Explorations of Vector Calculus using Virtual Reality
Calcflow Supplement[1], MATH 20E

Summer Lab II - Parametrized Curves and Surfaces

In this lab, you will explore parametrized functions, specifically curves and surfaces in 3-space. Parametrized functions are a key element to performing integration over more generalized domains. In single variable calculus, the integral is defined over a "straight line", or an interval of real numbers along a coordinate axis. When we make the jump into double integration, we integrate a function $z = f(x, y)$ over some region that lies in the xy-plane. What is important to note here, is that while these are very distinct types of integrals, the underlying idea is the same: perform the integration "over a region", or more formally expressed, over a set of real numbers or a set of pairs of real numbers. When we think about integration in this way, we see that these sets do not necessarily need to be straight lines or flat planes. If we have a path $\mathbf{c}\colon \mathbb{R} \to \mathbb{R}^3$, then we can collect the (x, y, z) triples produced by this path into a set. If we plot all of these points together, then we geometrically have formed a curve in 3-space. But this is still just a set, and so it is a viable region with which to integrate over. The same analog holds for surfaces in 3-space. IF we have a parametrized surface $\Phi\colon D \to \mathbb{R}^3$, then we can collect all of the (x, y, z) triples that make up this surface and collect them into a set. This set again forms a geometric shape in 3-space, but is mathematically still a set of points, and thus can be integrated over.

In class, you will encounter many exercises where you will actually compute line and surface integrals. However, in order to gain the conceptual understanding of what your computations actually mean, you need to understand the qualitative properties of parametrizations. Using Calcflow, we will explore tangent vectors and normal vectors to curves and surfaces, different parametrizations of the same geometric shape, and orientation of parametrizations.

Once you log in to a computer and load Calcflow, put your headset on and you will be looking at the main menu. Now, let's begin!

1. Drawing Curves in 3-Space

First, we will examine parametrized curves in 3-space. A paramatrized curve is a path $\mathbf{c}\colon [a, b] \to \mathbb{R}^3$. This is essentially a function that takes the real line, and brings in into space (while possibly twisting, turning, and bending it). We encounter parametrized curves when looking at path and line integrals. We will use VR to examine the geometry of a curves, and this will hopefully help you better understand the numbers you get from the actual computation of these integrals. From the Calcflow main menu, enter the `Parametrized Curve` module. Familiarize yourself with the environment before proceeding. In particular, note the calculator, the single real line (representing the domain/input), and the cube with xyz axes (representing the output).

[1] *Developed by Marc Loschen*

Let's start our discussion with a path that traces out a helix curve. Enter the following function into the calculator:

$$\mathbf{c} : [0, 2\pi] \to \mathbb{R}^3$$

$$\mathbf{c}(t) = (\cos(t), \sin(t), t) \tag{1}$$

Make sure that this path traces out a helix curve! Now, look at the real line (the domain). Grab the ball on the real line and move it from left to right. If we let t represent time, then this direction of motion would be as if the path were moving forward in time. Look at the graph as you move t forward, and you will see a corresponding point that is also moving. This shows how that point moves along the graph. If we also consider the graph to be drawn from 0 to 2π, then you can see how the helix itself it drawn, "from start to finish". Now, looking again at the graph, find the blue vector that is attached to the mapped point. This is the tangent vector to the graph. More specifically, for our path $\mathbf{c}(t)$, this blue vector represents $\mathbf{c}'(t)$. Make a note of what direction this vector is pointing in as you move across the domain.

Let us define *positive orientation* to be when the path travels in a counter-clockwise direction as t increases, from the minimum value to the maximum value. Then, this curve would have positive orientation.

Exercise 2.1: For a parametrized curve, give a definition of what *negative orientation* would be. You can simply give a 1-2 line description, using similar words as we did above.

Let's look at another curve. Enter the following function into the calculator:

$$\mathbf{p} : [0, 2\pi] \to \mathbb{R}^3$$

$$\mathbf{p}(t) = (\cos(2\pi - t), \sin(2\pi - t), 2\pi - t) \tag{2}$$

Examine the curve and test the output behavior with different inputs. Then, try the following exercise.

Exercise 2.2:

(a) What do you notice visually about this path $\mathbf{p}$ in relation to our first path $\mathbf{c}$? On your answer sheet, write down any visual similarities or differences that you see.

(b) Now, grab the ball in the domain and move it from left to right. Compared to the path $\mathbf{c}$, how is the path $\mathbf{p}$ traced out? What is the difference between the tangent vector for the path $\mathbf{p}$ and the tangent vector for the path $\mathbf{c}$.

(c) Would the path $\mathbf{p}$ have positive or negative orientation? Use the definition that we gave above and the definition you gave in the previous exercise to answer this question.

Curves (as well as surfaces) have infinitely many possible parametrizations that can describe them. Each parametrization has its own properties, some of which may or may not be helpful when working with a certain problem. One such quality that will vary from parametrization to parametrization is the orientation. Suppose we have a path $\mathbf{c}_1$ that traces out a certain geometric curve, and we find another path $\mathbf{c}_2$ that traces out the same curve. We say that a $\mathbf{c}_2$ is *orientation-preserving*, with respect to $\mathbf{c}_1$, if both paths have the same orientation. If the paths instead have opposite orientation, then we say $\mathbf{c}_2$ is *orientation-reversing*, with respect to $\mathbf{c}_1$.

