orientation represented by $dx^1 \wedge \ldots \wedge dx^p$, and $\star\sigma^p$ has an orientation represented by $dx^{p+1} \wedge \ldots \wedge dx^n$, then the orientation of $\star\star(\sigma^p))$ is chosen so that $dx^{p+1} \wedge \ldots \wedge dx^n \wedge dx^1 \wedge \ldots \wedge dx^p$ is consistent with the ambient volume form. Since by construction of $\star(\sigma^p)$, $dx^1 \wedge \ldots \wedge dx^n$ has an orientation consistent with the ambient volume form, we need only compare $dx^{p+1} \wedge \ldots \wedge dx^n \wedge dx^1 \wedge \ldots \wedge dx^p$ and $dx^1 \wedge \ldots \wedge dx^n$. Notice that it takes $n-p$ transpositions to get the dx^1 term in front of the $dx^{p+1} \wedge \ldots \wedge dx^n$ terms, and we need to do this p times for each term of $dx^1 \wedge \ldots \wedge dx^p$, so it follows that the sign is simply given by $(-1)^{p(n-p)}$, or equivalently,

$$\star\star(\sigma^p) = (-1)^{p(n-p)}\sigma^p. \tag{2.1}$$

A similar relationship holds if we use a dual cell instead of the primal simplex σ^p.

We can think of a cochain as being constructed out of a basis consisting of cosimplices or cocells with value 1 on a single simplex or cell, and 0 otherwise. The way to visualize this cosimplex is that it is associated with a differential form that has support on what we will refer to as the **support volume** associated with a given simplex or cell.

Definition 2.8. *The **support volume** of a simplex σ^p is a n-volume given by the convex hull of the geometric union of the simplex and its circumcentric dual. This is denoted by,*

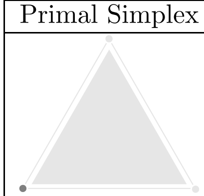
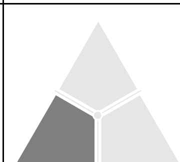
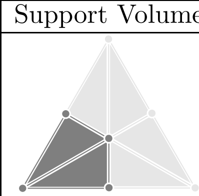
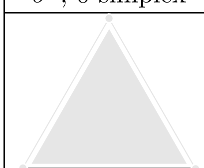
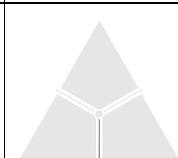
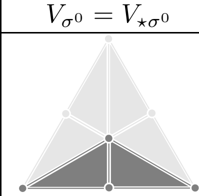
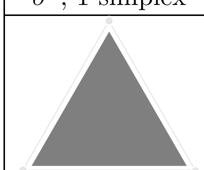
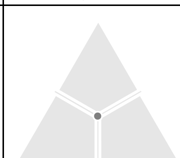
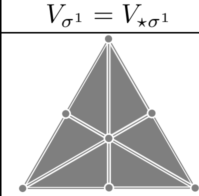
$$V_{\sigma^p} = \text{convexhull}(\sigma^p, \star\sigma^p) \cap |K|.$$

The intersection with $|K|$ is necessary to ensure that the support volume does not extend beyond the polytope $|K|$ which would otherwise occur if $|K|$ is nonconvex.

We extend the notion of a support volume to a dual cell $\star\sigma^p$ similarly,

$$V_{\star\sigma^p} = \text{convexhull}(\star\sigma^p, \star\star\sigma^p) \cap |K| = V_{\sigma^p}.$$

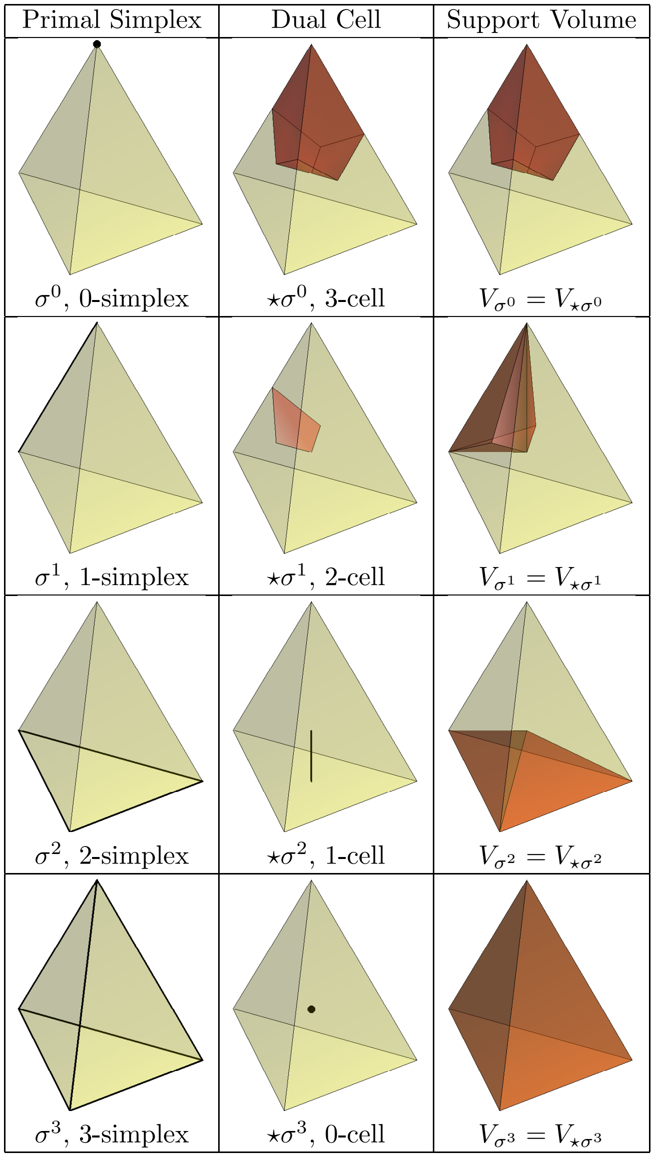
To clarify this definition, we will consider some examples of simplices, their dual cells, and the corresponding support volumes. In a 2-dimensional simplex, we have the following,

Primal Simplex	Dual Cell	Support Volume
 σ^0, 0-simplex	 $\star\sigma^0$, 2-cell	 $V_{\sigma^0} = V_{\star\sigma^0}$
 σ^1, 1-simplex	 $\star\sigma^1$, 1-cell	 $V_{\sigma^1} = V_{\star\sigma^1}$
 σ^2, 2-simplex	 $\star\sigma^2$, 0-cell	 $V_{\sigma^2} = V_{\star\sigma^2}$

This support volume has the nice property that at each dimension, it partitions the polytope $|K|$ into distinct non-intersecting regions associated with each individual p-simplex. For any two distinct p-simplices, the intersection of their corresponding support volumes have measure zero, and the union of the support volumes of all p-simplices recover the original polytope $|K|$.

Notice that from our construction that the support volume of a simplex and its dual cell are the same, which suggests that there is an identification between cochains on p-simplices and cochains on $(n-p)$-cells. This is indeed the case, and is a concept associated with the Hodge star for differential forms.

We now show examples of simplices, their dual cells, and the corresponding support volumes in 3-dimensions,



In our subsequent discussion, we will assume that we are given a simplicial complex K of dimension n in $\mathbb{R}^N$. Thus the highest dimension simplex in the complex is of dimension n and each 0-simplex (vertex) is in $\mathbb{R}^N$. One can obtain this for example by starting from 0-simplices, i.e. vertices, and then constructing a Delaunay triangulation using these as sites. Often our examples will be for two dimensional discrete surfaces in $\mathbb{R}^3$ made up of triangles (here $n = 2$ and $N = 3$) or three dimensional manifolds made of tetrahedra possibly embedded in a higher dimensional space.

The circumcentric dual of a primal simplicial complex is an example of a cell complex. The definition of a cell complex follows.

Definition 2.9. *A* ***cell complex*** $\star K$ *in* $\mathbb{R}^N$ *is a collection of cells in* $\mathbb{R}^N$ *such that,*

(1) *There is a partial ordering of cells in* $\star K$, $\hat{\sigma}^k \prec \hat{\sigma}^l$, *which is read as* $\hat{\sigma}^k$ *is a face of* $\hat{\sigma}^l$.
(2) *The intersection of any two cells in* $\star K$, *is either a face of each of them, or it is empty.*
(3) *The boundary of a cell is expressible as a sum of its proper faces.*

We will see in the next section that the notion of boundary in the circumcentric dual has to be modified slightly from the geometric notion of a boundary in order for the circumcentric dual to be made into a cell complex.

2.2. **Local and Global Embeddings.** While it is computationally more convenient to have a global embedding of the simplicial complex into a higher dimensional ambient space to account for non-flat manifolds, it suffices to have an abstract simplicial complex along with a local metric on vertices, which is local in the sense that distances between two vertices are only defined if they are part of a common n-simplex in the abstract simplicial complex. Then the local metric is a map $d : \{(v_0, v_1) \mid v_0, v_1 \in K^{(0)}, [v_0, v_1] \prec \sigma^n \in K\} \to \mathbb{R}$.

The axioms for a local metric are as follows,

Positive: $d(v_0, v_1) \geq 0$, and $d(v_0, v_0) = 0$, $\forall [v_0, v_1] \prec \sigma^n \in K$.
Strictly Positive: If $d(v_0, v_1) = 0$, then $v_0 = v_1$, $\forall [v_0, v_1] \prec \sigma^n \in K$.
Symmetry: $d(v_0, v_1) = d(v_1, v_0)$, $\forall [v_0, v_1] \prec \sigma^n \in K$.
Triangle Inequality: $d(v_0, v_2) \leq d(v_0, v_1) + d(v_1, v_2)$, $\forall [v_0, v_1, v_2] \prec \sigma^n \in K$.

This allows us to embed each n-simplex locally into $\mathbb{R}^n$, and thereby compute all the necessary metric dependent quantities in our formulation. For example, the volume of a k-dual cell will be computed as the sum of the k-volumes of the dual cell restricted to each n-simplex in its local embedding into $\mathbb{R}^n$.

This notion of local metrics and local embeddings is consistent with the point of view that exterior calculus is a local theory with operators that operate on objects in the tangent and cotangent space of a fixed point. The issue of comparing objects in different tangent spaces is addressed in the discrete theory of connections on principal bundles in Leok *et al.* [22].

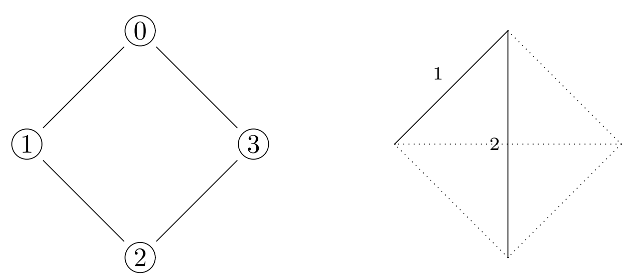
This also provides us with a criterion for evaluating a global embedding. The embedding should be such that the metric of the ambient space $\mathbb{R}^N$ restricted to the vertices of the complex, though of as points in $\mathbb{R}^N$, agrees with the local metric imposed on the abstract simplicial complex. A global embedding that satisfies this condition will produce the same numerical results in discrete exterior calculus as that obtained using the local embedding method.

It is essential that the metric condition we impose is local, since the notion of distances between points in a manifold which are far away is not a well-defined concept, nor is it particularly useful for embeddings. As the simple example below illustrates, there may not exist any global embeddings into Euclidean space that satisfies a metric constraint imposed for all possible pairs of vertices.

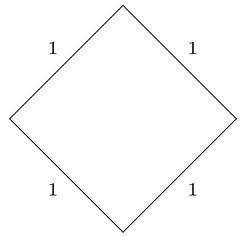
Example 2.4. *Consider a circle, with the distance between two points given by the minimal arc length. Consider a discretization given by 4 equidistant points on the circle, labelled $v_0, \dots, v_3$, with the metric distances as follows,*

$$d(v_i, v_{i+1}) = 1, d(v_i, v_{i+2}) = 2,$$

where the indices are evaluated modulo 4, and this distance function is extended to a metric on all pairs of vertices by symmetry. It is easy to verify that this distance function is indeed a metric on vertices.



If we only use the local metric constraint, then we only require that adjacent vertices are separated by 1*, and the following is an embedding of the simplicial complex into* $\mathbb{R}^2$*,*



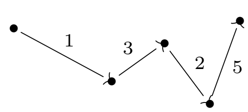
If however, we use the metric defined on all possible pairs of vertices, we see from considering v_0, v_1, v_2 *that since* $d(v_0, v_1) + d(v_1, v_2) = d(v_0, v_2)$*, and we are embedding this into a Euclidean space, it follows that* v_0, v_1, v_2 *are colinear.*

Similarly, by considering v_0, v_2, v_3*, we conclude that they are colinear as well, and that* v_1, v_3 *are coincident, which contradicts* $d(v_1, v_3) = 2$*. Thus we find that there does not exist a global embedding of the circle into Euclidean space if we require that the embedding is consistent with the metric on vertices defined for all possible pairs of vertices.*

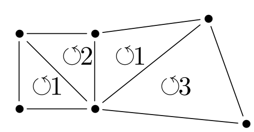
2.3. **Differential Forms and Exterior Derivative.** We will now define discrete differential forms. We will use some terms (which we will define) from algebraic topology but it will become clear by looking at the examples that one can gain a clear and working notion of what a discrete form is without any algebraic topology. We start with a few definitions for which more details can be found on page 26 and 27 of Munkres [28].

Definition 2.10. *Let* K *be a simplicial complex. We denote the free abelian group generated by a basis consisting of oriented* $p-$*simplices by,* $C_p(K; \mathbb{Z})$. *This is the space of finite formal sums of the* p*-simplices, with coefficients in* $\mathbb{Z}$*. Elements of* $C_p(K;\mathbb{Z})$ *are called* p***-chains****.*

Example 2.5.



$1 - \text{chain}$



$2 - \text{chain}$

We view discrete p-forms as maps from the space of p-chains to $\mathbb{R}$. Recalling that the space of p-chains is a group we require these maps that define the forms to be homomorphisms into the additive group $\mathbb{R}$. Thus discrete forms are what are called cochains in algebraic topology. We will define cochains below in the definition of forms but for more context and more details readers can refer to any algebraic topology text, for example page 251 of Munkres [28].

This point of view of forms as cochains is not new. The idea of defining forms as cochains appears for example in the works of Adams [3], Dezin [14], Hiptmair [16], Sen *et al.*[30]. Our point of departure is that the other authors go on to develop a theory of discrete exterior calculus of forms only by introducing interpolation of forms which we will be able to avoid. The formal definition of discrete forms follows.

Definition 2.11. *A **primal discrete** p**-form** α is a homomorphism from the chain group* $C_p(K;\mathbb{Z})$ *to the additive group* $\mathbb{R}$*. Thus a discrete* p*-form is an element of* $\mathrm{Hom}(C_p(K), \mathbb{R})$ *the space of **cochains**. This space becomes an abelian group if we add two homomorphisms by adding their values in* $\mathbb{R}$*. The standard notation for* $\mathrm{Hom}(C_p(K), \mathbb{R})$ *in algebraic topology is* $C^p(K;\mathbb{R})$*. But we will often use the notation* $\Omega^p_d(K)$ *for this space as a reminder that this is the space of discrete (hence the* d *subscript)* p*-forms on the simplicial complex* K*. Thus*

$$\Omega^p_d(K) := C^p(K;\mathbb{R}) = \mathrm{Hom}(C_p(K), \mathbb{R}) \,.$$

Note that by the above definition for p-chain $\sum_i a_i c_i^p$ (where $a_i \in \mathbb{Z}$) and a discrete p-form α

$$\alpha\left(\sum_i a_i c_i^p\right) = \sum_i a_i \alpha(c_i^p)$$

and for two discrete p-forms $\alpha, \beta \in \Omega_d^p(K)$ and p-chain $c \in C_p(K;\mathbb{Z})$

$$(\alpha + \beta)(c) = \alpha(c) + \beta(c)\,.$$

In the usual exterior calculus on smooth manifolds integration of k-forms on a k dimensional manifold is defined in terms of the familiar integration in $\mathbb{R}^k$. This is done roughly speaking by doing the integration in local coordinates and showing that the value is independent of the choice of coordinates due to the change of variables theorem in $\mathbb{R}^k$. For details on this see the first few pages of Chapter 7 of Abraham, Marsden and Ratiu [2]. We will not try to introduce the notion of integration of discrete forms on a simplicial complex. Instead the fundamental quantity that we will work with is the natural bilinear pairing of cochains and chains defined by evaluation. More formally we have the following definition.

Definition 2.12. *The* ***natural pairing*** *of a p-form α and a p-chain c is defined as the bilinear pairing*

$$\langle \alpha, c \rangle = \alpha(c).$$

As mentioned above, in discrete exterior calculus this natural pairing plays a role that integration of forms on chains plays in the usual exterior calculus on smooth manifolds. The two are related by a procedure done at the time of discretization. Indeed consider a simplicial triagulation K of a polyhedron in $\mathbb{R}^n$ i.e consider a "flat" discrete manifold. If we are discretizing a continuous problem we will have some smooth forms defined in the space $|K| \subset \mathbb{R}^n$. Consider such a smooth p-form α^p. In order to define the discrete form α_d^p corresponding to α^p one would integrate α^p on all the p-simplices in K. Then the evaluation of α_d^p on a p-simplex σ^p is defined by $\alpha_d^p(\sigma^p) := \int_{\sigma^p} \alpha^p$. Thus discretization is the only place where integration plays a role in our discrete exterior calculus.

In the case of a non-flat manifold, the situation is somewhat complicated by the fact that the continuous manifold, and the simplicial complex as geometric sets embedded in the ambient space do not coincide. A continuous differential form on the manifold can be discretized into the cochain representation by identifying the vertices of the simplicial complex with points on the manifold, and then using a local chart to identify k-simplices with k-volumes on the manifold.

There is the possibility of k-volumes overlapping even when their corresponding k-simplices do not intersect, and this introduces a discretization error that scales like the mesh size. One can alternatively construct geodesic boundary surfaces in an inductive fashion, which yields a partition of the manifold, but this can be computationally prohibitive to compute.

Now we can define the discrete exterior derivative which we will call $\mathbf{d}$ as in the usual exterior calculus. The discrete exterior derivative will be defined as the dual with respect to the natural pairing defined above, of the boundary operator which is defined below.

Definition 2.13. *The* ***boundary*** *operator $\partial_p : C_p(K;\mathbb{Z}) \to C_{p-1}(K;\mathbb{Z})$ is a homomorphism defined by defining it on a simplex $\sigma^p = [v_0, \ldots, v_p]$,*

$$\partial_p \sigma^p = \partial_p\left([v_0, v_1, \ldots, v_p]\right) = \sum_{i=0}^{p} (-1)^i \, [v_0, \ldots, \hat{v}_i, \ldots, v_p]$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_p]$ is the $(p-1)$-simplex obtained by omitting the vertex v_i. Note that $\partial_p \circ \partial_{p+1} = 0$.

Example 2.6. *Given an oriented triangle $[v_0, v_1, v_2]$ the boundary by the above definition is the chain $[v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ which are the 3 boundary edges of the triangle.*

Definition 2.14. *On a simplicial complex of dimension* n, *a* ***chain complex*** *is a collection of chain groups and homomorphisms* ∂_p *such that*

$$0 \longrightarrow C_n(K) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_1} C_0(K) \longrightarrow 0\,,$$

and $\partial_p \circ \partial_{p+1} = 0$.

Definition 2.15. *The* ***coboundary operator*** $\delta^p : C^p(K) \rightarrow C^{p+1}(K)$ *defined by duality to the boundary operator using the natural bilinear pairing between discrete forms and chains. Specifically, for a discrete form* $\alpha^p \in \Omega^p_d(K)$ *and a chain* $c_{p+1} \in C_{p+1}(K;\mathbb{Z})$ *we define* δ^p *by*

$$\langle \delta^p \alpha^p, c_{p+1} \rangle = \langle \alpha^p, \partial_{p+1} c_{p+1} \rangle \tag{2.2}$$

that is

$$\delta^p(\alpha^p) = \alpha^p \circ \partial_{p+1}\,.$$

This definition of the coboundary operator induces the ***cochain complex***,

$$0 \longleftarrow C^n(K) \xleftarrow{\delta^{n-1}} \cdots \xleftarrow{\delta^p} C^p(K) \xleftarrow{\delta^{p-1}} \cdots \xleftarrow{\delta^0} C^0(K) \longleftarrow 0\,,$$

where it is easy to see that $\delta^{p+1} \circ \delta^p = 0$.

Definition 2.16. *The* ***discrete exterior derivative*** *denoted by* $\mathbf{d} : \Omega^p_d(K) \rightarrow \Omega^{p+1}_d(K)$ *is defined to be the coboundary operator* δ^p.

Remark 2.1. *With the above definition of the exterior derivative* $\mathbf{d} : \Omega^p_d(K) \rightarrow \Omega^{p+1}_d(K)$ *and the relationship between the natural pairing and integration one can regard equation 2.2 as a discrete generalized Stokes' theorem. Thus given a* p*-chain* c *and a discrete* p*-form* α *the discrete Stokes' theorem which is true by definition states that,*

$$\langle \mathbf{d}\alpha, c \rangle = \langle \alpha, \partial c \rangle\ .$$

Furthermore, it also follows immediately that, $\mathbf{d}^{p+1}\mathbf{d}^p = 0$.

Remark 2.2. *Everything we have said above in terms of simplices and the simplicial complex* K *can be said in terms of the cells that are duals of simplices and elements of the dual complex* $\star K$. *One just has to be a little more careful in the definition of the boundary operator, and the definition below is well-defined on the dual cell complex. This gives us the notion of cochains of cells in the dual complex and these are the* ***dual discrete forms***.

