

Explorations of Vector Calculus using Virtual Reality
Calcflow Supplement¹, MATH 20E

Summer Lab I - Introduction to Calcflow, and the Change of Variables Theorem

Welcome to the Calcflow supplement for Math 20E! In this lab, you will learn the basics of working with Calcflow, and then explore some visual ideas of double integration and the Change of Variables Theorem. When you are ready, put on the headset and make sure you are correctly holding the controllers. If at any time you experience any difficulties using the software, please see a lab tutor.

Working with Calcflow

After starting up Calcflow, when you enter the virtual environment you should be looking at the Calcflow main menu. We are going to do a few simple exercises to get you familiar with the VR controls, and then we will move into more conceptual exercises.

Move your head around to see what the environment looks like, and to get an idea of the depth perception you currently have. Now, look down at your controllers. The controls look like hands! They also operate like your actual hand. Move your fingers around and observe how the controllers work. On either hand, if you point with your actual index finger, your virtual finger will also point. If you move any of your fingers away from the controllers, your virtual fingers will also move. Now, the button under your index finger is called the “trigger” button. The button under your middle finger is called the “grip” button. Make a fist, and press the grip button as you do so. Your virtual hand(s) should also make a fist. Also, take note of the pink rays that are coming out of your virtual hands. These will be used to interact with objects as well.

Now, let’s enter one of the modules and begin exploring Calcflow. Point one of your hands to the **Vector Addition and Cross Product** module, and then press the trigger button with your index finger. You will enter into a scene with two sets of coordinate axes containing vectors. Look at coordinate system on the right hand side. Point your hand at the cube containing these axes, making sure the pink ray is hitting the cube, and then press the grab button with your middle finger and use the joystick to pull the cube to you. You can also reach out with your hand and grab the cube directly, using the same motions as you would to pick up an object in real life (and making sure to press the grab button). Now, move your hands into the cube, and make the pink ray is not visible. Now, move each hand over a vector tip and grab each vector. You can move each vector, and the resultant sum vector will change accordingly. Experiment with different sums of these vectors. When you are finished, push the cube away from you.

¹*Developed by Marc Loschen*

Next, look at your left-hand side, and pull this cube to you. The vectors on these axes demonstrate the cross product. Recall that the cross product produces a vector quantity, and this quantity is orthogonal (i.e. perpendicular) to the two vectors being crossed. Fix the vector labeled **A** to point in one direction, by grabbing it just as you did with the previous two vectors. Now, using one hand, and without letting go, move the vector labeled **B** to be on top on the first vector, and then move **B** in a clockwise direction by 45 degrees. Notice the direction that the cross product is pointing. Now, let's try our first exercise.

Exercise 1.1: Grab only the vector **B** and position it over the vector **A** again. Now, move the vector 45 degrees in the counter-clockwise direction.

- (a) Where does the cross product point to? How is this different from moving **B** 45 degrees clockwise?
- (b) Given two vectors **a** and **b**, what can you infer about the relationship between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{b} \times \mathbf{a}$?

When you are done examining these vector operations, you will want to go back to the main menu. Look at your hand, and notice there is a circle on your wrist. Hover your right index finger over the white inner circle and wait for the entire circle to become white. A menu will appear next to your left hand. Find the button that says "Reset Scene". If you ever are in a situation where the scene is not working, click this button and you can reset your environment. To go back to the main menu, find the button that says "Home" and click it using the trigger. You will now be transported back to the Calcflow main menu.

Hover over the **Double Integral** module and click on it, using the trigger. You will now enter a scene that allows you to see what double integration visually looks like. In this scene, you should see the following: a "cube" containing a coordinate system and a graph, and a calculator for entering functions $z = f(x, y)$. Examine the default graph that appears. Using the same controls as above, bring the graph to you using the grip button, and then move your hands inside the cube containing the graph. Grab the red ball under the graph and move it. Notice the blue shading underneath? This represents the volume between the xy -plane and the function! Move the ball to the corner so that no blue shading is present. Now, move the ball along the x -axis, while keeping the y -value unchanged. You can find the axis label on the ends of the graphs (along the edges of the cube). Then, drag the ball along the y -axis. This would be the double integral of $z = f(x, y)$ where we integrate along x first, and then y . Now, do the opposite. Reset the ball, and move it along the y -axis, while keeping the x -value unchanged. Then, drag the ball along the x -axis. This would be the double integral of $z = f(x, y)$ where we integrate along y first, and then x . So for this function, we have just visually demonstrated that the order of integration does not matter! You will still get the same volume, regardless of the order of integration. Now, let's try an exercise.