We shall examine this idea in the next exercise. Enter the following function into the calculator:

$$\mathbf{r} : \left[\frac{\pi}{2}, \frac{5\pi}{2}\right] \to \mathbb{R}^3$$
$$\mathbf{r}(t) = \left(\sin(t), -\cos(t), t - \frac{\pi}{2}\right) \tag{3}$$

As before, experiment with the input/output relationships and then do the following exercise.

Exercise 2.3:

(a) What do you notice visually about this path $\mathbf{r}$ in relation to our first path $\mathbf{c}$? On your answer sheet, write down any visual similarities or differences that you see.

(b) Now, grab the ball in the domain and move it from left to right. Compared to the path $\mathbf{c}$, how is the path $\mathbf{r}$ traced out? What is the difference between the tangent vector for the path $\mathbf{r}$ and the tangent vector for the path $\mathbf{c}$?

(c) Would the path $\mathbf{r}$ be orientation-preserving or orientation-reversing, with respect to the path $\mathbf{c}$?

Hopefully this gives you a better understanding of path orientation for curves in space! At this point, move your right index finger over the white circle on your left wrist, and click on the home button from the menu that appears.

2. Parametrizing the Surface of a Sphere

Now, let's examine surface parametrizations. The first surface we will look at is a sphere. There are two commonly used parametrizations of a sphere: one based on spherical coordinates, and the other based on latitude-longitude. Both of these parametrizations create a sphere, but in much different ways. These parametrizations have different *orientations*, and thus we see a difference in their normal vectors. Before we examine these ideas in greater detail, let's graph each parametrization. From the main menu, enter the `Parametrized Surface` module and familiarize yourself with the environment before moving on. In particular, note the uv-plane (which shows the domain), the xyz-space (which shows the output), and the calculator.

The first surface we will examine is a sphere, specifically the unit sphere. Our first parametrization to look at will be one that is based on spherical coordinates. The spherical parametrization is given as follows:

$$\begin{aligned} \mathbf{\Phi} &: [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3 \\ \mathbf{\Phi}(u, v) &= (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)) \end{aligned} \tag{4}$$

Enter this parametrization into the calculator and examine the output graph. In the uv-plane, there is a red ball that you can grab. This represents a point in the domain, and if you move it you will see the corresponding point on the sphere move as well. Let's see how the sphere is formed. Just as with paths, if we start at the minimum values of the domain, and move to the maximum values, we can "trace out" the how the surface would be formed. Move the ball to the line $v = \pi/2$, then drag the ball from $u = 0$ to $u = 2\pi$. This allows us to see what the u parameter traces out. Next, fix the u-value and toggle the v values from 0 to π. This allows us to see what the v parameter traces out.

Exercise 2.4: Repeat the steps above, examining the u or v parameter one at a time. Write down on your answer sheet what each parameter traces out. That is, describe what part of the sphere u is mapping out and what part of the sphere v is mapping out.

When looking on the graph, notice that the mapped-to point has three vectors attached to it. These are the tangent and normal vectors. The tangent vectors are blue and green, corresponding to the blue or green coloring of u and v in the domain (look at the uv-plane to verify this). To get the normal vector to the surface, we take the cross product of these two tangent vectors. By convention, we cross the u tangent vector with the v tangent vector. In symbols, if we have $\mathbf{T}_u$ and $\mathbf{T}_v$ as our u and v tangent vectors, respectively, then $\mathbf{T}_u \times \mathbf{T}_v$ is our normal vector. We can use the normal vector to determine orientation. For a sphere, let's define a parametrization as having *positive orientation* if the normal vector points "outward", i.e. away from the origin.

Exercise 2.5:

(a) For a parametrization of a sphere, define what *negative orientation* would be. You can simply give a 1-2 line description, using similar words as we did above.

(b) Examine the normal vector for the spherical parametrization above. Based on what you see, does this parametrization have positive or negative orientation? Write down you answer and give a brief explanation for what you chose.

As we saw with parametrized curves, we can have many possible parametrizations for any given surface. In fact, another common parametrization for the unit sphere is called the Latitude-Longitude parametrization. Enter the following function into the calculator:

$$\mathbf{\Psi} : [-\pi, \pi] \times \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$$

$$\mathbf{\Psi}(u, v) = (\cos(u)\cos(v), \sin(u)\cos(v), \sin(v)) \tag{5}$$

Examine the output graph. This is also a sphere! Grab the ball in the uv-plane and move is throughout the domain. Pay attention to how the corresponding point on the output graph behaves. Examine how the u and v parameters map out the sphere, and then do the following exercise.

Exercise 2.6:

(a) As we did for the spherical parametrization, fix a value for v and move u from left to right, and then fix a value for u and move v up and down. Describe how the u and v parameters trace out the sphere for the Latitude-Longitude parametrization.