Definition 2.17. *The* ***dual boundary*** *operator* $\partial_p : C_p(\star K;\mathbb{Z}) \rightarrow C_{p-1}(\star K;\mathbb{Z})$ *is a homomorphism defined by defining it on a simplex* $\hat{\sigma}^p = \star\sigma^{n-p} = \star[v_0,\ldots,v_{n-p}]$,

$$\begin{aligned} \partial\hat{\sigma}^p &= \partial \star [v_0, ..., v_{n-p}] \\ &= \sum_{\sigma^{n-p+1} \succ \sigma^{n-p}} \star\sigma^{n-p+1}\,, \end{aligned}$$

where σ^{n-p+1} *is oriented so that the induced orientation on* σ^{n-p} *is consistent.*

2.4. **Hodge Star and Codifferential.** In the exterior calculus for smooth manifolds the Hodge star denoted $*$ is an isomorphism between the space of p-forms and $(n-p)$-forms. The Hodge star is useful in defining the adjoint of the exterior derivative and this is adjoint is called the codifferential. For the definition of Hodge star in the smooth case see page 411 of Abraham, Marsden and Ratiu [2]. The appearance of p and $n-p$ in the definition of Hodge star may be taken to be a hint that primal and dual meshes will play some role in the definition of a discrete Hodge star since the dual of a p-simplex is an $(n-p)$-cell. Indeed this is the case.

Definition 2.18. *The* ***discrete Hodge Star*** *is a map $*: C^p(K) \to C^{n-p}(\star K)$ defined by defining it over simplices and duals of simplices. For a p-simplex σ^p and a discrete p-form α^p*

$$\frac{1}{|\sigma^p|}\langle \alpha^p, \sigma^p \rangle = \frac{1}{|\star \sigma^p|}\langle * \alpha^p, \star \sigma^p \rangle.$$

The idea that the discrete Hodge star maps primal discrete forms to dual forms and vice versa is well known. See for example Sen *et al.* [30]. However, notice we now make use of the volume of these primal and dual meshes. But the definition we have given above does appear in the work of Hiptmair [19].

The definition implies that the primal and dual *averages* must be equal. This idea has already been introduced, not in the context of exterior calculus, but in an attempt at defining discrete differential geometry operators [27].

Remark 2.3. *Although we have defined the discrete Hodge star above we will show in Section 2.6 that if the discrete wedge product and inner product of discrete p-forms is defined first then the discrete Hodge star definition above indeed follows from those definitions.*

Lemma 2.1. *For a p-form α,*

$$* * \alpha = (-1)^{p(n-p)}\alpha \,.$$

Proof. The proof is a simple calculation using the property that for a simplex or a cell σ^p, $\star \star (\sigma^p) = (-1)^{p(n-p)}\sigma^p$ (equation 2.1). ☐

Definition 2.19. *Given a simplicial or a dual cell complex K the* ***discrete codifferential operator*** *$\delta : \Omega_d^{p+1}(K) \to \Omega_d^p(K)$ is defined by $\delta(\Omega_d^0(K)) = 0$ and on $p+1$ discrete forms to be*

$$\delta \beta = (-1)^{np+1} * \mathbf{d} * \beta \,.$$

With the discrete forms, Hodge star, $\mathbf{d}$ and δ defined so far we already have enough to do an interesting calculation involving the Laplace-Beltrami operator. But we will show this calculation in Section 2.7 after we have introduced discrete divergence and curl operators.

2.5. **Maps Between One Forms and Vector Fields.** Just as discrete forms come in two flavours of primal and dual (being cochains of primal chains or chains made up of cells from the dual mesh) discrete vector fields also come in two flavours.

Definition 2.20. *A* ***dual discrete vector field*** *X on a simplicial complex K is a map from the zero dimensional dual subcomplex $(\star K)^{(0)}$ (i.e the circumcenters of the primal n simplices) to $\mathbb{R}^N$ such that its value on each dual vertex is tangential to the corresponding primal n-simplex. We will denote the space of such vector fields by $\mathfrak{X}_d(\star K)$. The value of such a vector field is piecewise constant on the n-simplices of K. Thus we could just as well have called such vector fields primal and defined them as functions on the n-simplices of K.*

Definition 2.21. *Let K be a flat simplicial complex, i.e the dimension of K is the same as that of the embedding space. A* ***primal discrete vector field*** *X on a flat simplicial complex K is a map from the zero dimensional primal subcomplex $K^{(0)}$ (i.e the primal vertices) to $\mathbb{R}^N$. We will denote the space of such vector fields by $\mathfrak{X}_d(K)$. The value of such a vector field is piecewise constant on the dual n-cells of $\star K$. Thus we could just as well have called such vector fields dual and defined them as functions on the n cells of $\star K$.*

Remark 2.4. *In this paper we have defined the primal vector fields only for flat meshes. We will address the issue of non flat meshes in separate work.*

As in the smooth exterior calculus we want to define the flat ($\flat$) and sharp ($\sharp$) operators to relate forms to vector fields. This allows one to write various vector calculus identities in terms of exterior calculus.

Definition 2.22. *Given a simplicial complex K of dimension n, the **discrete flat operator on a dual vector field** is $\flat : \mathfrak{X}_d(\star K) \to \Omega^d(K)$ and is defined by its evaluation on a primal 1 simplex σ^1 by*

$$\langle X^\flat, \sigma^1 \rangle = \sum_{\sigma^n \succ \sigma^1} \frac{|\star \sigma^1 \cap \sigma^n|}{|\star \sigma^1|} X \cdot \vec{\sigma}^1$$

where $X \cdot \vec{\sigma}\ 1$ is the usual dot product of vectors in $\mathbb{R}^N$ and $\vec{\sigma}^1$ stands for the vector corresponding to σ^1 and with the same orientation. The sum is over all σ^n containing the edge σ^1.

2.6. **Wedge Product.** As in the smooth case, the wedge product we will construct is a way to build higher degree forms from lower degree ones. For information about the smooth case see the first few pages of Chapter 6 of Abraham, Marsden and Ratiu [2].

Definition 2.23. *Given a primal discrete k-form $\alpha^k \in \Omega_d^k(K)$ and a primal discrete l-form $\beta^l \in \Omega_d^l(K)$ the **discrete primal-primal wedge product** $\wedge : \Omega_d^k(K) \times \Omega_d^l(K) \to \Omega_d^{k+l}(K)$ defined by the evaluation on a $(k+l)$-simplex $\sigma^{k+l} = [v_0, \dots, v_{k+l}]$ as follows,*

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(k)}|}{|\sigma^{k+l}|} \alpha \smile \beta(\tau(\sigma^{k+l}))$$

where S_{k+l+1} is the permutation group and its elements are thought of as permutations of the numbers $0, \dots, k+l+1$. The notation $\tau(\sigma^{k+l})$ stands for the simplex $[v_{\tau(0)}, \dots, v_{\tau(k+l)}]$. Finally the notation $\alpha \smile \beta(\tau(\sigma^{k+l}))$ is borrowed from algebraic topology (see for example page 206 of Hatcher [15]) and is defined as

$$\alpha \smile \beta(\tau(\sigma^{k+l})) := \langle \alpha, \left[v_{\tau(0)}, \dots, v_{\tau(k)}\right] \rangle \langle \beta, \left[v_{\tau(k)}, \dots, v_{\tau(k+l)}\right] \rangle$$

Definition 2.24. *Given a dual discrete k-form $\hat{\alpha}^k \in \Omega_d^k(\star K)$ and a primal discrete l-form $\hat{\beta}^l \in \Omega_d^l(\star K)$ the **discrete dual-dual wedge product** $\wedge : \Omega_d^k(\star K) \times \Omega_d^l(\star K) \to \Omega_d^{k+l}(\star K)$ defined by the evaluation on a $(k+l)$-simplex $\hat{\sigma}^{k+l} = \star \sigma^{n-k-l} = \star[v_0, \dots, v_{n-k-l}]$ as follows.*

$$\begin{aligned}\langle \hat{\alpha}^k \wedge \hat{\beta}^l, \hat{\sigma}^{k+l} \rangle =& \langle \hat{\alpha}^k \wedge \hat{\beta}^l, \star \sigma^{n-k-l} \rangle \\ =& \sum_{\sigma^n \succ \sigma^{n-k-l}} \operatorname{sign}(\sigma^{n-k-l}, [v_{k+l}, \dots, v_n]) \sum_{\tau \in S_{k+l}} \operatorname{sign}(\tau) \\ & \cdot \langle \hat{\alpha}^k, \star[v_{\tau(0)}, \dots, v_{\tau(l-1)}, v_{k+l}, \dots, v_n] \rangle \langle \hat{\beta}^l, \star[v_{\tau(l)}, \dots, v_{\tau(k+l-1)}, v_{k+l}, \dots, v_n] \rangle\end{aligned}$$

where $\sigma^n = [v_0, \dots, v_n]$, and we have without loss of generality assumed that $\sigma^{n-k-l} = \pm[v_{k+l}, \dots, v_n]$.

Lemma 2.2. *The discrete wedge product $\wedge : C^k(K) \times C^l(K) \to C^{k+l}(K)$ is anti-commutative, i.e.,*

$$\alpha^k \wedge \beta^k = (-1)^{kl} \beta^l \wedge \alpha^k$$

Proof. We first rewrite the expression for the discrete wedge product using the following computation,

$$\begin{aligned}&\sum_{\bar{\tau} \in S_{k+l+1}} \operatorname{sign}(\bar{\tau}) |\sigma^{k+l} \cap \star v_{\bar{\tau}(k)}| \langle \alpha^k, \bar{\tau}[v_0, \dots, v_k] \rangle \beta^l, \bar{\tau}[v_k, \dots, v_{k+l}] \rangle \\ &= \sum_{\bar{\tau} \in S_{k+l+1}} (-1)^{k-1} \operatorname{sign}(\bar{\tau}) |\sigma^{k+l} \cap \star v_{\bar{\tau}(k)}|\ \langle \alpha^k, \bar{\tau}[v_1, \dots, v_0, v_k] \rangle \langle \beta^l, \bar{\tau}[v_k, \dots, v_{k+l}] \rangle \\ &= \sum_{\bar{\tau} \in S_{k+l+1}} (-1)^{k-1} \operatorname{sign}(\bar{\tau}) |\sigma^{k+l} \cap \star v_{\bar{\tau}\rho(0)}| \langle \alpha^k, \bar{\tau}\rho[v_1, \dots, v_k, v_0] \rangle \langle \beta^l, \bar{\tau}\rho[v_0, v_{k+1}, \dots, v_{k+l}] \rangle \\ &= \sum_{\bar{\tau} \in S_{k+l+1}} (-1)^{k-1} (-1)^k \operatorname{sign}(\bar{\tau}) |\sigma^{k+l} \cap \star v_{\bar{\tau}\rho(0)}| \langle \alpha^k, \bar{\tau}\rho[v_0, \dots, v_k] \rangle \langle \beta^l, \bar{\tau}\rho[v_0, v_{k+1}, \dots, v_{k+l}] \rangle \\ &= \sum_{\bar{\tau}\rho \in S_{k+l+1}\rho} (-1)^{k-1} (-1)^k (-1) \operatorname{sign}(\bar{\tau}\rho) |\sigma^{k+l} \cap \star v_{\bar{\tau}\rho(0)}| \langle \alpha^k, \bar{\tau}\rho[v_0, \dots, v_k] \rangle \langle \beta^l, \bar{\tau}\rho[v_0, v_{k+1}, \dots, v_{k+l}] \rangle\end{aligned}$$

$$= \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) |\sigma^{k+l} \cap \star v_{\tau(0)}| \langle \alpha^k, \tau[v_0, \dots, v_k] \rangle \langle \beta^l, \tau[v_0, v_{k+1}, \dots, v_{k+l}] \rangle$$

Here, we used the elementary fact from permutation group theory that a $k+1$ cycle can be written as the product of k transpositions, which accounts for the $(-1)^k$ factors. Also, ρ is a transposition of 0 and k. Then, the discrete wedge product can be rewritten as,

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, [v_{\tau(0)}, \dots, v_{\tau(k)}] \rangle \langle \beta^l, [v_{\tau(0),\tau(k+1)}, \dots, v_{\tau(k+l)}] \rangle.$$

For ease of notation, we denote $[v_0, \dots, v_k]$ by σ^k, and $[v_0, v_{k+1}, \dots, v_{k+l}]$ by σ^l. Then we have,

$$\langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \tau(\sigma^k) \rangle \langle \beta^l, \tau(\sigma^l) \rangle.$$

Furthermore, we denote $[v_0, v_{l+1} \dots, v_{k+l}]$ by $\bar{\sigma}^k$, and $[v_0, v_1, \dots, v_l]$ by $\bar{\sigma}^l$. Then,

$$\langle \beta^l \wedge \alpha^k, \sigma^{k+l} \rangle = \frac{1}{(k+l)!} \sum_{\bar{\tau} \in S_{k+l+1}} \operatorname{sign}(\bar{\tau}) \frac{|\sigma^{k+l} \cap \star v_{\bar{\tau}(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \bar{\tau}(\bar{\sigma}^k) \rangle \langle \beta^l, \bar{\tau}(\bar{\sigma}^l) \rangle.$$

Consider the permutation $\theta \in S_{k+l+1}$, given by,

$$\theta = \begin{pmatrix} 0 & 1 & \dots & k & k+1 & \dots & k+l \\ 0 & l+1 & \dots & k+l & 1 & \dots & l \end{pmatrix},$$

which has the property that,

$$\begin{aligned} \bar{\sigma}^k &= \theta(\sigma^k), \\ \bar{\sigma}^l &= \theta(\sigma^l). \end{aligned}$$

Then, we have,

$$\begin{aligned} \langle \beta^l \wedge \alpha^k, \sigma^{k+l} \rangle &= \frac{1}{(k+l)!} \sum_{\bar{\tau} \in S_{k+l+1}} \operatorname{sign}(\bar{\tau}) \frac{|\sigma^{k+l} \cap \star v_{\bar{\tau}(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \bar{\tau}(\bar{\sigma}^k) \rangle \langle \beta^l, \bar{\tau}(\bar{\sigma}^l) \rangle \\ &= \frac{1}{(k+l)!} \sum_{\bar{\tau} \in S_{k+l+1}} \operatorname{sign}(\bar{\tau}) \frac{|\sigma^{k+l} \cap \star v_{\bar{\tau}\theta(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \bar{\tau}\theta(\sigma^k) \rangle \langle \beta^l \bar{\tau}\theta(\sigma^l) \rangle \\ &= \frac{1}{(k+l)!} \sum_{\bar{\tau}\theta \in S_{k+l+1}\theta} \operatorname{sign}(\bar{\tau}\theta) \operatorname{sign}(\theta) \frac{|\sigma^{k+l} \cap \star v_{\bar{\tau}\theta(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \bar{\tau}\theta(\sigma^k) \rangle \langle \beta^l, \bar{\tau}\theta(\sigma^l) \rangle. \end{aligned}$$

By making the substitution, $\tau = \bar{\tau}\theta$, and noting that $S_{k+l+1}\theta = S_{k+l+1}$, we obtain,

$$\begin{aligned} \langle \beta^l \wedge \alpha^k, \sigma^{k+l} \rangle &= \operatorname{sign}(\theta) \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(0)}|}{|\sigma^{k+l}|} \langle \alpha^k, \tau(\sigma^k) \rangle \langle \beta^l, \tau(\sigma^l) \rangle \\ &= \operatorname{sign}(\theta) \langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle \end{aligned}$$

To obtain the desired result, we simply need to compute the sign of θ, which is given by,

$$\operatorname{sign}(\theta) = (-1)^{kl},$$

which follows from the observation that in order to move each of the last l vertices of σ^{k+l} forward, we require k transpositions with $v_1, \dots, v_k$. Therefore, we obtain,

$$\langle \beta^l \wedge \alpha^k, \sigma^{k+l} \rangle = \operatorname{sign}(\theta) \langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle = (-1)^{kl} \langle \alpha^k \wedge \beta^l, \sigma^{k+l} \rangle,$$

and,

$$\alpha^k \wedge \beta^l = (-1)^{kl} \beta^l \wedge \alpha^k.$$

□

Lemma 2.3. *The discrete wedge product satisfies the Leibniz rule,*

$$\mathbf{d}(\alpha^k \wedge \beta^l) = (\mathbf{d}\alpha^k) \wedge \beta^l + (-1)^k \alpha^k \wedge (\mathbf{d}\beta^l).$$

Proof. The proof of the Leibniz rule for discrete wedge products is directly analogous to the coboundary formula for the simplicial cup product on cochains, which can be found on page 206 of Hatcher [15]. This is because the discrete exterior derivative is precisely the coboundary operator.

The cup product satisfies the Leibniz rule for an given partial ordering of the vertices, and the permutations in the signed sum in the discrete wedge product correspond to different choices of partial ordering. We then obtain the Leibniz rule for the discrete wedge product by applying it term-wise for each choice of permutation.

Consider,

$$\begin{aligned}\langle (\mathbf{d}\alpha^k) \wedge \beta^l, \sigma^{k+l+1} \rangle = \sum_{i=0}^{k+1} (-1)^i \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(0)}|}{|\sigma^{k+l}|} \\ \cdot \langle \alpha^k, [v_{\tau(0)}, \dots, \hat{v}_i, \dots, v_{\tau(k+1)}] \rangle \langle \beta^l, [v_{\tau(k+1)}, \dots, v_{\tau(k+l+1)}] \rangle,\end{aligned}$$

and,

$$\begin{aligned}(-1)^k \langle \alpha^k \wedge (\mathbf{d}\beta^l), \sigma^{k+l+1} \rangle = &(-1)^k \sum_{i=k}^{k+l+1} (-1)^{i-k} \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l+1}} \operatorname{sign}(\tau) \frac{|\sigma^{k+l} \cap \star v_{\tau(0)}|}{|\sigma^{k+l}|} \\ &\cdot \langle \alpha^k, [v_{\tau(0)}, \dots, v_{\tau(k)}] \rangle \langle \beta^l, [v_{\tau(k)}, \dots, \hat{v}_i, \dots, v_{\tau(k+l+1)}] \rangle.\end{aligned}$$

The last set of terms, $i = k+1$, of the first expression cancels the first set of terms, $i = k$, of the second expression, and what remains is simply $\langle \alpha^k \wedge \beta^l, \partial \sigma^{k+l+1} \rangle$. Therefore, we can conclude that,

$$\langle (\mathbf{d}\alpha^k) \wedge \beta^l, \sigma^{k+l+1} \rangle + (-1)^k \langle \alpha^k \wedge (\mathbf{d}\beta^l), \sigma^{k+l+1} \rangle = \langle \alpha^k \wedge \beta^l, \partial \sigma^{k+l+1} \rangle = \langle \mathbf{d}(\alpha^k \wedge \beta^l), \sigma^{k+l+1} \rangle,$$

or simply that the Leibniz rule for discrete differential forms holds,

$$\mathbf{d}(\alpha^k \wedge \beta^l) = (\mathbf{d}\alpha^k) \wedge \beta^l + (-1)^k \alpha^k \wedge (\mathbf{d}\beta^l).$$

□

Lemma 2.4. *The discrete wedge product is associative for closed forms. That is to say, for* $\alpha^k \in C^k(K)$, $\beta^l \in C^l(K)$, $\gamma^m \in C^m(K)$, *such that* $\mathbf{d}\alpha^k = 0$, $\mathbf{d}\beta^l = 0$, $\mathbf{d}\gamma^m = 0$, *we have that,*

$$(\alpha^k \wedge \beta^l) \wedge \gamma^m = \alpha^k \wedge (\beta^l \wedge \gamma^m).$$

Proof.

$$\begin{aligned}&\langle (\alpha^k \wedge \beta^l) \wedge \gamma^m, \sigma^{k+l+m} \rangle \\ &\quad = \sum_{\tau \in S_{k+l+m+1}} \operatorname{sign}(\tau) \langle \alpha^k \wedge \beta^l, \tau[v_0, \dots, v_{k+l}] \rangle \langle \gamma^m, \tau[v_{k+l}, \dots, v_{k+l+m}] \rangle \\ &\quad = \sum_{\tau \in S_{k+l+m+1}} \sum_{\rho \in S_{k+l+1}} \operatorname{sign}(\tau) \operatorname{sign}(\rho) \langle \alpha^k, \rho\tau[v_0, \dots, v_k] \rangle \langle \beta^l, \rho\tau[v_k, \dots, v_{k+l}] \rangle \langle \gamma^m, \tau[v_{k+l}, \dots, v_{k+l+m}] \rangle\end{aligned}$$

Here, either $\rho\tau(k) = \tau(k+l)$, in which case all three permuted simplices share $v_{\tau(k+l)}$ as a common vertex, or we need to rewrite either $\langle \alpha^k, \rho\tau[v_0, \dots, v_k] \rangle$ or $\langle \beta^l, \rho\tau[v_k, \dots, v_{k+l}] \rangle$, using the fact that α^k and β^l are closed forms.

If $v_{\tau(k+l)} \notin \rho\tau[v_0, \dots, v_k]$, then we need to rewrite $\langle \alpha^k, \rho\tau[v_0, \dots, v_k] \rangle$ by considering the simplex obtained by adding the vertex $v_{\tau(k+l)}$ to $\rho\tau[v_0, \dots, v_k]$, which is $[v_{\tau(k+l)}, v_{\rho\tau(0)}, \dots, v_{\rho\tau(k)}]$. Then, since α^k is closed, we have that,

$$\begin{aligned}0 &= \langle \mathbf{d}\alpha^k, [v_{\tau(k+l)}, v_{\rho\tau(0)}, \dots, v_{\rho\tau(k)}]\rangle \\ &= \langle \alpha^k, \partial[v_{\tau(k+l)}, v_{\rho\tau(0)}, \dots, v_{\rho\tau(k)}]\rangle \\ &= \langle \alpha^k, [v_{\rho\tau(0)}, \dots, v_{\rho\tau(k)}]\rangle - \sum_{i=0}^{k}(-1)^i \langle \alpha^k, [v_{\tau(k+l)}, v_{\rho\tau(0)}, \dots, \hat{v}_{\rho\tau(i)} \dots, v_{\rho\tau(k)}]\rangle\end{aligned}$$

or equivalently,

$$\langle \alpha^k, [v_{\rho\tau(0)}, \dots, v_{\rho\tau(k)}]\rangle = \sum_{i=0}^{k}(-1)^i \langle \alpha^k, [v_{\tau(k+l)}, v_{\rho\tau(0)}, \dots, \hat{v}_{\rho\tau(i)} \dots, v_{\rho\tau(k)}]\rangle.$$

Notice that all the simplices in the sum, with the exception of the last one, will share two vertices, $v_{\tau(k+l)}$ and $v_{\rho\tau(k)}$ with $\rho\tau[v_k, \dots, v_{k+l}]$, and so their contribution in the triple wedge product will vanish due to the anti-symmetrized sum.