Exercise 1.2:

- (a) Enter the function $z = 0.25x^2 + 0.5y^2$ into the “Z=” field, and then set the bounds to be $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$. This will show us the double integral $\int_{y=-4}^4 \int_{x=-4}^4 (\frac{1}{4}x^2 + \frac{1}{2}y^2) dx dy$. The integrand itself gives us a “bowl” shaped graph. Grab the red ball and observe the shaded volume. Where is the volume located, inside the bowl or outside the bowl? Does this match what you thought it would be?
- (b) Next, enter the function $z = \sin(x) + \cos(y)$ and set the bounds to be $[0, 2\pi] \times [0, 2\pi]$. Drag the red ball to observe the shaded volume. Move the red ball to the “end” of the region (that is, to the 2π values). What do you notice about the volume? Why does this make sense?

Hopefully this gives you a bit more intuition as to what double integration physically represents. While it may be a simple concept after-the-fact, it is still a crucial aspect to remember as you study calculus!

This ends the tutorial portion of assignment 1. Next, we will examine Change of Variables in three dimensions. If you haven’t done so already, point to the white circle on your left hand until the menu appears, and select “Home” to return to the main menu.

The Change of Variables Theorem

We shall now explore the conceptual ideas behind the Change of Variables Theorem. This theorem provides a generalization of the u -substitution integration technique that you previously saw in single-variable calculus. In class, you will see examples of this theorem in two dimensions ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$) and three-dimensions ($\mathbb{R}^3 \rightarrow \mathbb{R}^3$), and your textbook homework will provide many examples of using this theorem to perform integration. However, an important part of change of variables involves determining the variable substitution(s) to make, as well as the bounds on these new variables. We can think of these variable substitutions as “mappings” from one set of variables to another. But in order for these mappings to be useful, we must be able to invert them. That is, if we use our mapping to transform a set of variables into a new set of variables, we should be able to reverse this action and return to our original variables. This is why we can claim equality of the two integral expressions in the Change of Variables Theorem (see below). In order to guarantee our mapping is invertible, the mapping must be one-to-one and onto. For two-dimensional mappings, this can easily be drawn on paper. However, it becomes increasingly unfeasible to use a two-dimensional space to figure out a three-dimensional change of variables, especially when the integrand is a complicated function to draw on paper. This is where the virtual reality environment

comes into play: you can use Calcflow to draw these change of variables mappings, and this will allow you to gain a better conceptual understanding of how the Change of Variables Theorem is transforming an integral. For reference, here is the three-dimensional version of the theorem.

Theorem (Change of Variables). *Let W and W^* be elementary regions in 3-space, and let $T: W^* \rightarrow W$ be of class C^1 . Suppose that T is one-to-one on W^* , and suppose that $W = T(W^*)$. Then for any integrable function $f: D \rightarrow \mathbb{R}$, we have*

$$\iiint_W f(x, y, z) \, dx \, dy \, dz = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| \, du \, dv \, dw$$

This theorem, and its two-dimensional analogue, are found in section 6.2 of the textbook. You are encouraged to read both of these theorems, as well as the examples in this section, before proceeding with this assignment.

We will first explore the geometry of maps from \mathbb{R}^3 to \mathbb{R}^3 , including the cylindrical and spherical coordinate transformations. Afterwards, we will examine a triple integral and determine an appropriate change of variables to use. From the main menu, point to the **3D Coordinate Transformation** module, and enter this scene. Take a moment to familiarize yourself with the environment, and then proceed to the next section.

1. The Geometry of Maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$

One-to-one and Onto Maps

Here, we briefly review the definitions of one-to-one and onto. For more detailed information, consult section 6.1 of your textbook.