(b) How are the Latitude-Longitude and spherical parametrizations different? If we set u and v to start 0, then how would each parametrization "trace out" the sphere?

(c) Does the Latitude-Longitude have positive or negative orientation? Give a brief justification, and use the output graph to help with your explanation.

So we have just found two surface parametrizations, both of which give us a sphere, but the functions themselves give the information differently. Do you think you can find another parametrization for a sphere?

3. Parametrizing a Plane in 3-Space

Next, we will look at parametrizing a plane in 3-space. Let P be a plane that contains the point (x_0, y_0, z_0) and is spanned by two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. Then the parametrization of this plane is given by:

$$\mathbf{\Phi} : D \to \mathbb{R}^3$$
$$\mathbf{\Phi}(u, v) = (x_0, y_0, z_0) + u\mathbf{a} + v\mathbf{b} = \big(x_0 + a_1u + b_1v, y_0 + a_2u + b_2v, z_0 + a_3u + b_3v\big)$$

Let's find the parametrization of a specific plane and then gather some qualitative information. Make sure you are still in the `Parametrized Surface` module.

Exercise 2.7: Let P be the plane containing the points $(0,0,0)$, $(1,2,3)$, and $(0,0,1)$.

(a) Find a parametrization for this plane, using the formula above. For the domain of each variable, let $u \in [-10,10]$ and $v \in [-10,10]$.

(b) Find the normal vector to this plane **by hand**. Does the normal vector point "upward" (positive z direction) or "downward" (negative z direction)?

(c) Enter your parametrization into the calculator, and visually inspect the surface that is produced. Does the normal vector behave as you expected?

Examining the surface above, you'll note that the domain was a square region. We essentially took a square in the xy-plane and projected it into xyz-space, turning it in the necessary directions. However, we often encounter surfaces that are planes, but not a full rectangular plane.

Let's take the plane that goes through the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. You can verify that this surface is represented by the equation $x+y+z=1$. We could also write this as $z = 1-x-y$, which means we can represent this surface as a graph $z = f(x,y)$. This leads us to the following parametrization:

$$\mathbf{\Phi} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$$

$$\mathbf{\Phi}(x,y) = (x, y, 1-x-y) \tag{6}$$

This is called the *graph parametrization*. The exact region of the plane that is expressed by this parametrization is dependent on the domain D. Note that this region is not necessarily a rectangular region.

Now, what if we only wanted the portion of this plane that was in the first octant? This gives us a triangular surface, with vertices at the points specified previously. In this case, we can project the surface into the xy-plane, and this will give us the domain of the parametrization. It turns out, that the domain D is also a triangle. We can represent D as the set $D = \{(x,y) \mid 0 \le y \le 1-x, 0 \le x \le 1\}$. While this is not the most complicated of domains to work with, it would still be nice to have a rectangular domain, especially if we then needed to integrate over this surface. The good news is, we can actually apply a coordinate transformation to achieve this! Consider the following 2D coordinate transformation:

$$T : [0,1] \times [0,1] \to D$$

$$T(u,v) = (u, v(1-u)) \tag{7}$$

This takes a square in the plane and maps it to a triangle in the plane, specifically the triangle oriented in the same way as our domain D. We can take this transformation and compose it with our original parametrization, giving us an entirely new parametrization! More precisely, let $\mathbf{\Psi} = \mathbf{\Phi} \circ T = \mathbf{\Phi}(T(u, v))$, so that we now have:

$$\mathbf{\Psi} : [0, 1] \times [0, 1] \to \mathbb{R}^3$$

$$\mathbf{\Psi}(u, v) = (u, v(1 - u), 1 - u - v + uv) \tag{8}$$

We now have a parametrization $\mathbf{\Psi}$ that takes the unit square $[0, 1] \times [0, 1]$ and maps it into the triangular planar surface we wanted. Take a moment to review what we just did, and then do the following exercise.

Exercise 2.8:

(a) On your answer sheet, draw the portion of $x + y + z = 1$ that lies in the first octant. It should look like a triangle. Be sure to clearly label everything.

(b) On your answer sheet, draw the region D of the parametrization $\mathbf{\Phi}$ above. Be sure to clearly label everything.

(c) Now, enter the parametrization $\mathbf{\Psi}$ into the Calcflow calculator and plot the surface, then examine the output graph. On your answer, describe what you see. Does the graph in Calcflow match your sketch in part (a)?

(d) Is the normal vector to the surface pointing upward (positive z direction) or downward (negative z direction)?

You just observed a very important conceptual point, that more than one parametrization can produce the same geometric surface. It is important to know which parametrization is the best choice for the problem at hand, e.g. when computing surface integrals. It is also important to note the behavior of the normal vectors, as we say that they determine a parametrization's orientation.

Hopefully you have a better visual grasp of parametrized curves and surfaces! Understanding the conceptual ideas behind parametrizations is important for examining the qualitative behavior of these functions. Building this intuition can greatly improve your physical understanding of line and surface integrals. You are encouraged to graph more parametrizations in Calcflow and have fun interacting with different surfaces!