Similarly, if $v_{\tau(k+l)} \notin \rho\tau[v_k, \dots, v_{k+l}]$, using the fact that β^l is closed yields,

$$\langle \alpha^k, [v_{\rho\tau(k)}, \dots, v_{\rho\tau(k+l)}]\rangle = \sum_{i=k}^{k+l}(-1)^{(i-k)} \langle \alpha^k, [v_{\tau(k+l)}, v_{\rho\tau(k)}, \dots, \hat{v}_{\rho\tau(i)} \dots, v_{\rho\tau(k+l)}]\rangle.$$

As before, all the simplices in the sum, with the exception of the last one, will share two vertices, $v_{\tau(k+l)}$ and $v_{\rho\tau(k)}$ with $\rho\tau[v_0, \dots, v_k]$, and so their contribution in the triple wedge product will vanish due to the anti-symmetrized sum.

This allows us to rewrite the triple wedge product in the case of closed forms as,

$$\begin{aligned}\langle (\alpha^k \wedge \beta^l) \wedge \gamma^m, \sigma^{k+l+m}\rangle = \sum_{i=0}^{k+l+m} \sum_{\tau \in S_{k+l+m}} & \operatorname{sign}(\rho_i\tau)\langle \alpha^k, \rho_i\tau[v_0, \dots, v_k]\rangle \\ & \cdot \langle \beta^l, \rho_i\tau[v_0, v_{k+1}, \dots, v_{k+l}]\rangle \langle \gamma^m, \rho_i\tau[v_0, v_{k+l+1}, \dots, v_{k+l+m}]\rangle\end{aligned}$$

where $\tau \in S_{k+l+m}$ is thought of as acting on the set $\{1, \dots, k+l+m\}$, and ρ_i is a transposition of 0 and i. A similar argument allows us to write $\alpha^k \wedge (\beta^l \wedge \gamma^m)$ in the same form, and therefore, the wedge product is associative for closed forms. □

Remark 2.5. *This lemma is significant since if we think of a constant continuous differential form, and discretize it to obtain a discrete differential form, this discrete form will be closed. As such, this lemma states that in the infinitesimal limit, the discrete wedge product we have defined will be associative.*

In practice, if we have a mesh with characteristic length Δx, then we will have,

$$\frac{1}{|\sigma^{k+l+m}|}\langle \alpha^k \wedge (\beta^l \wedge \gamma^m) - (\alpha^k \wedge \beta^l) \wedge \gamma^m, \sigma^{k+l+m}\rangle = \mathcal{O}(\Delta x),$$

which is to say that the average of the associativity defect is of the order of the mesh size, and therefore vanishes in the infinitesimal limit.

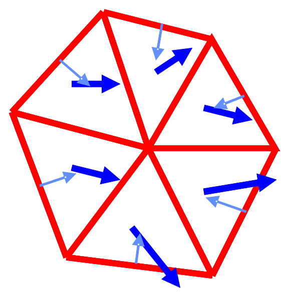
2.7. **Divergence, Curl and Laplace-Beltrami.** We will define the discrete divergence and curl by using the formulas defining them in the smooth exterior calculus. The divergence definition will be for arbitrary dimensions and the curl for $\mathbb{R}^3$. The resulting expressions involve operators that we have already defined and so we can actually perform some calculations to express these quantities in terms of geometric quantities.

We will show that the resulting expression in terms of geometric quantities is the same as that derived by variational means in Desbrun *et al.*[13].

Definition 2.25. *For a discrete dual vector field X the divergence* $\mathrm{div}(X)$ *is defined to be*

$$\mathrm{div}(\mathbf{X}) = -\delta X^\flat \,.$$

Remark 2.6. *The above definition is actually a theorem in smooth exterior calculus. See for example page 458 of Abraham, Marsden and Ratiu* [2].



As an example we will now compute the divergence of a discrete dual vector field on a two dimensional simplicial complex K. A similar derivation works in higher dimensions. Since $\mathrm{div}(X) = -\delta X^\flat$ it follows that $\mathrm{div}(X) = *\mathbf{d} * X^\flat$. Since this is a primal 0 form it can be evaluated on a 0 simplex σ^0. Using the definition of discrete Hodge star and discrete Stokes' theorem we get

$$\begin{aligned}
\frac{1}{|\sigma^0|}\langle \mathrm{div}(X), \sigma^0\rangle &= \frac{1}{|\star\sigma^0|}\langle * * \mathbf{d} * X^\flat, \star\sigma^0\rangle \\
&= \frac{1}{|\star\sigma^0|}\langle \mathbf{d} * X^\flat, \star\sigma^0\rangle \\
&= \frac{1}{|\star\sigma^0|}\langle *X^\flat, \partial(\star\sigma^0)\rangle
\end{aligned}$$

The second equality is by application of definition of Hodge star and the last one above is by application of discrete Stokes' theorem. But

$$\partial(\star\sigma^0) = \sum_{\sigma^1 \succ \sigma^0} \star\sigma^1$$

as shown in the figure. Thus

$$\begin{aligned}
\frac{1}{|\sigma^0|}\langle \mathrm{div}(X), \sigma^0\rangle &= \frac{1}{|\star\sigma^0|}\langle *X^\flat, \sum_{\sigma^1 \succ \sigma^0} \star\sigma^1\rangle \\
&= \frac{1}{|\star\sigma^0|}\sum_{\sigma^1 \succ \sigma^0}\langle *X^\flat, \star\sigma^1\rangle \\
&= \frac{1}{|\star\sigma^0|}\sum_{\sigma^1 \succ \sigma^0}\frac{|\star\sigma^1|}{|\sigma^1|}\langle X^\flat, \sigma^1\rangle \\
&= \frac{1}{|\star\sigma^0|}\sum_{\sigma^1 \succ \sigma^0}\frac{|\star\sigma^1|}{|\sigma^1|}\sum_{\sigma^2 \succ \sigma^1}\frac{|\star\sigma^1 \cap \sigma^2|}{|\star\sigma^1|} X \cdot \vec{\sigma}^1 \\
&= \frac{1}{|\star\sigma^0|}\sum_{\sigma^1 \succ \sigma^0}\sum_{\sigma^2 \succ \sigma^1}\frac{|\star\sigma^1 \cap \sigma^2|}{|\star\sigma^1|} X \cdot \vec{\sigma}^1 \\
&= \frac{1}{|\star\sigma^0|}\sum_{\sigma^1 \succ \sigma^0} |\star\sigma^1| \,(X \cdot \frac{\vec{\sigma}^1}{|\sigma^1|})\,.
\end{aligned}$$

2.7.1. *Laplace-Beltrami.* The Laplace-Beltrami operator is the generalization of the Laplacian to curved spaces. In the smooth case the Laplace-Beltrami operator on smooth functions is defined to be $\nabla^2 = \operatorname{div} \circ \operatorname{curl} = \delta d$. See for example page 459 of Abraham, Marsden and Ratiu [2]. Thus in the smooth case the Laplace-Beltrami on functions is a special case of the more general Laplace-deRham operator $\Delta : \Omega^k(M) \to \Omega^k(M)$ defined by $\Delta = \mathbf{d}\delta + \delta\mathbf{d}$.

As an example we can compute the Δf on a primal vertex σ^0 where $f \in \Omega_d^0(K)$ and K is a (not necessarily flat) triangle mesh in $\mathbb{R}^3$. This calculation is done below.

$$
\begin{aligned}
\frac{1}{|\sigma^0|}\langle \Delta f, \sigma^0 \rangle &= -\langle \delta \mathbf{d} f, \sigma^0 \rangle \\
&= -\langle * \mathbf{d} * \mathbf{d} f, \sigma^0 \rangle \\
&= -\frac{1}{|\star \sigma^0|}\langle \mathbf{d} * \mathbf{d} f, \star \sigma^0 \rangle \\
&= -\frac{1}{|\star \sigma^0|}\langle * \mathbf{d} f, \partial(\star \sigma^0) \rangle \\
&= -\frac{1}{|\star \sigma^0|}\langle * \mathbf{d} f, \sum_{\sigma^1 \succ \sigma^0} \star \sigma^1 \rangle \\
&= -\frac{1}{|\star \sigma^0|}\sum_{\sigma^1 \succ \sigma^0} \langle * \mathbf{d} f, \star \sigma^1 \rangle \\
&= -\frac{1}{|\star \sigma^0|}\sum_{\sigma^1 \succ \sigma^0} \frac{|\star \sigma^1|}{|\sigma^1|}\langle \mathbf{d} f, \sigma^1 \rangle \\
&= -\frac{1}{|\star \sigma^0|}\sum_{\sigma^1 \succ \sigma^0} \frac{|\star \sigma^1|}{|\sigma^1|}(f(v) - f(\sigma^0))
\end{aligned}
$$

where $\partial\sigma^1 = v - \sigma^0$. But the above is the same as the formula involving cotangents found by Meyer *et al.*[27] without using discrete exterior calculus.

2.8. **Contraction and Lie Derivative.** In this section we will discuss some more operators that involve vector fields, namely contraction, Lie derivatives and Lie brackets.

For contraction we will first define the usual smooth contraction algebraically by relating it to Hodge star and wedge products. This yields one potential approach to defining discrete contraction. However since in the discrete theory we are only concerned with integrals of forms we can use the very interesting notion of extrusion of a manifold by the flow of a vector field to define integral of contraction. We learned about this definition of contraction via extrusion from Bossavit [8] who goes on to define discrete extrusion in his paper. Thus he is able to obtain a definition of discrete contraction. Extrusion turns out to be a very nice way to define integrals of operators involving vector fields and we will show how to define integrals of Lie derivatives via extrusion, which will yield discrete Lie derivatives.

Definition 2.26. *Given a manifold M and S a k-dimensional submanifold of M and a vector field $X \in \mathfrak{X}(M)$ we call the manifold obtained by sweeping S along the flow of X for time t as the* ***extrusion*** *of S by X for time t and denote it by $E_X^t(S)$. The manifold S carried by the flow for time t will be denoted S_t.*

2.8.1. *Contraction (extrusion).*

Lemma 2.5.

$$
\int_S i_X \beta = \left.\frac{d}{dt}\right|_{t=0} \int_{E_X^t(S)} \beta
$$

Proof. Prove instead that
$$\int_0^t \left[\int_{S_\tau} i_X\beta\right] d\tau = \int_{E_X^t(S)} \beta\,.$$
Then by first fundamental theorem of calculus the desired result will follow. To prove the above simply take coordinates on S and carry them along with the flow and define the transversal coordinate to be the flow of X. □

2.8.2. *Contraction (algebraic).* Contraction is an operator that allows one to combine vector fields and forms. For a smooth manifold M the contraction of a vector field $X \in \mathfrak{X}(M)$ with a $(k+1)$-form $\alpha \in \Omega^{k+1}(M)$ is written as $\mathbf{i}_X\alpha$ and for vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$ the contraction in smooth exterior calculus is defined by
$$\mathbf{i}_X\alpha(X_1,\ldots,X_k) = \alpha(X, X_1, \ldots, X_k)\,.$$
We define contraction by using an identity that is true in smooth exterior calculus. Since we have not seen this identity we state it here with proof.

Lemma 2.6. *Given a smooth manifold M of dimension n and a vector field $X \in \mathfrak{X}(M)$ and a k-form $\alpha \in \Omega^k(M)$ we have that*
$$\mathbf{i}_X\alpha = (-1)^{k(n-k)} * (*\alpha \wedge X^\flat)\,.$$

Proof. Recall that for a smooth function $f \in \Omega^0(M)$ we have that $\mathbf{i}_X\alpha = f\mathbf{i}_X\alpha$. This and the multilinearity of α implies that it is enough to show the result in terms of basis elements. In particular let $\tau \in S_n$ be a permutation of the numbers $1, \ldots n$ such that $\tau(1) < \ldots < \tau(k)$ and $\tau(k+1) < \ldots < \tau(n)$. Let $X = e_{\tau(j)}$ for some $j \in 1, \ldots, n$. Then we have to show that
$$\mathbf{i}_{e_{\tau(j)}} e^{\tau(1)} \wedge \ldots \wedge e^{\tau(k)} = (-1)^{k(n-k)} * (*(e^{\tau(1)} \wedge \ldots \wedge e^{\tau(k)}) \wedge e^{\tau(j)})\,.$$
It is easy to see that the LHS is 0 if $j > k$ and
$$(-1)^{j-1}(e^{\tau(1)} \wedge \ldots \wedge \hat{e}^{\tau(j)} \ldots \wedge e^{\sigma(k)})$$
otherwise. Now on the RHS we have that
$$*(e^{\tau(1)} \wedge \ldots \wedge e^{\tau(k)}) = \operatorname{sign}(\tau)(e^{\tau(k+1)} \wedge \ldots \wedge e^{\tau(n)})\,.$$
Thus RHS is equal to
$$(-1)^{k(n-k)} \operatorname{sign}(\tau) * (e^{\tau(k+1)} \wedge \ldots \wedge e^{\tau(n)} \wedge e^{\tau(j)})$$
which is 0 as required if $j > k$. So assume that $1 \le j \le k$. Then RHS = LHS as required. □

2.8.3. *Lie Derivative (extrusion).*

Lemma 2.7.
$$\int_S \pounds_X\beta = \left.\frac{d}{dt}\right|_{t=0} \int_{S_t} \beta\,.$$

Proof.
$$F_t^*(\pounds_X\beta) = \frac{d}{dt}F_t^*\beta$$
$$\int_0^t F_\tau^*(\pounds_X\beta)d\tau = F_t^*\beta - \beta$$
$$\int_S\int_0^t F_\tau^*(\pounds_X\beta)d\tau = \int_S F_t^*\beta - \int_S\beta$$
$$\int_0^t \int_{S_\tau} \pounds_X\beta d\tau = \int_{S_t}\beta - \int_S\beta\,.$$

□

2.8.4. *Lie Derivative (algebraic).* We can now construct a discrete Lie derivative from the exterior derivative and the contraction operator using the Cartan magic formula,

$$\pounds_{\mathbf{X}}\omega = \mathbf{i_X d}\omega + \mathbf{d i_X}\omega.$$

2.9. **Discrete Poincaré Lemma.** In this section, we will prove the Discrete Poincaré lemma using the cocone construction. We will first consider the case of trivially star-shaped complexes, followed by logically star-shaped complexes, before generalizing the result to contractible complexes.

Definition 2.27. *Given a k-simplex $\sigma^k = [v_0, \ldots, v_k]$ we construct the* ***cone*** *with vertex w and base σ^k, as follows,*

$$w \diamond \sigma^k = [w, v_0, \ldots, v_k].$$

Lemma 2.8. *The geometric cone operator satisfies the following property,*

$$\partial(w \diamond \sigma^k) + w \diamond (\partial\sigma^k) = \sigma^k.$$

Proof. This is a standard result from simplicial algebraic topology. □

2.9.1. *Trivially Star-Shaped Complexes.*

Definition 2.28. *A complex K is called* ***trivially star-shaped*** *if there exists a vertex $w \in K^{(0)}$ such that for all $\sigma^k \in K$, the cone with vertex w and base σ^k is expressible as a chain in K. That is to say,*

$$\exists w \in K^{(0)} \mid \forall \sigma^k \in K, w \diamond \sigma^k \in C_{k+1}(K).$$

We can then denote the cone operation with respect to w as $p : C_k(K) \to C_{k+1}(K)$.

Lemma 2.9. *In trivially star-shaped complexes, the cone operator $p : C_k(K) \to C_{k+1}(K)$ satisfies the following identity,*

$$p\partial + \partial p = I,$$

at the level of chains.

Proof. Follows immediately from the identity for cones, and noting that the cone is well-defined at the level of chains on trivially star-shaped complexes. □

Definition 2.29. *The* ***cocone*** *operator $H : C^k(K) \to C^{k-1}(K)$ is defined by,*

$$\langle H\alpha^k, \sigma^{k-1}\rangle = \langle \alpha^k, p(\sigma^{k-1})\rangle.$$

This operator is well-defined on trivially star-shaped simplicial complexes.

Lemma 2.10. *The cocone operator $H : C^k(K) \to C^{k-1}(K)$ satisfies the following identity,*

$$H\mathbf{d} + \mathbf{d}H = I,$$

at the level of cochains.

Proof. A simple duality argument applied to the cone identity,

$$p\partial + \partial p = I,$$

yields the following,

$$\begin{aligned}
\langle \alpha^k, \sigma^k\rangle &= \langle \alpha^k, (p\partial + \partial p)\sigma^k\rangle \\
&= \langle \alpha^k, p\partial\sigma^k\rangle + \langle \alpha^k, \partial p\sigma^k\rangle \\
&= \langle H\alpha^k, \partial\sigma^k\rangle + \langle \mathbf{d}\alpha^k, p\sigma^k\rangle \\
&= \langle (\mathbf{d}H\alpha^k, \sigma^k\rangle + \langle H\mathbf{d}\alpha^k, \sigma^k\rangle
\end{aligned}$$

$$= \langle (\mathbf{d}H + H\mathbf{d})\alpha^k, \sigma^k \rangle.$$

Therefore,

$$H\mathbf{d} + \mathbf{d}H = I,$$

at the level of cochains. □

Corollary 2.11 (Discrete Poincaré Lemma for Trivially Star-shaped Complexes)**.** *Given a closed cochain α^k, that is to say, $\mathbf{d}\alpha^k = 0$, there exists a cochain β^{k-1} such that $\mathbf{d}\beta^{k-1} = \alpha^k$.*

Proof. Applying the identity for cochains,

$$H\mathbf{d} + \mathbf{d}H = I,$$

we have,

$$\langle \alpha^k, \sigma^k \rangle = \langle (H\mathbf{d} + \mathbf{d}H)\alpha^k, \sigma^k \rangle$$

but $\mathbf{d}\alpha^k = 0$, so,

$$= \langle \mathbf{d}(H\alpha^k), \sigma^k \rangle.$$

Therefore, $\beta^{k-1} = H\alpha^k$ is such that $\mathbf{d}\beta^{k-1} = \alpha^k$ at the level of cochains. □

Example 2.7. *We demonstrate the construction of the tetrahedralization of the cone of a $n-1$-simplex over the origin.*

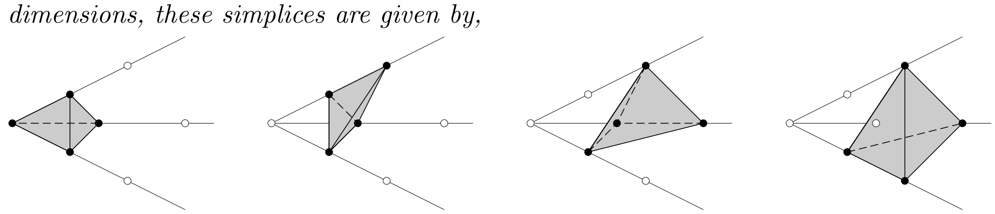
If we denote by v_i^k, the projection of the v_i vertex to the k-th concentric sphere, where the 0-th concentric sphere is simply the central point, then we fill up the cone $[c, v_1, ...v_n]$ with simplices as follows,

$$[v_1^0, v_1^1, \dots, v_n^1], [v_1^2, v_1^1, \dots, v_n^1], [v_1^2, v_2^2, v_2^1, \dots, v_n^1], \dots, [v_1^2, \dots, v_n^2, v_n^1].$$

Since S^{n-1} is orientable, we can use a consistent triangulation of S^{n-1} and these n-cones to consistently triangulate B^n such that the resulting triangulation is star-shaped.

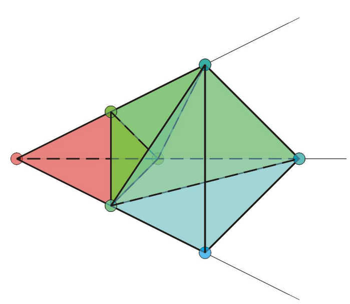
This fills up the region to the 1st concentric sphere, and we repeat the process by leapfrogging at the last vertex to add $[v_1^2, ..., v_n^2, v_n^3]$, and continuing the construction, to fill up the annulus between the 1st and 2nd concentric sphere. Thus, we can keep adding concentric shells to create an arbitrarily dense triangulation of a n-ball about the origin.

In three dimensions, these simplices are given by,



$[c, v_1^1, v_2^1, v_3^1]$, $[v_1^2, v_1^1, v_2^1, v_3^1]$, $[v_1^2, v_2^2, v_2^1, v_3^1]$, $[v_1^2, v_2^2, v_3^2, v_3^1]$.

Putting them together, we obtain,



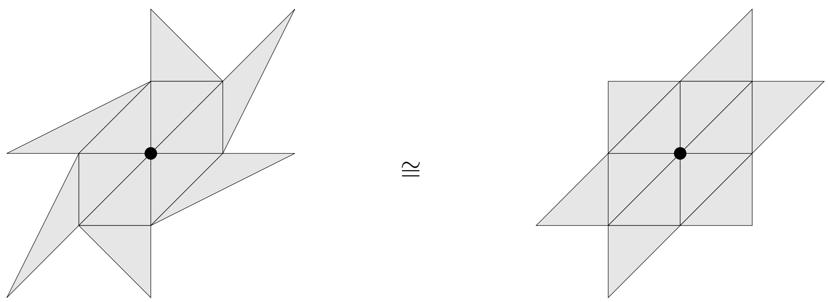
Triangulation of a 3-dimensional cone

This example is significant, since we have demonstrated that for any n-dimensional ball about a point, we can construct a trivially star-shaped triangulation of the ball, with arbitrarily high resolution. This allows us to recover the continuous Poincaré lemma in the limit of an infinitely fine mesh, using the discrete Poincaré lemma for trivially star-shaped complexes.