Let $T: D^* \rightarrow D$ be a transformation. We say that T is *one-to-one* if for any two points (u, v) and (u', v') in D^* , $T(u, v) = T(u', v')$ implies that $(u, v) = (u', v')$. Another way of saying this, is that every point in the range of T is only reachable from **one** point in the domain. We say that T is *onto* D if for every point $(x, y) \in D$, there is at least one point $(u, v) \in D^*$ such that $T(u, v) = (x, y)$. Another way of saying this, is that the entire range of T is reachable by at least one point from the domain. These are the technical definitions of one-to-one and onto, but what we wish to explore in this lab is the visual implications of these definitions.

Spherical Coordinates

Consider the transformation $T: [0, 8] \times [0, 2\pi] \times [0, \pi] \rightarrow \mathbb{R}^3$ given by

$$T(\rho, \theta, \phi) = (\rho \cos(\theta) \sin(\phi), \rho \sin(\theta) \sin(\phi), \rho \cos(\phi)) \quad (1)$$

This is the coordinate transformation that attempts to represent Cartesian coordinates, for example (x, y, z) , in terms of spherical coordinates (ρ, θ, ϕ) . According to the Change

of Variables Theorem, we need the transformation to be one-to-one and onto. However, as presently defined, T has neither of these properties. Let's examine why, and see if we can fix this.

Enter the above transformation into the calculator as follows:

$$T(u, v, w) = (u \cos(v) \sin(w), u \sin(v) \sin(w), u \cos(w))$$

with $u \in [0, 8]$, $v \in [0, 2\pi]$, and $w \in [0, \pi]$. (Note that we have made the following substitutions: $u = \rho$, $v = \theta$, $w = \phi$.) Hit the “=” button, and check that a ball (solid version of a sphere) has been graphed in the output axes. Notice that there is a red ball in the input coordinate axes. This red ball represents a point in the domain, and as it moves, a corresponding red ball in the output coordinate system moves according to the transformation rules. Using the grabber tool, drag the ball so that the u value is zero. Then, drag the ball along the v and w axes. The red ball in the output graph will disappear into the solid, but above the axes you should see the point image point as a triple (x, y, z) . You should notice that, no matter how you change the values of v or w , the corresponding point on the output graph does not move. Hence, all of these points get mapped to the same point on the sphere, making the mapping not one-to-one. One way we can remedy this is to simply not allow u to be zero. On the calculator, notice that the domain for each of the input variables is set (by default) to have closed brackets. To not allow u to be zero, we can simply change the bracket to be a parenthesis, indicating that the interval is open at zero. Click on the bracket on the calculator to change it to a parenthesis. Now, no matter what value u takes on, we will not have this issue of many points getting mapped to a single point on the sphere. However, there are still other issues with our mapping. Set v to be zero. Then move the point along the u or w directions. Make a note of what the corresponding point on the sphere is doing. Now, set v to be 2π . Move the u and w values along the same directions as you just did. You should notice that the point on the ball has the same behavior, whether v is 0 or 2π . Thus, $T(u, 0, w) = T(u, 2\pi, w)$, and so T is again not one-to-one. To fix this, we can make the domain of the v variable open at either endpoint of the interval. Choose either 0 or 2π on the θ domain, and change the interval to be open at that point. We are almost done making our transformation one-to-one! We just have one more input to adjust.

Exercise 1.3: Examine the w input variable of the transformation above. What changes do you need to make to the domain of w to allow T to be completely one-to-one? Hint: examine both endpoints!
On your answer sheet, write down the domain of each variable u, v, w .

Now, we have spherical coordinate transformation that is one-to-one! But, we are not quite ready to use this in the Change of Variables Theorem yet. We need T to be both one-to-one AND onto. As it is currently defined, the actual output of T is not all of \mathbb{R}^3 . We need to modify the codomain of our transformation.