2.9.2. *Logically Star-Shaped Complexes.*

Definition 2.30. *A simplicial complex L is* ***logically star-shaped*** *if it is isomorphic at the level of an abstract simplicial complex to a trivially star-shaped complex K.*

Example 2.8. *We see two simplicial complexes which are clearly isomorphic as abstract simplicial complexes.*



Trivially star-shaped complex Logically star-shaped complex

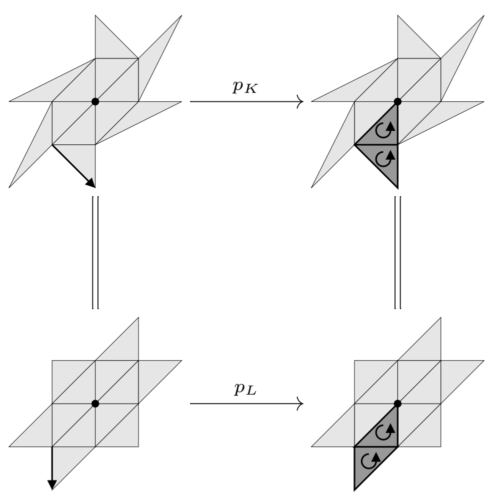
Definition 2.31. *The* ***logical cone*** *operator $p : C^k(L) \to C^{k+1}(L)$ is defined by making the following diagram commute,*

$$\begin{array}{ccc} C^k(K) & \xrightarrow{p_K} & C^{k+1}(K) \\ \| & & \| \\ C^k(L) & \xrightarrow{p_L} & C^{k+1}(L) \end{array}$$

Which is to say that given the isomorphism $\varphi : K \to L$, we define,

$$p_L = \varphi \circ p_K \circ \varphi^{-1}.$$

Example 2.9. *We show an example of the construction of the logical cone operator.*



This definition of the logical cone operator results in the identities for the cone and cocone operator to follow from the trivially star-shaped case, and we record the results as follows.

Lemma 2.12. *In logically star-shaped complexes, the logical cone operator satisfies the following identity,*

$$p\partial + \partial p = I,$$

at the level of chains.

Proof. Follows immediately by pushing forward the result for trivially star-shaped complexes using the isomorphism. □

Lemma 2.13. *In logically star-shaped complexes, the logical cocone operator satisfies the following identity,*

$$H\mathbf{d} + \mathbf{d}H = I,$$

at the level of cochains.

Proof. Follows immediately by pushing forward the result for trivially star-shaped complexes using the isomorphism. □

Similarly, we have a Discrete Poincaré Lemma for logically star-shaped complexes.

Corollary 2.14 (Discrete Poincaré Lemma for Logically Star-shaped Complexes)**.** *Given a closed cochain* α^k*, that is to say,* $\mathbf{d}\alpha^k = 0$*, there exists a cochain* β^{k-1} *such that* $\mathbf{d}\beta^{k-1} = \alpha^k$*.*

Proof. Follows from the above lemma using the proof for the trivially star-shaped case. □

2.9.3. *Contractible Complexes.* For arbitrary contractible complexes, we construct a generalized cone operator such that it satisfies the identity,

$$p\partial + \partial p = I,$$

which is the crucial property of the cone operator, from the point of view of proving the Discrete Poincaré Lemma.

The trivial cone construction gives a clue as to how to proceed in the construction of a generalized cone operator. Notice that if a σ^{k+1} is a term in $p(\sigma^k)$, then $p(\sigma^{k+1}) = \varnothing$. This suggests how we can use the cone identity to inductively construct the generalized cone operator.

To define $p(\sigma^k)$, we consider $\sigma^{k+1} \succ \sigma^k$, such that σ^{k+1} and σ^k are consistently oriented. We apply $p\partial + \partial p$ to σ^{k+1}. Then we have,

$$\sigma^{k+1} = p(\sigma^k) + p(\partial\sigma^{k+1} - \sigma^k) + \partial p(\sigma^{k+1}),$$

if we set $p(\sigma^{k+1}) = \varnothing$,

$$\begin{aligned}\sigma^{k+1} &= p(\sigma^k) + p(\partial\sigma^{k+1} - \sigma^k) + \partial(\varnothing)\\ &= p(\sigma^k) + p(\partial\sigma^{k+1} - \sigma^k).\end{aligned}$$

Rearranging, we have,

$$p(\sigma^k) = \sigma^{k+1} - p(\partial\sigma^{k+1} - \sigma^k),$$

and,

$$p(\sigma^{k+1}) = \varnothing.$$

We are done, so long as the simplices in the chain $\partial\sigma^{k+1} - \sigma^k$ already have p defined on it. This then reduces to enumerating the simplices in such a way that in the right hand side of the equation, we never evoke terms that are undefined.

We now introduce a method of augmenting a complex so that the enumeration condition is always satisfied.

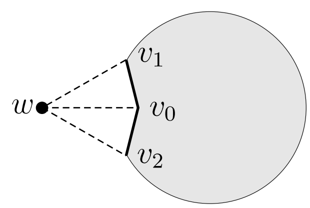
Definition 2.32. *Given a* n*-complex* K*, consider a* $(n-1)$*-chain* c_{n-1} *that is contained on the boundary of* K*, and is included in the one-ring of some vertex on* ∂K*. Then, the* ***one-ring cone augmentation*** *of* K *is the complex obtained by adding the* n*-cone* $w \diamond c_{n-1}$*, and all its faces to the complex.*

Definition 2.33. *A complex is* ***generalized star-shaped*** *if it can be constructed by repeatedly applying the one-ring augmentation procedure.*

We will explicitly show in Examples 2.10, 2.13 how to enumerate the vertices in 2 and 3-dimensions. And in Examples 2.12, 2.14, we will introduce regular triangulations of $\mathbb{R}^2$ and $\mathbb{R}^3$ that can be constructed by inductive one-ring cone augmentation.

Remark 2.7. *Notice that a non-contractible complex cannot be constructed by inductive one-ring cone augmentation, since it will involve adding a cone to a vertex that has two disjoint base chains. This prevents us from enumerating the simplices in such as way that all the terms in $\partial\sigma^{k+1} - \sigma^k$ have had p defined on them, and we see in Example 2.15 how this causes the cone identity, and hence the discrete Poincaré lemma to break.*

Example 2.10. *In 2-dimensions, the 1-ring condition implies that the base of the cone consists of either one or two 1-simplices. To aid in visualization, consider the following diagram,*



One-ring cone augmentation of a complex in 2-dimensions

In the case of one 1-simplex, $[v_0, v_1]$, when we augment using the cone construction with the new vertex w, we define,

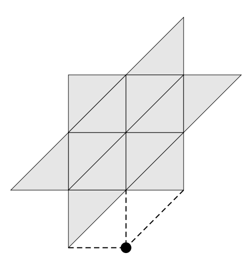
$$p([w]) = [v_0, w] + p([v_0]), \qquad p([v_0, w]) = \varnothing,$$
$$p([v_1, w]) = [v_0, v_1, w] - p([v_0, v_1]), \qquad p([v_0, v_1, w]) = \varnothing.$$

In the case of two 1-simplices, $[v_0, v_1]$, $[v_0, v_2]$, we have,

$$p([w]) = [v_0, w] + p([v_0]), \qquad p([v_0, w]) = \varnothing,$$
$$p([v_1, w]) = [v_0, v_1, w] - p([v_0, v_1]), \qquad p([v_0, v_1, w]) = \varnothing,$$
$$p([v_2, w]) = [v_0, v_2, w] - p([v_0, v_2]), \qquad p([v_0, v_2, w]) = \varnothing.$$

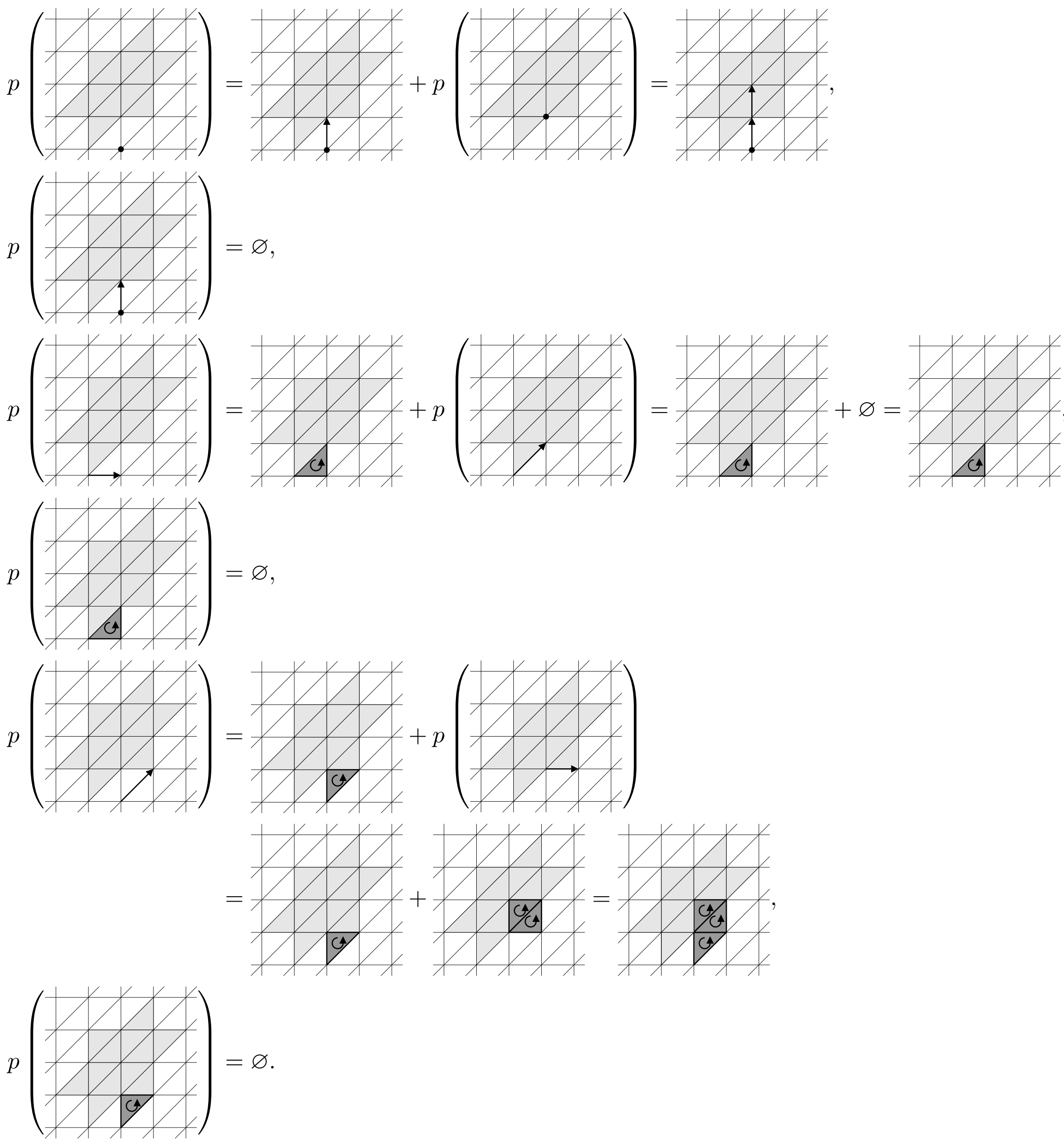
Example 2.11. *We will now explicitly utilize the one-ring cone augmentation procedure to compute the generalized operator for part of a regular 2-dimensional triangulation that is not logically star-shaped.*

As a preliminary, we shall consider a logically star-shaped complex, and augment with a new vertex.

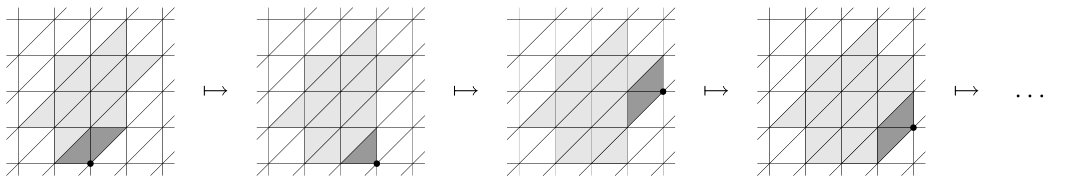


Logically star-shaped complex augmented by cone

We use the logical cone operator for the subcomplex that is logically star-shaped, and the augmentation rules in the example above for the newly introduced simplices. This yields,

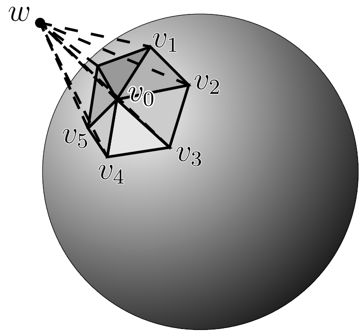


Example 2.12. *Clearly, the regular 2-dimensional triangulation can be obtained by the successive application of the one-ring cone augmentation procedure, as the following sequence illustrates,*



which means that the Discrete Poincaré lemma can be extended to the entire regular triangulation of the plane.

Example 2.13. *We consider the case of augmentation in 3-dimensions. Denote by v_0 the center of the 1-ring on the 2-surface, to which we are augmenting the new vertex w. The other vertices of the 1-ring are enumerated in order $v_1, \dots, v_m$. To aid in visualization, consider the following diagram,*



One-ring cone augmentation of a complex in 3-dimensions

If the 1-ring does not go completely around v_0, we shall denote the missing term by $[v_0, v_1, v_m]$.

k=0,

$$p([w]) = [v_0, w] + p([v_0]), \qquad p([v_0, w]) = \varnothing,$$

k=1,

$$\begin{aligned} p([v_1, w]) &= [v_0, v_1, w] - p([v_0, v_1]), & p([v_0, v_1, w]) &= \varnothing, \\ p([v_m, w]) &= [v_0, v_m, w] - p([v_0, v_m]), & p([v_0, v_m, w]) &= \varnothing, \end{aligned}$$

k=2,

$$\begin{aligned} p([v_1, v_2, w]) &= [v_0, v_1, v_2, w] + p([v_0, v_1, v_2]), & p([v_1, v_2, w]) &= \varnothing, \\ p([v_{m-1}, v_m, w]) &= [v_0, v_{m-1}, v_m, w] + p([v_0, v_{m-1}, v_m]), & p([v_{m-1}, v_m, w]) &= \varnothing. \end{aligned}$$

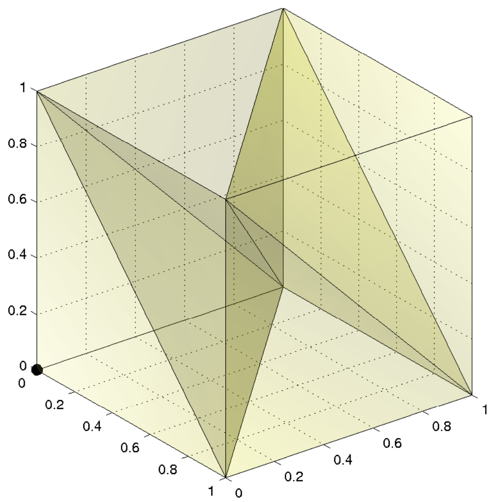
If it does go around completely,

$$p([v_m, v_1, w]) = [v_0, v_m, v_1, w] + p([v_0, v_m, v_1]), \qquad p([v_0, v_m, v_1, w]) = \varnothing.$$

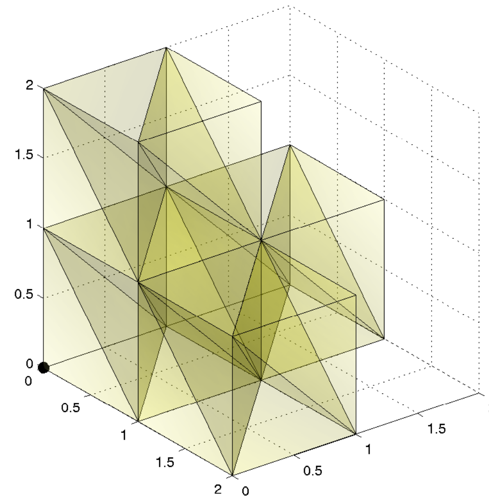
Example 2.14. *We provide a tetrahedralization of the unit cube that can be tiled to yield a regular tetrahedralization of $\mathbb{R}^3$. The 3-simplices are as follows,*

$$[v_{000}, v_{001}, v_{010}, v_{10}], [v_{001}, v_{010}, v_{100}, v_{101}], [v_{001}, v_{010}, v_{011}, v_{101}],$$
$$[v_{010}, v_{100}, v_{101}, v_{110}], [v_{010}, v_{011}, v_{101}, v_{110}], [v_{011}, v_{101}, v_{110}, v_{111}].$$

The tetrahedralization of the unit cube can be visualized as follows,



Tileable tetrahedralization of the unit cube

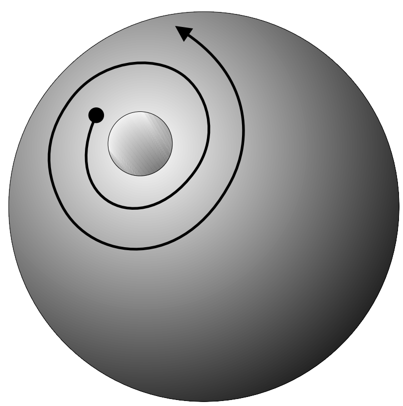


Partial tiling

Since this regular tetrahedralization can be constructed by the successive application of the one-ring cone augmentation procedure, the Discrete Poincaré lemma can be extended to the entire regular tetrahedralization of $\mathbb{R}^3$.

In higher dimensions, we can extend the construction of the generalized cone operator inductively using the one-ring cone augmentation by choosing an appropriate enumeration of the base chain. The base chain will topologically be given by the cone of S^{n-2} (with possibly an open $n-2$ ball removed) with respect to the central point.

By spiraling around S^{n-2}, starting from around the boundary of the $n-2$ ball, and covering the rest of S^{n-2}, we obtain the higher dimensional generalization of the procedure we have taken in Examples 2.10, 2.13.



Spiral enumeration of S^{n-2}, $n = 4$

Notice that $n = 2$ is distinguished, since $S^{2-2} = S^0$ is disjoint, which is why in the 2-dimensional case, we were not able to use the spiraling technique to enumerate the simplices.

Since we have constructed the generalized cone operator such that the cone identity holds, we have,

Lemma 2.15. *In generalized star-shaped complexes, the generalized cone operator satisfies the following identity,*

$$p\partial + \partial p = I,$$

at the level of chains.

Proof. By construction. □

Lemma 2.16. *In generalized star-shaped complexes, the generalized cocone operator satisfies the following identity,*

$$H\mathbf{d} + \mathbf{d}H = I,$$

at the level of cochains.

Proof. Follows immediately from applying the proof in the trivially star-shaped case, and using the identity in the previous lemma. □

Similarly, we have a Discrete Poincaré Lemma for generalized star-shaped complexes.

Corollary 2.17 (Discrete Poincaré Lemma for Generalized Star-shaped Complexes)**.** *Given a closed cochain α^k, that is to say, $\mathbf{d}\alpha^k = 0$, there exists a cochain β^{k-1} such that $\mathbf{d}\beta^{k-1} = \alpha^k$.*

Proof. Follows from the above lemma using the proof for the trivially star-shaped case. □

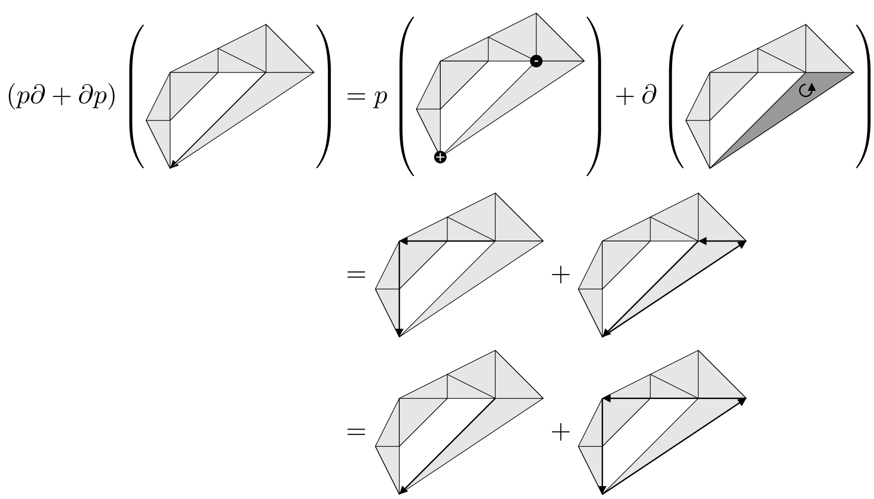
Example 2.15. *We will consider an example of how the Poincaré lemma fails in the case when the complex is not contractible. Consider the following trivially star-shaped complex, and augment by one vertex so as to make the region non-contractible.*

Trivially star-shaped complex   Non-contractible complex

Now we attempt to verify the identity,

$$p\partial + \partial p = I,$$

and we will see how this is only true up to a chain that is homotopic to the inner boundary.



Since the second term cannot be expressed as the boundary of a 2-chain, it will contribute a non-trivial effect, even on closed discrete forms, and therefore the Poincaré lemma breaks.

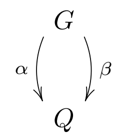
2.10. **Groupoid Interpretation of Discrete Variational Mechanics.** The groupoid interpretation of discrete mechanics is most clearly illustrated if we consider the discretization of trajectories on TQ in two stages. Given a curve $\gamma : \mathbb{R}^+ \to TQ$, we consider a discrete sampling given by,

$$g_i = \gamma(ih) \in TQ$$

We then approximate TQ by $Q \times Q$, and associate to g_i two elements in Q. We denote this by,

$$g_i \mapsto (q_i^0, q_i^1).$$

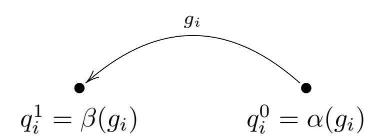
Or equivalently, in the language of groupoids [9, 31], we have,



where α is the **source** map, and β is the **target** map. Then,

$$g_i \mapsto (\alpha(g_i), \beta(g_i)) = (q_i^0, q_i^1).$$

This can be visualized as,

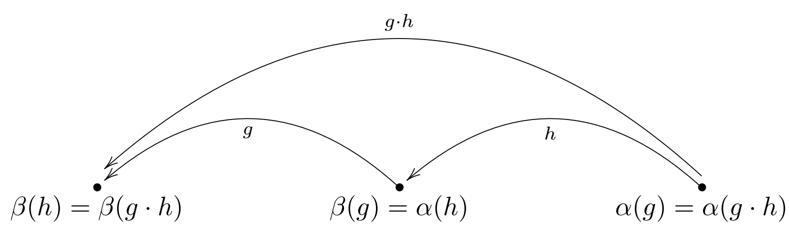


A product $\cdot : G^{(2)} \to G$ is defined on the set of composable pairs,

$$G^{(2)} := \{(g, h) \in G \times G \mid \beta(g) = \alpha(h)\}.$$

The element $g \cdot h$ is defined by,

$$\alpha(g \cdot h) = \alpha(g),$$
$$\beta(g \cdot h) = \beta(h).$$



The set of composable pairs is the discrete analogue of the set of second order curves on TQ. A curve $\gamma : \mathbb{R}^+ \to TQ$ is said to be second order if there exists a curve $q : \mathbb{R}^+ \to Q$, such that,

$$\gamma(t) = (q(t), \dot{q}(t)).$$

The corresponding condition for discrete curves is that given a sequence of points in $Q \times Q$, $(q_1^0, q_1^1), \ldots, (q_p^0, q_p^1)$, we require that,

$$q_i^1 = q_{i+1}^0.$$

This implies that the discrete curve on $Q \times Q$ is derived from a $p+1$ pointed curve $(q_0, \ldots, q_p)$ on Q, where,

$$q_i = \begin{cases} q_{i+1}^0, & \text{if } 0 \leq i < p; \\ q_i^1, & \text{if } i = p. \end{cases}$$

This condition has a direct equivalent in groupoids,

$$\beta(g_i) = q_i^1 = q_{i+1}^0 = \alpha(g_{i+1}).$$

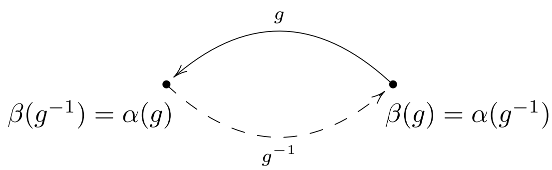
and in general, we have a hierarchy of sets, given by,

$$G^{(p)} := \{(g_1, \dots, g_p) \in G^p \mid \beta(g_i) = \alpha(g_{i+1})\},$$

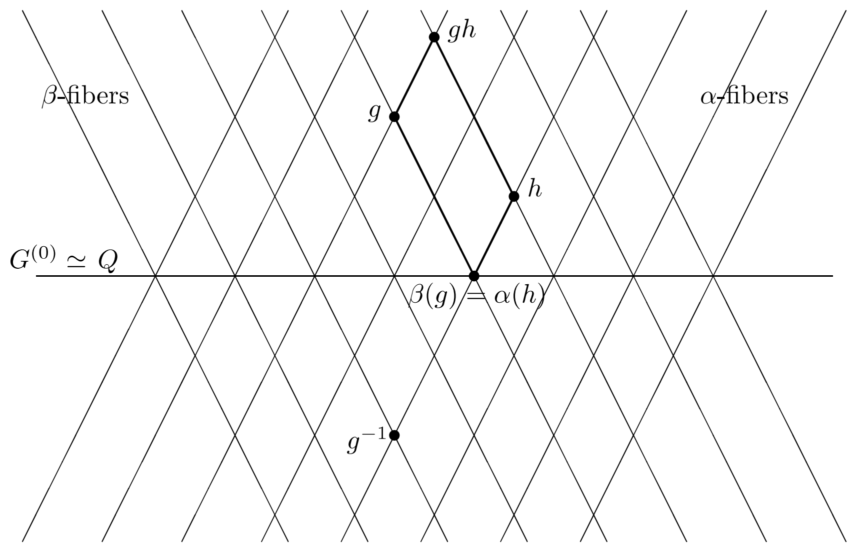
where $G^{(0)} \simeq Q$.

In addition, the inverse of a groupoid element is defined by the following,

$$\alpha(g^{-1}) = \beta(g),$$
$$\beta(g^{-1}) = \alpha(g).$$



2.10.1. *Visualizing Groupoids.* In summary, composition of groupoid elements, and the inverse of groupoid elements can be illustrated by the following diagram.



2.11. **Discrete Diffeomorphisms and Discrete Flows.**

Definition 2.34. *Given a complex K embedded in V, and its corresponding abstract simplicial complex M, a* ***discrete diffeomorphism*** *$\varphi \in \mathrm{Diff}_d(M)$ is a pair of simplicial complexes K_1, K_2 which are realizations of M in the ambient space V. This is denoted by $\varphi(M) = (K_1, K_2)$.*

Definition 2.35. *A* ***one-parameter family of discrete diffeomorphisms*** *is a map $\varphi : I \to \mathrm{Diff}_d(M)$, such that,*

$$\pi_1(\varphi(t)) = \pi_1(\varphi(s)), \qquad \forall s, t \in I.$$

Since we are concerned with evolving equations represented by these discrete diffeomorphisms, and mesh degeneracy causes the numerics to fail, we introduce the notion of non-degenerate discrete diffeomorphisms,

Definition 2.36. *A* ***non-degenerate discrete diffeomorphism*** *$\varphi = (K_1, K_2)$ is such that K_1 and K_2 are non-degenerate realizations of the abstract simplicial complex M in the ambient space V.*

Notice that it is sufficient to define the discrete diffeomorphism on the vertices of the abstract complex M, since we can extend it to the entire complex by the relation,

$$\varphi([v_0,...,v_k]) = ([\pi_1\varphi(v_0),...,\pi_1\varphi(v_k)],[\pi_2\varphi(v_0),...,\pi_2\varphi(v_k)]).$$

If $X \in K^{(0)}$ is a material vertex of the manifold, corresponding to the abstract vertex w, that is to say, $\pi_1\varphi_t(w) = X, \forall t \in I$, the corresponding trajectory followed by X in space is $x = \pi_2\varphi_t(w)$. Then the **material velocity** $V(X,t)$ is given by,

$$V(\pi_1(w),t) = \left.\frac{\partial \pi_2\varphi_s(w)}{\partial s}\right|_{s=t},$$

and the **spatial velocity** $v(x,t)$ is given by,

$$v(\pi_2(w),t) = V(\pi_1(w),t) = \left.\frac{\partial \varphi_s(\varphi_t^{-1}(x))}{\partial s}\right|_{s=t}.$$

The material velocity field can be thought of as a discrete vector field with the vectors based at the vertices of K, which is to say that $D\varphi_t \in \mathfrak{X}(K)$, is a discrete primal vector field. Notice that φ_t on K induces a map $\hat{\varphi}_t$ on the vertices of the dual $\hat{K}$, by the following,

$$\hat{\varphi}_t(c[v_0,\ldots,v_n]) = (c[\pi_1\varphi_t(v_0),\ldots,\pi_1\varphi_t(v_n)], c[\pi_2\varphi_t(v_0),\ldots,\pi_2\varphi_t(v_n)]).$$

Similarly then, $D\hat{\varphi}_t \in \mathfrak{X}(\hat{K})$, is a discrete dual vector field.

At first glance, this seems like a cumbersome way to define a one-parameter family of discrete diffeomorphisms, and one may be tempted to think of extending φ_t to the ambient space. We would then be thinking of $\varphi_t : V \to V$. This is undesirable since given φ_t and ψ_s which are non-degenerate flows, their composition $\varphi_t \circ \psi_s$, which is defined, may result in a degenerate mesh when applied to K. Thus, non-degenerate flows are not closed under this notion of composition.

If we adopt groupoid composition instead at the level of vertices, we can always be sure that if we compose two nondegenerate discrete diffeomorphisms, they will remain a nondegenerate discrete diffeomorphism.

The pullback of a form $\alpha^k \in C^k(L)$, under the map $f : K \to L$ is defined so that the **change of variables formula** holds,

$$\langle f^*\alpha^k, \sigma^k\rangle = \langle \alpha^k, f\sigma^k\rangle,$$

where $\sigma^k \in K$.

The space of discrete diffeomorphisms naturally has the structure of a pair groupoid. The discrete analogue of $T\operatorname{Diff}(M)$ from the point of view of temporal discretization is the pair groupoid $\operatorname{Diff}(M) \times \operatorname{Diff}(M)$. In addition, we discretize $\operatorname{Diff}(M)$ using $\operatorname{Diff}_d(M)$, which is in turn a pair groupoid involving realizations of an abstract simplicial complex in an ambient space.

2.12. **Remeshing Cochains and Multigrid Extensions.** It is sometimes desirable, particularly in the context of multigrid, multiscale, and multiresolution computations to be able to represent a discrete differential form which is given as a cochain on a prescribed mesh as one which is supported on a new mesh. Given a differential form $\omega^k \in \Omega^k(K)$, and a new mesh M such that $|K| = |M|$, we can define it at the level of cosimplices,

$$\forall \tau^k \in M^{(k)}, \qquad \langle \omega^k, \tau^k\rangle = \sum_{\sigma^k \in K^{(k)}} \operatorname{sgn}(\tau^k,\sigma^k)\frac{|V_{\tau^k} \cap V_{\sigma^k}|}{|V_{\sigma^k}|}\langle \omega^k, \sigma^k\rangle,$$

and extend this by linearity to cochains. Here, $\operatorname{sgn}(\tau^k,\sigma^k)$ is $+1$ if the orientation of τ^k and σ^k are consistent, and -1 otherwise. Since k-skeletons of meshes that are not related by subdivision may not have nontrivial intersections, intersections of support volumes are used in the remeshing formula, as opposed to intersections of the k-simplices.

We denote this transformation at the level of cochains as, $T_{K,M} : C^k(K) \to C^k(M)$. This has the natural property that if we have a k-volume U^k that can be represented as a chain in either the complex K or the complex M, that is to say, $U^k = \sigma_1^k + \ldots + \sigma_l^k = \tau_1^k + \ldots + \tau_l^k$, then we have,

$$\begin{aligned}\langle \omega^k, \tau_1^k + \ldots + \tau_m^k \rangle &= \sum_{i=1}^m \langle \omega, \tau_i^k \rangle = \sum_{i=1}^m \sum_{\sigma^k \in K^{(k)}} \operatorname{sgn}(\tau_i^k, \sigma^k) \frac{|V_{\tau_i^k} \cap V_{\sigma^k}|}{|V_{\sigma^k}|} \langle \omega^k, \sigma^k \rangle \\ &= \sum_{i=1}^m \sum_{j=1}^l \operatorname{sgn}(\tau_i^k, \sigma_j^k) \frac{|V_{\tau_i^k} \cap V_{\sigma_j^k}|}{|V_{\sigma_j^k}|} \langle \omega^k, \sigma_j^k \rangle = \sum_{j=1}^l \sum_{i=1}^m \operatorname{sgn}(\tau_i^k, \sigma_j^k) \frac{|V_{\tau_i^k} \cap V_{\sigma_j^k}|}{|V_{\sigma_j^k}|} \langle \omega^k, \sigma_j^k \rangle \\ &= \sum_{j=1}^l \langle \omega^k, \sigma_j^k \rangle = \langle \omega^k, \sigma_1^k + \ldots + \sigma_l^k \rangle. \end{aligned}$$

Which is to say that the integral of the differential form over U^k is well-defined, and independent of the representation of the differential form.

Note that, in particular, if we choose to coarsen the mesh, the value the form takes on a cell in the coarser mesh is simply the sum of the value the form takes on the old cells of the fine mesh which make up the new cell in the coarser mesh.

Non-flat manifolds. The case of non-flat manifolds presents a challenge in remeshing akin to that encountered in the discretization of differential forms. In particular, if the two meshes represent different discretizations of a non-flat manifold, they will in general correspond to different polyhedral regions in the embedding space and not have the same support region.

We assume that our discretization of the manifold is sufficiently fine that for every simplex, all its vertices are contained in some chart. Then, by using these local charts, we can identify support volumes in the computational domain with n-volumes in the manifold, and thereby make sense of the remeshing formula.

3. Discrete Theory of Connections on Principal Bundles

3.1. Homological Algebra and Connections.

3.1.1. *Equivalent representations of split exact sequences.* Recall that given a short exact sequence,

$$0 \longrightarrow A_1 \underset{k}{\overset{f}{\rightleftarrows}} B \underset{h}{\overset{g}{\rightleftarrows}} A_2 \longrightarrow 0\ ,$$

applying the Five Lemma allows us to conclude that the following conditions are equivalent:

- There is a homomorphism $h : A_2 \to B$ with $g \circ h = 1_{A_2}$;
- There is a homomorphism $k : B \to A_1$ with $k \circ f = 1_{A_1}$;
- The given sequence is isomorphic (with identity maps on A_1 and A_2) to the direct sum short exact sequence,

  $$0 \longrightarrow A_1 \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} A_1 \oplus A_2 \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} A_2 \longrightarrow 0\ ,$$

  and in particular, $B \cong A_1 \oplus A_2$.

3.1.2. *Exact Sequences of Vector Bundles.* Given a principal bundle $\pi : Q \to Q/G$ we have the following exact sequence of vector bundles over Q,

$$0 \longrightarrow VQ \longrightarrow TQ \xrightarrow{\pi_*} \pi^* T(Q/G) \longrightarrow 0\ ,$$

where VQ is the vertical subbundle of TQ.

The quotient modulo G of the above sequence yields an exact sequence of vector bundles over Q/G. The resulting sequence is called the **Atiyah sequence** [4].

3.1.3. *Principal and Associated Bundles.* Recall the following definitions, which will be relevant in the subsequent sections.

Definition 3.1. *A* ***Principal Bundle*** *is a manifold Q with a free left action $G \times Q \to Q$ of a Lie group G, such that the natural projection $\pi : Q \to Q/G$ is a submersion.*

Definition 3.2. *An* ***Associated Bundle*** *$\tilde{M}$ is given by the fibre product of the base manifold Q with the manifold M, both viewed as principal bundles over G,*

$$\tilde{M} = Q \times_G M = (Q \times M)/G.$$

In particular, two associated bundles arise when considering the continuous and discrete Atiyah sequence of a principal bundle:

- $\tilde{\mathfrak{g}}$, where the action of G on $Q \times \mathfrak{g}$ is given by $g(q, \xi) = (gq, Ad_g \xi)$, and $\pi_{\mathfrak{g}} : \tilde{\mathfrak{g}} \to Q/G$ is given by $\pi_{\mathfrak{g}}([q, \xi]_G) = \pi(q)$.
- $\tilde{G}$, where the action of G on $Q \times G$ is given by $g(q, h) = (gq, ghg^{-1})$, and $\pi_G : \tilde{G} \to Q/G$ is given by $\pi_G([q, g]_G) = \pi(q)$.

3.1.4. *Atiyah sequence of a Principal Bundle.* The continuous Atiyah sequence is represented by the following diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{\mathfrak{g}} & \underset{(\pi_1,\mathcal{A})}{\overset{i}{\rightleftarrows}} & TQ/G & \underset{X^h}{\overset{\pi_*}{\rightleftarrows}} & TS & \longrightarrow & 0 \\
 & & \Big\| 1_{\tilde{\mathfrak{g}}} & & \Big\downarrow \alpha_{\mathcal{A}} & & \Big\| 1_{TS} & & \\
0 & \longrightarrow & \tilde{\mathfrak{g}} & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & \tilde{\mathfrak{g}} \oplus TS & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & TS & \longrightarrow & 0
\end{array}$$

where the relevant maps are given by,

- $i : (Q \times \mathfrak{g})/G \to TQ/G$, where,
$$i([q, \xi]_G) = [\xi_Q(q)]_G.$$
- $\pi_* : TQ/G \to TS$, where,
$$\pi_*([v_q]_G) = T\pi(v_q).$$

and where $\tilde{\mathfrak{g}}$ is the adjoint bundle, which is the associated bundle given by $Q \times_G \mathfrak{g}$.

3.1.5. *Discrete Atiyah sequence of a Principal Bundle.* The discrete Atiyah sequence is represented by the following diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{G} & \underset{(\pi_1,\mathcal{A}_d)}{\overset{i}{\rightleftarrows}} & (Q \times Q)/G & \underset{(\cdot,\cdot)^h}{\overset{(\pi,\pi)}{\rightleftarrows}} & S \times S & \longrightarrow & 0 \\
 & & \Big\| 1_{\tilde{G}} & & \Big\downarrow \alpha_{\mathcal{A}_d} & & \Big\| 1_{S \times S} & & \\
0 & \longrightarrow & \tilde{G} & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & \tilde{G} \oplus (S \times S) & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & S \times S & \longrightarrow & 0
\end{array}$$

where the relevant maps are given by,

- $i : \tilde{G} \to (Q \times Q)/G$, where,
$$i([q, g]_G) = [q, gq]_G.$$
- $(\pi, \pi) : (Q \times Q)/G \to S \times S$, where,
$$(\pi, \pi)([q_0, q_1]_G) = (\pi q_0, \pi q_1).$$

and where $\tilde{G}$ is the associated bundle $Q \times_G G$.

3.1.6. *Equivalent Representations of a Discrete Connection.* There are four methods of representing a discrete connection,

- Maps on the un-quotiented sequence

  - Discrete Connection Form $\mathcal{A}_d : Q \times Q \to G$.
  - Discrete Horizontal Lift $(\cdot,\cdot)_q^h : S \times S \to Q \times Q$.
- Maps defining a splitting
  - $(\pi_1, \mathcal{A}_d) : (Q \times Q)/G \to \tilde{G}$,
    related to the discrete connection form.
  - $(\cdot,\cdot)^h : S \times S \to (Q \times Q)/G$,
    related to the discrete horizontal lift.

We can relate the two sets of representations in the following way:

- The maps on the un-quotiented sequence induce splittings.
- The splittings extend by equivariance to recover the un-quotiented maps.

3.2. **Discrete Connections.** We introduce a discrete Lie group valued connection, $\mathcal{A}_d : Q \times Q \to G$ which is that natural generalization of the Lie algebra valued connection on tangent bundles, $\mathcal{A} : TQ \to \mathfrak{g}$.

Definition 3.3. *A* ***discrete connection*** *is a map, $\mathcal{A}_d : Q \times Q \to G$, which satisfies the following properties,*

(1) *The map is G-equivariant, that is,*
$$\mathcal{A}_d \circ L_g = I_g \circ \mathcal{A}_d,$$
which is the discrete analogue of the G-equivariance of the continuous connection, $\mathcal{A} \circ L_g = Ad_g \circ \mathcal{A}$.

(2) *The map induces a splitting of the Discrete Atiyah sequence, that is,*
$$\mathcal{A}_d(q_0, gq_0) = g,$$
which is the discrete analogue of $\mathcal{A}(\xi_Q) = \xi$.

(3) *The map is antisymmetric, that is,*
$$\mathcal{A}_d(q_0, q_1) = (\mathcal{A}_d(q_1, q_0))^{-1}$$
which is the discrete analogue of the condition $\mathcal{A}(-v_q) = -\mathcal{A}(v_q)$.

The second condition is equivalent to the map recovering the Discrete Euler–Poincaré connection when restricted to a G-fibre, that is, $\mathcal{A}_d(x, g_0, x, g_1) = g_1 g_0^{-1}$. In particular, it follows that the map is trivial when restricted to the diagonal space, that is, $\mathcal{A}_d(q, q) = e$, which can also be seen from the third condition.

Remark 3.1. *The properties of a discrete connection are discrete analogues of the properties of a continuous connection in the sense that if a discrete connection has a given property, the corresponding continuous connection which is induced in the infinitesimal limit has the analogous continuous property.*

Remark 3.2. *In general, a discrete connection is only defined for pairs of points in $Q \times Q$ which project to pairs of points in $S \times S$ that are sufficiently close to the diagonal. A discrete connection is only globally defined for a trivial bundle.*

Definition 3.4. *A* ***flat discrete connection*** *is a discrete connection that further satisfies the property that the map is a homomorphism from the pair groupoid $Q \times Q$ to the group G, that is, $\mathcal{A}_d(q_0, q_2) = \mathcal{A}_d((q_1, q_2) \cdot (q_0, q_1)) = \mathcal{A}_d(q_1, q_2) \cdot \mathcal{A}_d(q_0, q_1)$.*

Example 3.1. *As an example, we construct the natural discrete analogue of the mechanical connection $\mathcal{A} : TQ \to \mathfrak{g}$ by the following procedure.*

(1) *Given the point $(q_0, q_1) \in Q \times Q$, we construct the geodesic path $q_{01} : [0,1] \to Q$ with respect to the kinetic energy metric, such that $q_{01}(0) = q_0$, and $q_{01}(1) = q_1$.*
(2) *Project the geodesic path to the shape space, $x_{01}(t) \equiv \pi q_{01}(t)$, to obtain the curve x_{01} on S.*
(3) *Taking the horizontal lift of x_{01} to Q using the connection $\mathcal{A}$ yields $\tilde{q}_{01}$.*
(4) *There is a unique $g \in G$ such that $q_{01}(1) = g \cdot \tilde{q}_{01}(1)$.*
(5) *Define $\mathcal{A}_d(q_0, q_1) = g$.*

This discrete connection is consistent with the classical notion of a connection in the limit that q_1 approaches q_0, in the usual sense in which discrete mechanics on $Q \times Q$ converges to continuous Lagrangian mechanics on TQ.

This notion of a discrete connection is motivated by the desire to construct a global diffeomorphism between $(Q \times Q)/G \to S$ and $(S \times S) \oplus \tilde{G} \to S$. This is the discrete analogue of the identification between $TQ/G \to Q/G$ and $T(Q/G) \oplus \tilde{\mathfrak{g}} \to Q/G$ which is the context for Lagrangian Reduction in Cendra, Marsden, Ratiu [11].

Remark 3.3. *The discrete adjoint bundle $\tilde{G}$ is the associated bundle one obtains when $M = G$, and ρ_g acts by conjugation.*

Remark 3.4. *The action of G on $Q \times Q$ is by the diagonal action, and the action of $G \times G$ on $Q \times Q$ is component-wise.*

Lemma 3.1. *The map $\alpha_{\mathcal{A}_d} : (Q \times Q)/G \to (S \times S) \oplus \tilde{G}$ defined by*

$$\alpha_{\mathcal{A}_d}([q_0, q_1]_G) = (\pi q_0, \pi q_1) \oplus [q_0, \mathcal{A}_d(q_0, q_1)]_G.$$

is a well-defined bundle isomorphism. The inverse of α_{A_d} is given by

$$\alpha^{-1}_{\mathcal{A}_d}((x_0, x_1) \oplus [q, g]_G) = [(e, g) \cdot (x_0, x_1)^h_q]_G.$$

for any $q \in Q$ such that $\pi q = x_0$.