Exercise 1.4: Let $S \subset \mathbb{R}^3$ be the image of our mapping (1) above. Give an explicit definition for the set S . You may use words, set-builder notation, etc. Then, write down the the final transformation we've established for our one-to-one and onto mapping. Your answer should look like this:

$$T: A \rightarrow B$$

$$T(\rho, \theta, \phi) = (x, y, z)$$

where you need to write down what A and B are, as well as write x, y, z as functions of ρ, θ, ϕ .

Congratulations! Now you have successfully determined a coordinate transformation that could be used in the Change of Variables Theorem for a triple integral.

Cylindrical Coordinates

Consider the following transformation

$$T: [0, 5] \times [0, 2\pi] \times [-5, 5] \rightarrow \mathbb{R}^3$$

$$T(u, v, w) = (u \cos(v), u \sin(v), w) \quad (2)$$

This transformation attempts to represent Cartesian coordinates in terms of cylindrical coordinates. Let's investigate whether this transformation satisfies the conditions for use in the Change of Variables Theorem. Make sure you are still in the **3D Coordinate Transformation** module. Using the calculator, enter the transformation as given above. Make sure to set all of the intervals to be closed, for the input variables u, v, w . Examine the structure of the domain and range before proceeding with the next exercise.

Exercise 1.5

- As before, grab the ball in the domain space and move it around to see the corresponding point on the image. Where does the mapping (2) fail to be one-to-one? (Hint: check the endpoints!)
- Adjust the intervals to be open, where needed, and then write down on your answer what the new "one-to-one" domain is.
- Finally, determine what image set this transformation maps onto, and express this set mathematically. Write this down on your answer sheet.

While we have just seen the “standard” ways of expressing the spherical and cylindrical coordinate transformations, these are not the only combination of variable substitutions we could make. Let’s examine a modified cylindrical coordinate transformation and see what properties this might have in relation to our transformation above.

Exercise 1.6 Consider the cylindrical coordinate transformation (2) above. What would happen if we switched the y and z components? That is, assume our transformation was now

$$\begin{aligned} T^*: [0, 5] \times [0, 2\pi] \times [-5, 5] &\rightarrow \mathbb{R}^3 \\ T^*(u, v, w) &= (u \cos(v), w, u \sin(v)) \end{aligned} \quad (3)$$

What would the image of this transformation look like? When would you want to use this transformation instead of the “standard” cylindrical transformation as given above?

Hopefully you have a better understanding now of what makes a function truly one-to-one and onto. Developing intuition in higher space and help solidify the the concepts behind the integration, and can lead to a more informed understanding of your computations!

2. Triple Integrals and Change of Variables

Now the we’ve looked at one-to-one and onto maps from $\mathbb{R}^3 \rightarrow \mathbb{R}^3$, let’s apply our knowledge to solving triple integrals. Consider the following triple integral:

$$\iiint_W e^{-(x+y+z)^{3/2}} dV$$

where W is the region bound by $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

This looks fairly complicated to compute directly. Let’s try to apply the Change of Variables Theorem here. To do this, we need a transformation from a simpler region to the region of integration of our triple integral, and we need this transformation to be one-to-one and onto. Let’s start with the following mapping.

$$\begin{aligned} T: [0, 1] \times [0, 1] \times [0, 1] &\rightarrow W \subset \mathbb{R}^3 \\ T(u, v, w) &= (u(1 - v), uv(1 - w), uvw) \end{aligned}$$

Exercise 1.7:

- (a) Enter the transformation above into the calculator. Then, determine the modifications that need to be made in order for the mapping to be one-to-one. Write this down on your answer sheet.
- (b) Next, determine the set W . Describe visually what this shape is, and write it down on your answer sheet.
- (c) Finally, write down the new transformation on your answer sheet. Use this to transform the triple integral using the Change of Variables Theorem. You do not need to solve the integral, just write down on your answer sheet the expression for the integral.

It can often be hard to find a suitable change of variables to compute an integral. The Changes of Variables Theorem does not apply to all integrals, but even the ones where the theorem does apply often do not lend themselves to obvious variable substitutions. Building your intuition of three-dimensional space, as well as using tools such as Calcflow, can help you determine the correct transformations needed to compute double and triple integrals. Having this conceptual understanding can then allow you to develop intuition for higher dimensions, and solve even more complicated multiple integrals!