Proof. To show that $\alpha_{\mathcal{A}_d}$ is well-defined, note that for any $g \in G$, we have,

$$(\pi g q_0, \pi g q_1) = (\pi q_0, \pi q_1),$$

and also,

$$\begin{aligned}[g q_0, \mathcal{A}_d(g q_0, g q_1)]_G &= [g q_0, g \mathcal{A}_d(q_0, q_1) g^{-1}]_G \\ &= [q_0, \mathcal{A}_d(q_0, q_1)]_G\end{aligned}$$

Then we see that,

$$\alpha_{\mathcal{A}_d}([g q_0, g q_1]_G) = \alpha_{\mathcal{A}_d}([q_0, q_1]_G).$$

To show that $\alpha^{-1}_{\mathcal{A}_d}$ is well-defined, note that for any $k \in G$,

$$(x_0, x_1)^h_{kq} = k \cdot (x_0, x_1)^h_q,$$

and that,

$$\begin{aligned}\alpha^{-1}_{\mathcal{A}_d}((x_0, x_1) \oplus [kq, kgk^{-1}]_G) &= [(e, kgk^{-1}) \cdot (x_0, x_1)^h_{kq}]_G \\ &= [(e, kgk^{-1}) \cdot k \cdot (x_0, x_1)^h_q]_G \\ &= [(ek, kgk^{-1}k) \cdot (x_0, x_1)^h_q]_G \\ &= [(ke, kg) \cdot (x_0, x_1)^h_q]_G \\ &= [k \cdot (e, g) \cdot (x_0, x_1)^h_q]_G \\ &= [(e, g) \cdot (x_0, x_1)^h_q]_G \\ &= \alpha^{-1}_{\mathcal{A}_d}((x_0, x_1) \oplus [q, g]_G).\end{aligned}$$

□

3.2.1. *Local Representation of the Discrete Connection Form.* Since the discrete connection form can be thought of as comparing group fibre quantities at different base points, we have the natural relationship,

$$\mathcal{A}_d(g q_0, h q_1) = h \mathcal{A}_d(q_0, q_1) g^{-1}.$$

In a local trivialization, this corresponds to,

$$\mathcal{A}_d(x_0, g_0, x_1, g_1) = g_1 \mathcal{A}_d(x_0, e, x_1, e) g_0^{-1}.$$

We define,

$$A(x_0, x_1) = \mathcal{A}_d(x_0, e, x_1, e),$$

which yields the local representation of the discrete connection form,

$$\mathcal{A}_d(x_0, g_0, x_1, g_1) = g_1 A(x_0, x_1) g_0^{-1}.$$

3.2.2. *Splittings from Discrete Connection Forms.* The G-equivariant G-valued discrete connection 1-form $\mathcal{A}_d : Q \times Q \to G$ induces a splitting of the discrete Atiyah sequence,

$$0 \longrightarrow \tilde{G} \longrightarrow (Q \times Q)/G \longrightarrow S \times S \longrightarrow 0$$

where the map $\varphi : (Q \times Q)/G \to \tilde{G}$ is given by

$$\varphi([q_0, q_1]_G) = [q_0, \mathcal{A}_d(q_0, q_1)]_G.$$

This expression is well-defined, as the following computation shows,

$$\begin{aligned} \varphi([gq_0, gq_1]_G) &= [gq_0, \mathcal{A}_d(gq_0, gq_1)]_G \\ &= [gq_0, g\mathcal{A}_d(q_0, q_1) g^{-1}]_G \\ &= \varphi([q_0, q_1]_G). \end{aligned}$$

3.2.3. *Discrete Connections Forms from Splittings.* Given $[q_0, q_1]_G \in (Q \times Q)/G$, we obtain from the splitting of the discrete Atiyah sequence an element, $[q, g]_G \in \tilde{G}$. Viewing $[q, g]_G$ as a subset of $Q \times G$, consider the unique $\tilde{g}$ such that $(q_0, \tilde{g}) \in [q, g]_G \subset Q \times G$. Then, we define,

$$\mathcal{A}_d(x_0, e, x_1, g_0^{-1} g_1) = \tilde{g}.$$

We extend this definition to the whole of $Q \times Q$ by equivariance,

$$\mathcal{A}_d(x_0, g_0, x_1, g_1) = g_0 \tilde{g} g_0^{-1}.$$

Proof. To show that the $\mathcal{A}_d$ satisfies the axioms of a discrete connection form, we first note that equivariance follows from the construction.

Since we have a splitting, it follows that $\varphi([q, gq]_G) = [q, g]_G$, as φ composed with the map from $\tilde{G}$ to $(Q \times Q)/G$ is the identity on $\tilde{G}$. Using a local trivialization, we have,

$$\begin{aligned} [q_0, g]_G &= \varphi([q_0, gq_0]_G) \\ &= \varphi([(x_0, e), (x_0, g_0^{-1} g g_0)]_G) \\ &= [(x_0, e), \tilde{g}]_G \\ &= [(x_0, g_0), g_0 \tilde{g} g_0^{-1}]_G. \end{aligned}$$

Then, by definition,

$$\mathcal{A}_d((x_0, e), (x_0, g_0^{-1} g g_0)) = \tilde{g},$$

and furthermore, $g = g_0 \tilde{g} g_0^{-1}$. From this, we conclude that,

$$\begin{aligned} \mathcal{A}_d(q_0, gq_0) &= \mathcal{A}_d((x_0, g_0), (x_0, gg_0)) \\ &= g_0 \mathcal{A}_d((x_0, e), (x_0, g_0^{-1} g g_0)) g_0^{-1} \\ &= g_0 \tilde{g} g_0^{-1} \\ &= g. \end{aligned}$$

Therefore, we have that $\mathcal{A}_d(q_0, gq_0) = g$, which together with equivariance implies that $\mathcal{A}_d$ is a discrete connection form. □

3.2.4. *Splittings and Discrete Horizontal Lifts.* From the equivalent conditions for a splitting of a short exact sequence, we see that the splitting induced by a discrete connection yields a **discrete horizontal lift** $(\cdot,\cdot)^h : S \times S \to (Q \times Q)/G$. This follows from the application of the five-lemma to the following diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{G} & \underset{(\pi_1,\mathcal{A}_d)}{\overset{(q,gq)}{\rightleftarrows}} & (Q\times Q)/G & \underset{(\cdot,\cdot)^h}{\overset{(\pi,\pi)}{\rightleftarrows}} & S\times S & \longrightarrow & 0 \\
 & & \Big\| {\scriptstyle 1_{\tilde{G}}} & & \Big\downarrow {\scriptstyle \alpha_{\mathcal{A}_d}} & & \Big\| {\scriptstyle 1_{S\times S}} & & \\
0 & \longrightarrow & \tilde{G} & \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} & \tilde{G}\oplus(S\times S) & \underset{i_2}{\overset{\pi_2}{\rightleftarrows}} & S\times S & \longrightarrow & 0
\end{array}$$

Here, we see how the identification between $(Q\times Q)/G$ and $(S\times S)\oplus\tilde{G}$ are naturally related to the discrete connection and the discrete horizontal lift. The relation between the two representations of the splitting of the Discrete Atiyah sequence can be made more explicit, as we will describe later in this subsection.

We denote by $(x_0, x_1)^h_{q_0}$ the unique element in $(x_0, x_1)^h$ thought of as a subset of $Q\times Q$, such that the first component is q_0. This is the horizontal lift of the point $(x_0, x_1) \in S \times S$ where the base point in $Q \hookrightarrow Q\times Q$ is specified.

More explicitly, given a splitting of the Discrete Atiyah sequence of the form, $(\cdot,\cdot)^h : S\times S \to (Q\times Q)/G$, we can construct the map $(\pi_1, \mathcal{A}_d) : (Q \times Q)/G \to \tilde{G}$. Given $[q_0, q_1]_G \in (Q \times Q)/G$, we represent this in a local trivialization as, $[q_0, q_1]_G = [x_0, e, x_1, g_0^{-1}g_1]_G$. We have that $(x_0, x_1)^h_{(x_0,e)} = (x_0, e, x_1, h)$. Then, we define,

$$(\pi_1, \mathcal{A}_d)([q_0, q_1]_G) = [(x_0, e), g_0^{-1}g_1h^{-1}]_G.$$

Conversely, given a splitting of the Discrete Atiyah sequence of the form, $(\pi_1, \mathcal{A}_d) : (Q \times Q)/G \to \tilde{G}$, we can construct the map $(\cdot,\cdot)^h : S \times S \to (Q \times Q)/G$. Given $(x_0, x_1) \in S \times S$, we consider $(\pi_1, \mathcal{A}_d)(x_0, e, x_1, e)$ and view it as a subset of $Q\times G$. There is a unique element $(x_0, e, h) \in (\pi_1, \mathcal{A}_d)(x_0, e, x_1, e) \subset Q \times G$. Then, we define,

$$(x_0, x_1)^h = [(x_0, e), (x_1, h^{-1})]_G.$$

Definition 3.5. *The* ***discrete horizontal lift****, $(\cdot,\cdot)^h_{q_0} : S \times S \to Q \times Q$, and the* ***discrete connection****, $\mathcal{A}_d : Q \times Q \to G$, satisfies the following relationship,*

$$(e, \mathcal{A}_d(q_0, q_1)) \cdot (\pi(q_0, q_1))^h_{q_0} = (q_0, q_1),$$

for any $(q_0, q_1) \in Q \times Q$.

By G-equivariance, a choice of the discrete horizontal lift automatically induces a discrete connection, and the converse is also true.

Remark 3.5. *This relationship is the discrete analogue of splitting the tangent vector into a horizontal and vertical part. The first term is a vertical element, and the second term is a horizontal element.*

We obtain the vertical part by taking the equivalent of $i_1 \circ \pi_1$, and the horizontal part by taking the equivalent of $i_2 \circ \pi_2$, in the discrete Atiyah sequence.

3.2.5. *Discrete Horizontal and Vertical Spaces.* Consider the following split exact sequence,

$$0 \longrightarrow A_1 \underset{k}{\overset{f}{\rightleftarrows}} B \underset{h}{\overset{g}{\rightleftarrows}} A_2 \longrightarrow 0$$

We can decompose any element in B into a A_1 and A_2 term by considering the following isomorphism,

$$B \cong f \circ k(B) \oplus h \circ g(B).$$

Similarly, in the discrete Atiyah sequence, we can decompose an element of $(Q\times Q)/G$ into a horizontal and vertical piece by performing the analogous construction on the split exact sequence,

$$0 \longrightarrow \tilde{G} \underset{(\pi_1,\mathcal{A}_d)}{\overset{(q,gq)}{\rightleftarrows}} (Q\times Q)/G \underset{(\cdot,\cdot)^h}{\overset{(\pi,\pi)}{\rightleftarrows}} S\times S \longrightarrow 0$$

This allows us to define horizontal and vertical spaces associated with the pair groupoid $Q\times Q$.

Definition 3.6. *The* ***horizontal space*** *is given by,*

$$\mathrm{Hor}_q = \{(q,q')\in Q\times Q \mid \mathcal{A}_d(q,q')=e\}.$$

This is the discrete analogue of the statement $\mathrm{Hor}_q=\{v_q\in TQ\mid \mathcal{A}(v_q)=0\}$.

Definition 3.7. *The* ***vertical space*** *is given by,*

$$\begin{aligned}\mathrm{Ver}_q &= \{(q,q')\in Q\times Q \mid (\pi,\pi)(q,q')=e_{S\times S}\}\\ &= \{i_q(g)\mid g\in G\}.\end{aligned}$$

This is the discrete analogue of the statement $\mathrm{Ver}_q=\{v_q\in TQ\mid \pi_*(v_q)=0\}=\{\xi_Q\mid \xi\in\mathfrak{g}\}$.

In particular, we can decompose an element of $Q\times Q$ into a horizontal and vertical component.

Definition 3.8. *The* ***horizontal component*** *of* $(q_0,q_1)\in Q\times Q$ *is given by,*

$$\mathrm{hor}(q_0,q_1)=((\cdot,\cdot)^h\circ(\pi,\pi))(q_0,q_1)=(\pi q_0,\pi q_1)^h_{q_0}.$$

Definition 3.9. *The* ***vertical component*** *of* $(q_0,q_1)\in Q\times Q$ *is given by,*

$$\mathrm{ver}(q_0,q_1)=((q,gq)\circ(\pi_1,\mathcal{A}_d))(q_0,q_1)=(q_0,\mathcal{A}_d(q_0,q_1)q_0).$$

Definition 3.10. *Given a point* $q\in Q$*, we can construct the map* $i_q:G\to Q\times Q$*, which is given by,*

$$i_q(g)=(q,gq).$$

Given this definition, we can rewrite the third axiom for a discrete connection as,

$$\mathcal{A}_d\circ i_q=1_G.$$

Furthermore, the vertical component can be written as,

$$\mathrm{ver}(q_0,q_1)=i_{q_0}(\mathcal{A}_d(q_0,q_1)).$$

Lemma 3.2. i_q *is a homomorphism.*

Proof. We compute,

$$\begin{aligned} i_q(g)\cdot i_q(h) &= (q,gq)\cdot(q,hq)\\ &= (e, I_{\mathcal{A}_d(q,gq)}\mathcal{A}_d(q,hq))(q,gq)\\ &= (e,I_g h)(q,gq)\\ &= (q,ghq)\\ &= i_q(gh).\end{aligned}$$

Therefore, i_q is a homomorphism. □

The action of G and the homomorphism i_{q_0} naturally induces the composition of an arbitrary element $(q_0,q_1)\in Q\times Q$ with a vertical element.

Definition 3.11. *The composition of an arbitrary element* $(q_0,q_1)\in Q\times Q$ *with a vertical element is given by,*

$$(q_0,q_1)\cdot i_{q_0}(g)=(e,g)(q_0,q_1)=(q_0,gq_1).$$

Lemma 3.3. *The horizontal component can be expressed as,*

$$\operatorname{hor}(q_0,q_1) = (q_0,q_1)\cdot i_{q_0}((\mathcal{A}_d(q_0,q_1))^{-1}).$$

Proof.

$$\begin{aligned}(q_0,q_1)\cdot i_{q_0}(\mathcal{A}_d(q_0,q_1)^{-1}) &= (q_0,q_1)\cdot(q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_0)\\ &= (e,(\mathcal{A}_d(q_0,q_1))^{-1})(q_0,q_1)\\ &= (q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_1).\end{aligned}$$

Clearly, $(\pi,\pi)(q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_1) = (\pi,\pi)(q_0,q_1)$. Furthermore,

$$\mathcal{A}_d(q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_1) = (\mathcal{A}_d(q_0,q_1))^{-1}\mathcal{A}_d(q_0,q_1) = e.$$

Therefore, by definition, $(q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_1) = \operatorname{hor}(q_0,q_1)$. □

Lemma 3.4. *The horizontal and vertical operators satisfy the following identity,*

$$\operatorname{hor}(q_0,q_1)\cdot\operatorname{ver}(q_0,q_1) = (q_0,q_1).$$

Proof.

$$\begin{aligned}\operatorname{hor}(q_0,q_1)\cdot\operatorname{ver}(q_0,q_1) &= ((q_0,q_1)\cdot i_{q_0}((\mathcal{A}_d(q_0,q_1))^{-1}))\cdot i_{q_0}(\mathcal{A}_d(q_0,q_1))\\ &= (e,\mathcal{A}_d(q_0,q_1))(e,(\mathcal{A}_d(q_0,q_1))^{-1})(q_0,q_1)\\ &= (e,\mathcal{A}_d(q_0,q_1))(q_0,(\mathcal{A}_d(q_0,q_1))^{-1}q_1)\\ &= (q_0,\mathcal{A}_d(q_0,q_1)(\mathcal{A}_d(q_0,q_1))^{-1}q_1)\\ &= (q_0,q_1),\end{aligned}$$

as desired. □

3.2.6. *Discrete Horizontal Lifts from Discrete Connection Forms.* We wish to construct a discrete horizontal lift $(\cdot,\cdot)^h : S\times S\to(Q\times Q)/G$, given a discrete connection $\mathcal{A}_d : Q\times Q\to G$. The discrete horizontal lift is defined by,

$$(x_0,x_1)^h = [\pi^{-1}(x_0,x_1)\cap\mathcal{A}_d^{-1}(e)]_G.$$

Furthermore, the discrete horizontal lift satisfies the following relation,

$$(e,\mathcal{A}_d(q_0,q_1))\cdot(\pi(q_0,q_1))^h_{q_0} = (q_0,q_1).$$

The horizontal lift can be expressed in a local trivialization, where $q_0 = (x_0,g_0)$, using the local expression for the discrete connection,

$$(x_0,x_1)^h_{q_0} = (x_0,g_0,x_1,g_0(A(x_0,x_1))^{-1}).$$

Proof. We will show that this operation is a well-defined function to the quotient space. Let $\mathcal{A}_d : Q\times Q\to G$ be a chosen principal discrete connection. Using a local trivialization, the connection can be described by,

$$\mathcal{A}_d(x_0,g_0,x_1,g_1) = g_1\cdot A(x_0,x_1)\cdot g_0^{-1}$$

where $A : S\times S\to G$ is the restriction of $\mathcal{A}_d$ to $S\times S$. $S\times S$ is identified with $S\times\{e\}\times S\times\{e\}$ which is in turn identified with $Q\times Q$. A simple computation shows that this local representation is G-equivariant, as desired.

Using the local representation of the discrete connection in the local trivialization, we have,

$$\begin{aligned}&\mathcal{A}_d^{-1}(e)\cap\pi^{-1}(x_0,x_1)\\ &\qquad= \{(\tilde{x}_0,g,\tilde{x}_1,g\cdot(A(\tilde{x}_0,\tilde{x}_1))^{-1}) \mid \tilde{x}_0,\tilde{x}_1\in S, g\in G\}\\ &\qquad\qquad\cap\{(x_0,h_0,x_1,h_1)\mid h_0,h_1\in G\}\\ &\qquad= \{(x_0,g,x_1,g\cdot(A(x_0,x_1))^{-1})\mid h\in G\}\end{aligned}$$

$$= G \cdot (x_0, e, x_1, (A(x_0, x_1))^{-1}),$$

which is a well-defined element of $(Q \times Q)/G$. Since this is true in a local trivialization, and both the discrete connection and projection operators are globally defined, this inverse coset is globally well-defined as an element of $(Q \times Q)/G$.

In particular, the computation above allows us to obtain a local expression for the discrete horizontal lift in terms of the local representation of the discrete connection. That is,

$$\begin{aligned}(x_0, x_1)^h_{(x_0,e)} &= (x_0, e, x_1, (A(x_0, x_1))^{-1}),\\ (x_0, x_1)^h &= [(x_0, e, x_1, (A(x_0, x_1))^{-1})]_G.\end{aligned}$$

By the properties of the discrete horizontal lift, this extends to $\pi^{-1}(x_0, x_1) \subset Q \times Q$,

$$\begin{aligned}(x_0, x_1)^h_{(x_0,g)} &= (x_0, x_1)^h_{g(x_0,e)}\\ &= g \cdot (x_0, x_1)^h_{(x_0,e)}\\ &= g(x_0, e, x_1, (A(x_0, x_1))^{-1})\\ &= (x_0, g, x_1, g(A(x_0, x_1))^{-1}).\end{aligned}$$

To prove the second claim, we have in the local trivialization of $Q \times Q$, $(q_0, q_1) = (x_0, g_0, x_1, g_1)$. Then, by the result above,

$$(\pi(q_0, q_1))^h_{q_0} = (\pi(q_0, q_1))^h_{(x_0,g_0)} = (x_0, g_0, x_1, g_0(A(x_0, x_1))^{-1}).$$

Also, by the local representation of the discrete connection,

$$\mathcal{A}_d(q_0, q_1) = g_1 A(x_0, x_1) g_0^{-1}.$$

Therefore,

$$\begin{aligned}(e, \mathcal{A}_d(q_0, q_1)) \cdot (\pi(q_0, q_1))^h_{q_0} &= (e, g_1 A(x_0, x_1) g_0^{-1}) \cdot (x_0, g_0, x_1, g_0(A(x_0, x_1))^{-1})\\ &= (x_0, g_0, x_1, (g_1 A(x_0, x_1) g_0^{-1})(g_0(A(x_0, x_1))^{-1}))\\ &= (x_0, g_0, x_1, g_1)\\ &= (q_0, q_1),\end{aligned}$$

as claimed. □

3.2.7. *Discrete Connection Forms from Discrete Horizontal Lifts.* Given a horizontal lift $(\cdot, \cdot)^h : S \times S \to (Q \times Q)/G$, we wish to construct a discrete connection from $\mathcal{A}_d : Q \times Q \to G$. Then, the discrete connection is defined by the following relation,

$$(e, \mathcal{A}_d(q_0, q_1)) \cdot (\pi(q_0, q_1))^h_{q_0} = (q_0, q_1).$$

Proof. To show that this construction is well-defined, we note that $\pi_1(q_0, q_1) = \pi_1(\pi(q_0, q_1))^h_{q_0}$ by the construction of $(\cdot, \cdot)^h_{q_0}$ from $(\cdot, \cdot)^h : S \times S \to (Q \times Q)/G$. Furthermore, $\pi_2(q_0, q_1)$ and $\pi_2(\pi(q_0, q_1))^h_{q_0}$ are in the same fibre of the principal bundle $\pi : Q \to Q/G$ and are therefore related by a unique element $g \in G$. Since this element is unique, $\mathcal{A}_d(q_0, q_1)$ is uniquely defined by the relation. □

3.2.8. *Continuous Connections from Discrete Connections.* Given a discrete G-valued connection 1-form $\mathcal{A}_d : Q \times Q \to G$, we associate with it a continuous $\mathfrak{g}$-valued connection 1-form $\mathcal{A} : TQ \to \mathfrak{g}$ by the following construction,

$$\mathcal{A}([q(\cdot)]) = [\mathcal{A}_d(q(0), q(\cdot))],$$

where $[\cdot]$ denotes the equivalence class of curves associated with a tangent vector.

More explicitly, given $v_q \in TQ$, we consider an associated curve $q : [0,1] \to Q$, and construct the curve $g : [0,1] \to G$, given by,

$$g(t) = \mathcal{A}_d(q(0), q(t)).$$

Then,

$$\mathcal{A}(v_q) = \frac{d}{dt}\bigg|_{t=0} g(t).$$

Definition 3.12. *We define a* ***vertical variation*** *of the point* $(q_0, q_1) \in Q \times Q$ *which projects to* $(x_0, x_1) \in S \times S$*. Given a curve* $q_1^\epsilon : [0,1] \to Q$*, such that* $q_1^\epsilon(0) = q_1$*, define* $g \equiv \mathcal{A}_d(q_0, q_1)$*, and consider the mapping,*

$$q_1^\epsilon \mapsto g \cdot (\pi(q_0, q_1^\epsilon))_{q_0}^h$$

Then,

$$\operatorname{ver} \delta q = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} g \cdot (\pi(q_0, q_1^\epsilon))_{q_0}^h.$$

Example 3.2. *It is illustrative to consider the notion of a discrete connection and the isomorphism in the degenerate case when* $Q = G$*, which is the context of Discrete Euler–Poincaré reduction. Here, the isomorphism is between* $(G \times G)/G$ *and* G*, and the connection* $\mathcal{A}_d : G \times G \to G$ *is given by,*

$$\mathcal{A}_d(g_0, g_1) = g_1 \cdot g_0^{-1}.$$

3.3. **Extending the Composition on the Pair Groupoid.** Recall that we can define the composition of an element $(q_0, q_1) \in Q \times Q$ with a vertical element (q_0, gq_0) by,

$$(q_0, q_1) \cdot (q_0, gq_0) = (q_0, gq_1).$$

The choice of a discrete connection allows us to further extend the composition, in a manner that is relevant in describing the curvature of a discrete connection.

The decomposition of an element of $Q \times Q$ into a horizontal and vertical piece naturally suggests a generalization of the composition operation on $Q \times Q$ (viewed as a pair groupoid), by using the discrete connection and the principal bundle structure of Q.

We wish to define a composition on $Q \times Q$ such that the composition of $(\tilde{q}_0, \tilde{q}_1) \cdot (q_0, q_1)$ is defined whenever $\pi q_1 = \pi \tilde{q}_0$. To be consistent with the pure group case, we require that our composition satisfy,

$$\mathcal{A}_d((\tilde{q}_0, \tilde{q}_1) \cdot (q_0, q_1)) = \mathcal{A}_d(\tilde{q}_0, \tilde{q}_1) \cdot \mathcal{A}_d(q_0, q_1).$$

This says that the connection remains a homomorphism from $Q \times Q$ to G when the composition is augmented.

Furthermore, we require that the composition projects to the pair groupoid composition on $S \times S$, which is to say,

$$(\pi, \pi)((\tilde{q}_0, \tilde{q}_1) \cdot (q_0, q_1)) = \pi(\tilde{q}_0, \tilde{q}_1) \cdot \pi(q_0, q_1) = (\pi q_0, \pi \tilde{q}_1).$$

These two conditions imply that in a local trivialization, we have,

$$(\tilde{q}_0, \tilde{q}_1) \cdot (q_0, q_1) = (x_0, g_0, \tilde{x}_1, h),$$

Where h satisfies the identity,

$$hA(x_0, \tilde{x}_1) g_0^{-1} = \mathcal{A}_d((\tilde{q}_0, \tilde{q}_1) \cdot (q_0, q_1)) = \mathcal{A}_d(\tilde{q}_0, \tilde{q}_1) \cdot \mathcal{A}_d(q_0, q_1).$$

Since $\mathcal{A}_d$ is a homomorphism, we have,

$$\begin{aligned} A(x_0, \tilde{x}_1) &= A(\tilde{x}_0, \tilde{x}_1) \cdot A(x_0, x_1) \\ &= (\tilde{g}_1^{-1} \mathcal{A}_d(\tilde{q}_0, \tilde{q}_1) \tilde{g}_0)(g_1^{-1} \mathcal{A}_d(q_0, q_1) g_0) \\ &= \tilde{g}_1^{-1} \mathcal{A}_d(\tilde{q}_0, \tilde{q}_1)(\tilde{g}_0 g_1^{-1}) \mathcal{A}_d(q_0, q_1) g_0 \\ &= \tilde{g}_1^{-1} \mathcal{A}_d(\tilde{q}_0, \tilde{q}_1) \mathcal{A}_d(q_1, \tilde{q}_0) \mathcal{A}_d(q_0, q_1) g_0. \end{aligned}$$

From this, we conclude that,

$$\begin{aligned} h &= \mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)\mathcal{A}_d(q_0,q_1)g_0g_0^{-1}(\mathcal{A}_d(q_0,q_1))^{-1}(\mathcal{A}_d(q_1,\tilde{q}_0))^{-1}(\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1))^{-1}\tilde{g}_1 \\ &= (I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}\mathcal{A}_d(\tilde{q}_0,q_1))\tilde{g}_1. \end{aligned}$$

We can then write the composition in a more intrinsic form,

$$(\tilde{q}_0,\tilde{q}_1)\cdot(q_0,q_1) = (e, I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}\mathcal{A}_d(\tilde{q}_0,q_1))\cdot(q_0,\tilde{q}_1).$$

Example 3.3. *Consider the pure group case, in which case, our formula reduces to,*

$$\begin{aligned} (\tilde{g}_0,\tilde{g}_1)\cdot(g_0,g_1) &= (g_0, I_{\mathcal{A}_d(\tilde{g}_0,\tilde{g}_1)}\mathcal{A}_d(\tilde{g}_0,q_1)\tilde{g}_1) \\ &= (g_0,(\tilde{g}_1\tilde{g}_0^{-1})(g_1\tilde{g}_0^{-1})(\tilde{g}_0\tilde{g}_1^{-1})\tilde{g}_1) \\ &= (g_0,\tilde{g}_1\tilde{g}_0^{-1}g_1), \end{aligned}$$

which is consistent with the composition on G when it is identified with $(G\times G)/G$ under the Discrete Euler–Poincaré connection $(g_0,g_1)\mapsto g_1g_0^{-1}$ since,

$$\begin{aligned} (\tilde{g}_1\tilde{g}_0^{-1})\cdot(g_1g_0^{-1}) &= [\tilde{g}_0,\tilde{g}_1]_G\cdot[g_0,g_1]_G \\ &= [g_0,\mathcal{A}_d(\tilde{g}_0,\tilde{g}_1)g_1]_G \\ &= [g_0,\tilde{g}_1\tilde{g}_0^{-1}g_1]_G \\ &= \tilde{g}_1\tilde{g}_0^{-1}g_1g_0^{-1}. \end{aligned}$$

Example 3.4. *Furthermore, if $\tilde{q}_0 = q_1$, then, $\mathcal{A}_d(\tilde{q}_0,q_1) = e = I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}e = I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}\mathcal{A}_d(\tilde{q}_0,q_1)$, and therefore,*

$$\begin{aligned} (\tilde{q}_0,\tilde{q}_1)\cdot(q_0,q_1) &= (e, I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}\mathcal{A}_d(\tilde{q}_0,q_1))\cdot(q_0,\tilde{q}_1) \\ &= (e,e)\cdot(q_0,\tilde{q}_1) \\ &= (q_0,\tilde{q}_1), \end{aligned}$$

which is the usual composition on the pair groupoid $Q\times Q$. In particular, when the group is trivial, the composition reduces to the pair groupoid composition.

Lemma 3.5. *The composition $\cdot : (Q\times Q)\times(Q\times Q)\to(Q\times Q)$ is G-equivariant, that is,*

$$(g\tilde{q}_0,g\tilde{q}_1)\cdot(gq_0,gq_1) = g\cdot((\tilde{q}_0,\tilde{q}_1)\cdot(q_0,q_1)).$$

Furthermore, the composition induces a well-defined quotient composition $\cdot : ((Q\times Q)\times(Q\times Q))/G \to (Q\times Q)/G$.

Proof. Given $g\in G$, we consider,

$$\begin{aligned} &(g\tilde{q}_0,g\tilde{q}_1)\cdot(gq_0,gq_1) \\ &\quad= (e, I_{\mathcal{A}_d(g\tilde{q}_0,g\tilde{q}_1)}\mathcal{A}_d(g\tilde{q}_0,gq_1))\cdot(gq_0,g\tilde{q}_1) \\ &\quad= (e, I_{g\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)g^{-1}}g\mathcal{A}_d(\tilde{q}_0,q_1)g^{-1})\cdot(gq_0,g\tilde{q}_1) \\ &\quad= (e,(g\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)g^{-1})(g\mathcal{A}_d(\tilde{q}_0,q_1)g^{-1})(g\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)g^{-1})^{-1})\cdot(gq_0,g\tilde{q}_1) \\ &\quad= (gq_0, g\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)\mathcal{A}_d(\tilde{q}_0,q_1)(\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1))^{-1}\tilde{q}_1) \\ &\quad= g\cdot(e, I_{\mathcal{A}_d(\tilde{q}_0,\tilde{q}_1)}\mathcal{A}_d(\tilde{q}_0,q_1))\cdot(q_0,\tilde{q}_1) \\ &\quad= g\cdot((\tilde{q}_0,\tilde{q}_1)\cdot(q_0,q_1)), \end{aligned}$$

where we used the equivariance of the discrete connection. It follows that the composition is equivariant. Furthermore,

$$[(g\tilde{q}_0,g\tilde{q}_1)\cdot(gq_0,gq_1)]_G = [(\tilde{q}_0,\tilde{q}_1)\cdot(q_0,q_1)]_G,$$

which means that $\cdot : ((Q\times Q)\times(Q\times Q))/G\to(Q\times Q)/G$ is well-defined. □

Corollary 3.6. *The composition of n-terms is G-equivariant. That is to say,*

$$\begin{aligned}(gq_0^n, gq_1^n)\cdot(gq_0^{n-1}, gq_1^{n-1})\cdot\ldots\cdot(gq_0^2, gq_1^2)\cdot(gq_0^1, gq_1^1)\\ = g\cdot((q_0^n, q_1^n)\cdot(q_0^{n-1}, q_1^{n-1})\cdot\ldots\cdot(q_0^2, q_1^2)\cdot(q_0^1, q_1^1)).\end{aligned}$$

Proof. The result follows by induction on the previous lemma. □

We find in general that the extended composition that we have constructed on the pair groupoid is not associative, and the associativity defect about a loop in the shape space is related to the curvature, and may also yield the discrete analogue of the expression giving the geometric phase in terms of a loop integral in shape space of the curvature of the connection.

Lemma 3.7. *When the connection $\mathcal{A}_d$ is flat, the composition $\cdot : (Q\times Q)\times(Q\times Q)\to(Q\times Q)$ is associative. That is,*

$$((q_0^2, q_1^2)\cdot(q_0^1, q_1^1))\cdot(q_0^0, q_1^0) = (q_0^2, q_1^2)\cdot((q_0^1, q_1^1)\cdot(q_0^0, q_1^0)).$$

Proof. We will repeatedly use that for a flat connection, $\mathcal{A}_d$ is a homomorphism from $Q\times Q$ to G, where the composition on $Q\times Q$ is the extended version we constructed in this subsection.

We first compute,

$$\begin{aligned}(q_0^0, \hat{q}) &\equiv ((q_0^1, q_1^1)\cdot(q_0^0, q_1^0))\\ &= (q_0^0, \mathcal{A}_d(q_0^1, q_1^1)\mathcal{A}_d(q_0^1, q_1^0)\mathcal{A}_d(q_1^1, q_0^1)q_1^1).\end{aligned}$$

Then, the right-hand side is given by,

$$\begin{aligned}&(q_0^2, q_1^2)\cdot((q_0^1, q_1^1)\cdot(q_0^0, q_1^0))\\ &\quad= (q_0^2, q_1^2)\cdot(q_0^0, \hat{q})\\ &\quad= (q_0^0, \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^2, \hat{q})\mathcal{A}_d(q_1^2, q_0^2)q_1^2)\\ &\quad= (q_0^0, \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^1, q_1^1)\mathcal{A}_d(q_0^0, q_1^0)\mathcal{A}_d(q_0^2, q_0^0)\mathcal{A}_d(q_1^2, q_0^2)q_1^2)\\ &\quad= (q_0^0, \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^1, q_1^1)\mathcal{A}_d(q_1^2, q_1^0)q_1^2).\end{aligned}$$

Similarly,

$$\begin{aligned}(q_0^1, \tilde{q}) &\equiv ((q_0^2, q_1^2)\cdot(q_0^1, q_1^1))\\ &= (q_0^1, \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^2, q_1^1)\mathcal{A}_d(q_1^2, q_0^2)q_1^2).\end{aligned}$$

Then,

$$\begin{aligned}((q_0^2, q_1^2)\cdot(q_0^1, q_1^1))\cdot(q_0^0, q_1^0) &= (q_0^1, \tilde{q})\cdot(q_0^0, q_1^0)\\ &= (q_0^0, \mathcal{A}_d(q_0^1, \tilde{q})\mathcal{A}_d(q_0^1, q_1^0)\mathcal{A}_d(\tilde{q}, q_0^1)\tilde{q}).\end{aligned}$$

We note that,

$$\begin{aligned}\mathcal{A}_d(q_0^1, \tilde{q}) &= \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^2, q_1^1)\mathcal{A}_d(q_1^2, q_0^2)\mathcal{A}_d(q_0^1, q_1^2)\\ &= \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^1, q_1^1),\end{aligned}$$

and,

$$\begin{aligned}\mathcal{A}_d(\tilde{q}, q_0^1)\tilde{q} &= \mathcal{A}_d(q_1^1, q_0^1)\mathcal{A}_d(q_1^2, q_0^2)\mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^2, q_1^1)\mathcal{A}_d(q_1^2, q_0^2)q_1^2\\ &= \mathcal{A}_d(q_1^2, q_0^1)q_1^2.\end{aligned}$$

Then, left-hand side is given by,

$$\begin{aligned}&((q_0^2, q_1^2)\cdot(q_0^1, q_1^1))\cdot(q_0^0, q_1^0)\\ &\quad= (q_0^0, \mathcal{A}_d(q_0^1, \tilde{q})\cdot\mathcal{A}_d(q_0^1, q_1^0)\cdot(\mathcal{A}_d(\tilde{q}, q_0^1)\tilde{q}))\end{aligned}$$

$$= (q_0^0, (\mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^1, q_1^1)) \cdot \mathcal{A}_d(q_0^1, q_1^0) \cdot (\mathcal{A}_d(q_1^2, q_0^1)q_1^2))$$
$$= (q_0^0, \mathcal{A}_d(q_0^2, q_1^2)\mathcal{A}_d(q_0^1, q_1^1)\mathcal{A}_d(q_1^2, q_1^0)q_1^2).$$

Therefore,

$$((q_0^2, q_1^2) \cdot (q_0^1, q_1^1)) \cdot (q_0^0, q_1^0) = (q_0^2, q_1^2) \cdot ((q_0^1, q_1^1) \cdot (q_0^0, q_1^0)),$$

and the composition is associative. □

3.4. **Connection-like Structures on Higher Order Tangent Bundles.** Given a continuous connection, we can construct connection-like structures on higher order tangent bundles. This construction is described in detail in Lemma 3.2.1 of Cendra, Marsden, Ratiu [11]. In particular, given a connection $\mathcal{A} : TQ \to \mathfrak{g}$, we obtain a well-defined map, $\mathcal{A}^k : T^{(k)}Q \to k\mathfrak{g}$.

As we will see later, these connection-like structures on higher order tangent bundles will provide an intrinsic method of characterizing the order of approximation of a continuous connection by a discrete connection.

We will describe the discrete analogue of this construction. To begin, the discrete analogue of the kth order tangent bundle $T^{(k)}Q$ is $k+1$ copies of Q, namely Q^{k+1}. Intermediate spaces between $T^{(k)}Q$ and Q^{k+1} arise in the general theory of multispaces, proposed by Olver.

The discrete analogue of tangent lifts and their higher order analogues are obtained by componentwise application of the map, since a tangent lift of a map is computed by applying the map to a representative curve, and taking its equivalence class. Therefore, given a map $f : M \to N$, we have the naturally induced map,

$$T^{(k)}f : M^{k+1} \to N^{k+1} \quad \text{given by} \quad T^{(k)}f(m_0, \ldots, m_k) = (f(m_0), \ldots, f(m_k)).$$

And in particular, the group action is lifted to the diagonal group action on the product space.

The discrete connection can be extended to Q^{k+1} in the natural way, $\mathcal{A}_d^k : Q^{k+1} \to \oplus_{l=0}^{k-1} G \equiv kG$,

$$\mathcal{A}_d^k(q_0, \ldots, q_k) = \oplus_{l=0}^{k-1} \mathcal{A}_d(q_l, q_{l+1}).$$

Similarly, we can define the map from Q^{k+1} to the Whitney sum of k copies of the conjugate bundle $\tilde{G}$ by,

$$Q^{k+1} \to k\tilde{G} \quad \text{by} \quad (q_0, \ldots, q_k) \mapsto \oplus_{l=0}^{k-1} [q_0, \mathcal{A}_d(q_l, q_{l+1})]_G.$$

In a natural way, we have the following lemma.

Lemma 3.8. *The map*

$$\alpha_{\mathcal{A}_d^k} : Q^{k+1} \to (Q/G)^{k+1} \times_{Q/G} k\tilde{G}$$

defined by

$$\alpha_{\mathcal{A}_d^k}(q_0, \ldots, q_k) = (\pi q_0, \ldots, \pi q_k) \times_{Q/G} \oplus_{l=0}^{k-1} [q_0, \mathcal{A}_d(q_l, q_{l+1})]_G.$$

is a well defined bundle isomorphism. The inverse of $\alpha_{\mathcal{A}_d^k}$ *is given by*

$$\alpha_{\mathcal{A}_d^k}^{-1} \left((x_0, \ldots, x_k) \times_{Q/G} \oplus_{l=0}^{k-1} [q_l, g_l]_G \right)$$
$$= [(e, g_0, g_1 g_0, \ldots, g_{k-1} \cdots g_0)) \cdot (x_0, \ldots, x_k)_{q_0}^h]_G,$$

where $(x_0, \ldots, x_k)_{q_0}^h = (\bar{q}_0, \ldots, \bar{q}_k)$ *is defined by the conditions:*

$$\bar{q}_0 = q_0,$$
$$\pi \bar{q}_l = x_l,$$
$$\mathcal{A}_d(\bar{q}_l, \bar{q}_{l+1}) = e.$$

Remark 3.6. *In a local trivialization, where* $q_0 = (h_0, x_0)$, *we have* $\bar{q}_{l+1} = \big((A(x_l, x_{l+1}) \cdots A(x_0, x_1))^{-1} h_0, x_{l+1} \big)$.

3.5. **Computational Aspects.**

3.5.1. *Exact Discrete Connection.* It is interesting from the point of view of computation to construct an **exact discrete connection** associated with a prescribed continuous connection, so that we can make sense of the statement that a given discrete connection is a p-th order approximation of a continuous connection.

3.5.2. *Additional Structure.* To do this, we require that the configuration manifold Q be endowed with a G-invariant Riemannian metric, with the property that the associated exponential map,

$$\exp : T_q Q \to Q,$$

is consistent with the group action, in the sense that,

$$\exp(\xi_Q(q)) = \exp(\xi) \cdot q.$$

We extend the exponential to $Q \times Q$ as follows,

$$\begin{aligned} \overline{\exp} : T_q Q &\to Q \times Q, \\ v_q &\mapsto (q, \exp(v_q)), \end{aligned}$$

and denote the inverse by $\overline{\log} : Q \times Q \to T_q Q$.

3.5.3. *Construction.* Then, we define the exact discrete connection as follows:

Definition 3.13. *The* ***Exact Discrete Connection*** $\mathcal{A}_d^E$ *associated with a prescribed continuous connection* $\mathcal{A} : TQ \to \mathfrak{g}$ *is given by,*

$$\mathcal{A}_d^E(q_0, q_1) = \exp(\mathcal{A}(\overline{\log}(q_0, q_1))).$$

This construction is more clearly illustrated in the following diagram,

$$\overbrace{Q \times Q \xrightarrow[\overline{\log}]{} TQ \xrightarrow[\mathcal{A}]{} \mathfrak{g} \xrightarrow[\exp]{} G}^{\mathcal{A}_d^E}$$

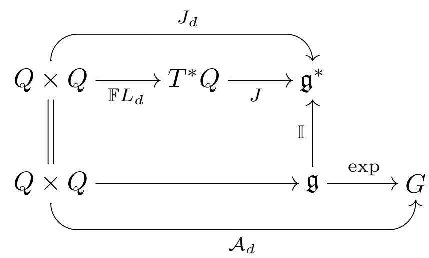
3.5.4. *Properties.* The exact discrete connection has the following properties,

- Equivariance of the exact discrete connection arises from the fact that each of the composed maps is equivariant.
- The splitting condition,
$$\mathcal{A}_d^E(i_q(g)) = g,$$
arises from the compatibility condition,
$$\exp(\xi_Q(q)) = \exp(\xi) \cdot q.$$

Example 3.5 (Discrete Mechanical Connection)**.** *The continuous mechanical connection is defined by the following diagram,*

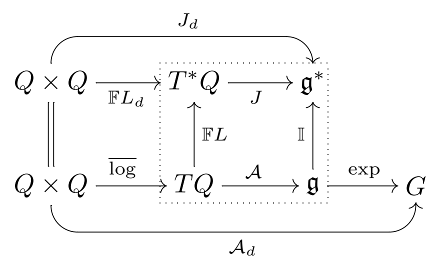
$$\begin{CD} T^*Q @>J>> \mathfrak{g}^* \\ @A{\mathbb{F}L}AA @AA{\mathbb{I}}A \\ TQ @>>{\mathcal{A}}> \mathfrak{g} \end{CD}$$

Correspondingly, the discrete mechanical connection is defined by the following diagram,



By construction, $\mathcal{A}_d(q_0, q_1) = e$ *implies that* $J_d(q_0, q_1) = 0$*, or equivalently, the horizontal space corresponds to the zero discrete momentum section of the pair groupoid.*

This is consistent with our notion of an exact discrete mechanical connection as the following diagram illustrates,



where the portion in the dotted box recovers the continuous mechanical connection.

In checking G*-equivariance, we use the equivariance of* $\exp : \mathfrak{g} \to G$, $J_d : Q \times Q \to \mathfrak{g}^*$*, and the equivariance of* $\mathbb{I}$ *in the sense of a map* $\mathbb{I} : Q \to L(\mathfrak{g}, \mathfrak{g}^*)$*, namely* $\mathbb{I}(gq) \cdot \mathrm{Ad}_g \xi = \mathrm{Ad}^*_{g^{-1}} \mathbb{I}(q) \cdot \xi$.

3.5.5. *Order of Approximation of a Connection.* We have the necessary constructions to consider the order to which a discrete connection approximates a continuous connection. There are two equivalent ways of defining the order of approximation of a continuous connection by a discrete connection, the first is more analytical, and is given by the order of convergence in an appropriate norm on the group.

Definition 3.14 (Order of Connection, Analytic)**.** *A discrete connection* $\mathcal{A}_d$ *is a* k*-th order discrete connection if,* k *is the maximum integer for which,*

$$\exists 0 < c < \infty,$$
$$\exists h_0 > 0,$$

such that,

$$\sup_{\substack{v_q \in TQ, \\ |v_q| = 1}} \|\mathcal{A}_d^E(q, \exp(hv_q))(\mathcal{A}_d(q, \exp(hv_q)))^{-1}\| \leq ch^{k+1}, \quad \forall h < h_0.$$

The second definition is more intrinsic, and is related to considering the infinitesimal limit of a discrete connection to connection-like structures on higher order tangent bundles, without the need for the introduction of the exact discrete connection.

Recall from §3.2.8 that we can construct a continuous connection from a discrete connection by the following construction,

$$\mathcal{A}([q(\cdot)]) = [\mathcal{A}_d(q(0), q(\cdot))],$$

and given $\mathcal{A}_d^k : Q^{k+1} \to kG$, we can obtain the continuous limit $\mathcal{A}^k : T^{(k)}Q \to k\mathfrak{g}$ in a similar fashion.

Definition 3.15 (Order of Connection, Intrinsic). *A discrete connection $\mathcal{A}_d$ is a k-th order approximation to $\mathcal{A}$ if, k is the maximum integer for which the diagram holds,*

$$\begin{array}{ccc} \mathcal{A}_d : Q \times Q \to G & & \mathcal{A} : TQ \to \mathfrak{g} \\ \Downarrow & & \Downarrow \\ \mathcal{A}_d^k : Q^{k+1} \to kG & \dashrightarrow & \mathcal{A}^k : T^{(k)}Q \to k\mathfrak{g} \end{array}$$

4. Applications

This section will sketch some of the applications of the mathematical machinery of discrete connections and discrete exterior calculus to problems in computational geometric mechanics, geometric control theory, and discrete Riemannian geometry.

4.1. Discrete Lagrangian Reduction.

Lagrangian reduction, which is the Lagrangian analogue of Poisson reduction on the Hamiltonian side, is associated with the reduction of Hamilton's variational principle for systems with symmetry.

The variation of the action integral associated with a variation in the curve can be expressed in terms of the Euler–Lagrange operator $\mathcal{EL} : T^{(2)}Q \to T^*Q$. When the Lagrangian is G-invariant, the associated Euler–Lagrange operator is G-equivariant, and this induced a reduced Euler–Lagrange operator, $[\mathcal{EL}]_G : T^{(2)}Q/G \to T^*Q/G$. The choice of a connection allows us to construct intrinsic coordinates on $T^{(2)}/G$ and T^*Q/G, and the representation of the reduced Euler–Lagrange operator in these coordinates correspond to the Lagrange–Poincaré operator $\mathcal{LP} : T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}} \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$.

The reduced equations obtained by reduction tend to have non-canonical symplectic structures. As such, naïvely applying standard symplectic algorithms to reduced equations have undesirable consequences for the long time behavior of the simulation, since it preserves the canonical symplectic form on the reduced space, as opposed to the reduced (non-canonical) symplectic form that is invariant under the reduced dynamics.

This sends an important cautionary message, that it is important to understand the reduction of discrete variational mechanics, since applying standard numerical algorithms to the reduced equations obtained from continuous reduction theory may not yield the desired results, inasmuch as long term stability is concerned.

Discrete connections on principal bundles provide the appropriate geometric structure to construct a discrete analogue of Lagrangian Reduction. We first introduce the Discrete Euler–Lagrange operator, which is constructed as follows.

4.1.1. *Discrete Euler–Lagrange operator.* The discrete Euler–Lagrange operator, $\mathcal{EL}_d : Q^3 \to T^*Q$ satisfies the following property,

$$\mathbf{d}\mathfrak{S}_d(L_d) \cdot \delta\mathbf{q} = \sum \mathcal{EL}_d(L_d)(q_{k-1}, q_k, q_{k+1}) \cdot \delta q_k.$$

In coordinates, the discrete Euler–Lagrange operator has the form,

$$[D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1})]\, dq_k.$$

4.1.2. *Discrete Lagrange–Poincaré operator.* The map $\mathcal{EL}_d(L_d) : Q^3 \to T^*Q$, being G-equivariant, induces a quotient map

$$[\mathcal{EL}_d(L_d)]_G : Q^3/G \to T^*Q/G,$$

which depends only on the reduced discrete Lagrangian $l_d : (Q \times Q)/G \to \mathbb{R}$. We can therefore identify $[\mathcal{EL}_d(L_d)]_G$ with an operator $\mathcal{EL}_d(l_d)$ which we call the **reduced discrete Euler–Lagrange operator**.

If in addition to the principal bundle structure, we have a discrete principal connection as described in the previous section, we can identify

$$Q^3/G \quad \text{with} \quad (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}).$$

Furthermore, each discrete G-valued connection 1-form $\mathcal{A}_d : Q \times Q \to G$ induces in the infinitesimal limit a continuous $\mathfrak{g}$-valued connection 1-form $\mathcal{A} : TQ \to \mathfrak{g}$. This continuous principal connection allows us to identify

$$T^*Q/G \quad \text{with} \quad T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*.$$

The discrete Lagrange–Poincaré operator $\mathcal{LP}_d(l_d) : (Q/G)^3 \times_{Q/G} (\tilde{G} \oplus \tilde{G}) \to T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$ is obtained from the reduced discrete Euler–Lagrange operator by making the identifications obtained from the discrete connection structure.

The splitting of the range space of $\mathcal{LP}_d(l_d)$ as a direct product (as in §3.3 of Cendra, Marsden, Ratiu [11]) naturally induces a decomposition of the discrete Lagrange–Poincaré operator

$$\mathcal{LP}_d(l_d) = \mathrm{Hor}(\mathcal{LP}_d(l_d)) \oplus \mathrm{Ver}(\mathcal{LP}_d(l_d)),$$

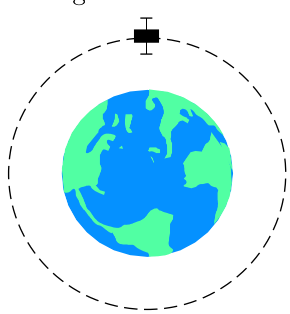
and this allows the discrete reduced equations to be decomposed in horizontal and vertical equations.

4.2. **Geometric Control Theory and Formations.** There are well established control algorithms for actuating a control system to achieve a desired reference configuration. In many problems of practical interest, the actuation of the mechanical system decomposes into shape and group variables in a natural fashion.

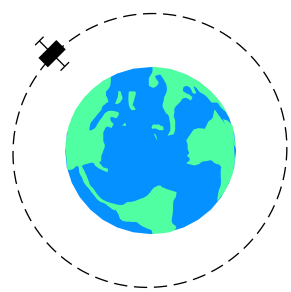
A canonical example of this is a satellite in motion about the Earth, where the orientation of the satellite is controlled by internal rotors through the use of holonomy and geometric phases, and the position is controlled by chemical propulsion.

In this example, the configuration space is $SE(3)$, the group is $SO(3)$, and the shape space is $\mathbb{R}^3$. When given an initial and desired configuration, it is desirable in computing the control inputs to decompose the relative motion into a shape component and a group component, so that they can be individually actuated.

To illustrate why it may not be desirable from a control theoretic point of view to decompose the space using a trivial connection, consider a satellite that is in a tidally locked orbit about the Earth, with the initial and desired configuration as follows.



Initial Configuration



Desired Configuration

Here, if we choose a trivial connection, then the relative group element would be a rotation by $\pi/4$, but this choice is undesirable, since the motion is tidally locked, and moving the center of mass to the new location would result in a shift in the orientation by precisely the desired amount. In this case, the optimal control input should therefore only actuate in the shape variables, and the relative group element assigned to this pair of configurations should be the identity element.

As such, the extension of mechanically relevant connections to pairs of points in the configuration space with finite separation, through the use of a discrete connection, can be of value in geometric control theory.

Similarly, in the case of formations, discrete connections allow for the orientation coordination problem to be handled in a more efficient manner, by taking into account the dynamic coupling of the shape and group motions, through the use of the discrete mechanical connection.

To aid in the construction of the exact discrete mechanical connection for the problems with configuration spaces involving the Special Euclidean group, which includes satellite clusters, we recall the representation of

$SE(3)$ as a semidirect product of $SO(3)$ and $\mathbb{R}^3$, as well as the exp and log operators which go between the Lie group and its Lie algebra. This allows us to then exponentiate the continuous mechanical connection, as shown in Example 3.5.

4.2.1. *Representation of SE(3).* The **Special Euclidean Group** $SE(3)$ is the Lie group consisting of isometries of $\mathbb{R}^3$. Using homogeneous coordinates, we can represent $SE(3)$ as follows,

$$SE(3) = \left\{ \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in GL(4,\mathbb{R}) \mid R \in SO(3), p \in \mathbb{R}^3 \right\}$$

with the action on $\mathbb{R}^3$ given by the usual matrix-vector product when we identify $\mathbb{R}^3$ with the section $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. In particular, given,

$$g = \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \in SE(3),$$

and $q \in \mathbb{R}^3$, we have,

$$g \cdot q = Rq + p,$$

or as a matrix-vector product,

$$\begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} Rq + p \\ 1 \end{pmatrix}.$$

The Lie algebra of $SE(3)$ is given by,

$$\mathfrak{se}(3) = \left\{ \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} \in M_4(\mathbb{R}) \mid \hat{\omega} \in \mathfrak{so}(3), v \in \mathbb{R}^3 \right\},$$

where $\hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3)$, given by,

$$\hat{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}.$$

4.2.2. *Exponentials and Logarithms.* The exponential map, $\exp : \mathfrak{se}(3) \to SE(3)$, is given by,

$$\exp \begin{pmatrix} \hat{\omega} & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \exp(\hat{\omega}) & Av \\ 0 & 1 \end{pmatrix},$$

where,

$$A = I + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\hat{\omega} + \frac{\|\omega\| - \sin\|\omega\|}{\|\omega\|^3}\hat{\omega}^2,$$

and $\exp(\hat{\omega})$ is given by the Rodriguez' formula,

$$\exp(\hat{\omega}) = I + \frac{\sin\|\omega\|}{\|\omega\|}\hat{\omega} + \frac{1 - \cos\|\omega\|}{\|\omega\|^2}\hat{\omega}^2.$$

The logarithm, $\log : SE(3) \to \mathfrak{se}(3)$, is given by,

$$\log \begin{pmatrix} R & p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \log(R) & A^{-1}p \\ 0 & 0 \end{pmatrix},$$

where,

$$\log(R) = \frac{\phi}{2\sin\phi}(R - R^T) \equiv \hat{\omega},$$

and ϕ satisfies,

$$\operatorname{Tr}(R) = 1 - 2\cos\phi, \qquad |\phi| < \pi,$$

and where,

$$A^{-1} = I - \frac{1}{2}\hat{\omega} + \frac{2\sin\phi - \phi(1 + \cos\phi)}{2\phi^2\sin\phi}\hat{\omega}^2.$$

4.3. **Discrete Levi-Civita Connection.** Vector bundle connections can be cast in the language of connections on principal bundles by considering the frame bundle consisting of oriented orthonormal frames over the manifold M, which is a principal $SO(n)$ bundle, as originally proposed by Cartan [10].

To construct our model of a discrete Riemannian manifold, we first trivialize the frame bundle to yield $SO(n) \times M$. Then, $Q = SO(n) \times M$, and $G = SO(n)$.

Here, we introduce the notion of a semidiscretized principal bundle, where the shape space $S = Q/G$ is discretized as a simplicial complex K, and the structure group G remains continuous. In this context then, the semidiscretization of the trivialization of the frame bundle is given by $Q = SO(n) \times K$.

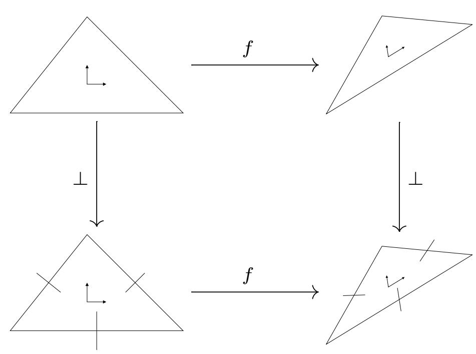
A discrete connection is $\mathcal{A}_d : Q \times Q \to G$, and we can construct a candidate for the Levi-Civita connection on a simplicial complex K, using the discrete analogue of the frame bundle described above.

We now introduce the notion of a discrete Riemannian manifold.

Definition 4.1. *A* ***discrete dual Riemannian manifold*** *is a simplical complex where each n-simplex σ^n is endowed with a constant Riemannian metric tensor g, such that the restriction of the metric tensor to a common face with an adjacent n-simplex is consistent.*

This is referred to as a discrete dual Riemannian manifold as we can equivalently think of associating a Riemannian metric tensor to each dual vertex, and as we shall see, by adopting Cartan's method of orthogonal frames [10], the connection is a $SO(n)$ valued discrete dual one-form, and the curvature is a $SO(n)$ valued discrete dual two-form.

For each n-simplex σ^n, consider an invertible transformation f of $\mathbb{R}^n$ such that $f^*g = I$. In the orthonormal space, we have a normal operator that maps a $n-1$ dimensional subspace to a generator of the orthogonal complement, denoted by $\perp$. Then, we obtain a normal operator on the faces of σ^n by making the following diagram commute.



The coordinate axes in the diagram represent the normalized eigenvectors of the metric, scaled by their respective eigenvalues.

The local representation of the discrete connection is given by,

$$\mathcal{A}_d((\sigma_0^n, R_0), (\sigma_1^n, R_1)) = R_1 A(\sigma_0^n, \sigma_1^n) R_0^{-1},$$

and so the discrete connection is uniquely defined if we specify $A(\sigma_0^n, \sigma_1^n)$, where σ_0^n and σ_1^n are adjacent n simplices. Since they are adjacent, they share a $n-1$ simplex, denoted σ^{n-1}. In particular, this can then be thought of as a $SO(n)$-valued discrete dual one-form, since to each dual one-cell, $\star\sigma^{n-1}$, we associate an element of $SO(n)$.

This element of $SO(n)$ is computed as follows,

(1) In each of the n-simplices, we have a normal direction associated with σ^{n-1}, denoted by,

$$\perp (\sigma^{n-1}, \sigma_i^n) \in \mathbb{R}^n.$$

(2) If these two normal directions are parallel, then we set,

$$\langle A, \star\sigma^{n-1}\rangle = I,$$

otherwise, we continue.

(3) Construct the $n-2$ dimensional hyperplane P^{n-2}, given by the orthogonal complement to the span of the two normal directions.

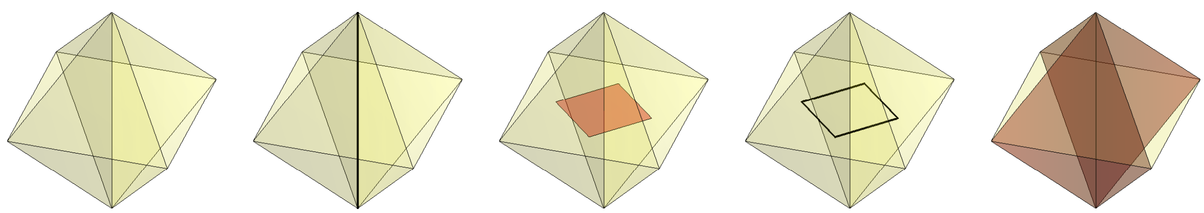
$$P^{n-2} = \perp (\text{span}(\perp (\sigma^{n-1}, \sigma_0^n), \perp (\sigma^{n-1}, \sigma_1^n))).$$

(4) If $\star\sigma^{n-1}$ is oriented from σ_0^n to σ_1^n, then we set,

$$\langle A, \star\sigma^{n-1}\rangle = \{R \in SO(n) \mid R|_{P^{n-2}} = I_{P^{n-2}}, R(\perp (\sigma^{n-1}, \sigma_0^n)) = \perp (\sigma^{n-1}, \sigma_1^n)\}.$$

The curvature of this discrete Levi-Civita connection is then a $SO(n)$-valued discrete dual two-form. There is however the curious property that the boundary operator for a dual cell complex may not necessarily agree with the standard notion of boundary, since that may not be expressible in terms of a chain in the dual cell complex. This is primarily an issue on the boundary of the simplicial complex, and if we are in the interior, this is not a problem.

Since curvature is a dual two-form, it is associated with the dual of a codimension-2 simplex, given by $\star\sigma^{n-2}$. Consider the following example,



Simplicial Complex, K — $n-2$ primal simplex, σ^{n-2} — 2 dual cell, $\star\sigma^{n-2}$ — 1 dual chain, $\partial \star \sigma^{n-2}$ — $n-1$ primal chain, $\star\partial \star \sigma^{n-2}$

The curvature $\mathcal{B}$ of the discrete Levi-Civita connection is given by,

$$\langle \mathcal{B}, \star\sigma^{n-2}\rangle = \langle \mathbf{d}A, \star\sigma^{n-2}\rangle = \langle A, \partial \star \sigma^{n-2}\rangle.$$

As can be seen from the geometric region $\star\partial \star \sigma^{n-2}$, the curvature associated with $\star\sigma^{n-2}$ is given by the ordered product of the connection associated with the dual cells, $\star\sigma^{n-1}$, where $\sigma^{n-1} \succ \sigma^{n-2}$. This also suggests that we can also think of the discrete connection as a primal $n-1$ form, and the curvature as a primal $n-2$ form, where the curvature is obtained from the connection using the codifferential.

When the group G is nonabelian, we see that the curvature is only defined up to conjugation, since we need to specify a dual vertex on the 1-dual chain $\partial\star\sigma^{n-2}$ from which to start composing group elements. To make this well defined, we can adopt the approach used in defining the simplicial cup product, and assume that there is a partial ordering on dual vertices, which would make the curvature unambiguous.

As a quality measure for simplicial triangulations of a Riemannian manifold, having the curvature defined up to conjugation may be sufficient if we have a norm on $SO(n)$ which is invariant under conjugation. As an example, taking the logarithm to get an element of the Lie algebra $so(n)$, and then using a norm on this vector space yields a conjugation invariant norm on $SO(n)$. This allows us to detect regions of the mesh with high curvature, and selectively subdivide the triangulation in such regions.

Similarly, we can define a discrete primal Riemannian manifold, where the Riemannian metric tensor is associated with primal vertices, and the connection is a G valued primal one-form, and the curvature is a G valued primal two-form.

4.3.1. *Abstract Simplicial Complex with a Local Metric.* Recall that in the §2.2, we introduced the notion of an abstract simplicial complex with a local metric defined on pairs of vertices that are adjacent.

In this situation, we can compute the curvature around a loop in mesh using local embeddings. We start with an initial n-simplex, which we endow with an orthonormal frame. By locally embedding adjacent n-simplices into Euclidean space, and parallel transporting the orthonormal frame, we will eventually transport the frame back to the initial simplex.

The relative orientation between the original frame and the transported frame yields the integral of the curvature of the surface which is bounded by the traversed curve. This results from a simple application of the Generalized Stokes' theorem, and the fact that the curvature is given by the exterior derivative of the connection form.

5. Conclusions and Future Work

We have focused on the construction of a discrete theory of exterior calculus and connections that exhibit desirable properties at the discrete level. It would certainly be interesting from both a theoretical, and applications oriented, perspective to analyze the convergence properties of these discrete theories, and the manner in which the discrete properties limit to their continuous counterparts.

The proper method of addressing the dynamics of non-flat simplicial complexes, including the appropriate notion of a tangent space at a vertex, remains to be considered. Similar considerations arise for manifolds with boundaries. In particular, dealing with boundary conditions that include slippage can pose challenges due to the geometric constraints on mesh movement at the boundary. By contrast, non-slip boundary conditions are easy to impose, since this corresponds to requiring that vertices on the boundary remain fixed.

The applications to discrete variational mechanics requires further exploration. In particular, it would be insightful to understand geometric conservation laws such as the Kelvin circulation theorem in fluids, in the context of discrete exterior calculus. Similarly, in Lagrangian reduction, the curvature of the connection arises as a magnetic term in the symplectic form, and discrete notions of curvature remain a topic of interest.

In elasticity and solid mechanics, as well as fluids, stress tensors provide a means of encoding the internal energy of the system. While differential forms are of interest in a large class of problems, more general tensor fields remain of relevance to geometric mechanics, and discrete analogues of such objects remain an open challenge.

In developing some of the mathematical tools that geometric mechanics has come to rely upon, it is hoped that the success of geometric mechanics will be repeated at the numerical level in the construction of geometrically informed computational algorithms.

Acknowledgements

I would like to acknowledge my advisor, Jerrold Marsden, for getting me interested in geometric mechanics, and its relation to computational methods. The section on discrete exterior calculus represents joint work with my collaborators, Mathieu Desbrun, Anil Hirani, and Jerrold Marsden. This work would not have been possible without discussions with, insights, and much needed motivation from them. The section on discrete connections was the product of multiple discussions with Jerrold Marsden and Alan Weinstein, and was motivated by applications to Discrete Lagrangian Reduction. In addition, the links with discrete variational mechanics and asynchronous variational integrators were helped by useful discussions with Matthew West.

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