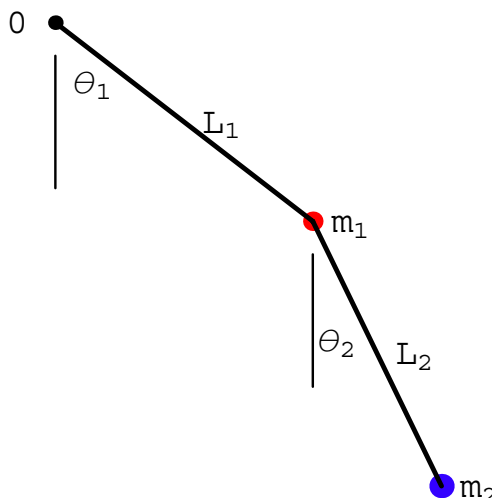


Double pendulum

We'll use the double pendulum as an example of a system governed by a non-linear pair of coupled differential equations, exhibiting chaotic behavior.

A double pendulum consists of a bob of mass m_1 attached to a fixed point O by a rigid massless wire of length L_1 to which is attached a second bob of mass m_2 by a rigid massless wire of length L_2 . Denote the angles made by the respective wires to the vertical by θ_1 , θ_2 .

```
Show[Graphics[{PointSize[.03], Point[{0, 100}],
  Hue[1.], PointSize[.04], Point[{20, 70}],
  Hue[.7], PointSize[.05], Point[{30, 30}],
  RGBColor[0, 0, 0], Thickness[.005], Line[{{0, 95}, {0, 75}}],
  Line[{{20, 65}, {20, 45}}],
  Thickness[.01], Line[{{0, 100}, {20, 70}}],
  Line[{{30, 30}, {20, 70}}],
  Text["O", {-3, 100}],
  Text[" $\theta_1$ ", {3, 91}],
  Text[" $\theta_2$ ", {22, 52}],
  Text[" $m_1$ ", {23, 70}],
  Text[" $m_2$ ", {33, 30}],
  Text[" $L_1$ ", {13, 85}],
  Text[" $L_2$ ", {28, 50}]
}, PlotRegion -> {{0.1, 0.9}, {0.1, 0.9}},
  AspectRatio -> 1, TextStyle -> {FontSize -> 14}
];
```



The dynamics of the double pendulum are given by the following differential equations (of Euler-Lagrange) for θ_1 and θ_2 . We change notation to avoid subscripts in the variables, using instead $u = \theta_1$ and $v = \theta_2$. Primes here denote differentiation with respect to time t .

$$\text{eq1} = (m_1 + m_2) L_1^2 u''[t] + m_2 L_1 L_2 v''[t] \text{Cos}[u[t] - v[t]] + m_2 L_1 L_2 (v'[t])^2 \text{Sin}[u[t] - v[t]] + L_1 g (m_1 + m_2) \text{Sin}[u[t]] == 0$$

$$g \text{Sin}[u[t]] L_1 (m_1 + m_2) + \text{Sin}[u[t] - v[t]] L_1 L_2 m_2 v'[t]^2 + L_1^2 (m_1 + m_2) u''[t] + \text{Cos}[u[t] - v[t]] L_1 L_2 m_2 v''[t] == 0$$

$$\text{eq2} = m_2 L_2^2 v''[t] + m_2 L_1 L_2 u''[t] \text{Cos}[u[t] - v[t]] - m_2 L_1 L_2 (u'[t])^2 \text{Sin}[u[t] - v[t]] + L_2 g m_2 \text{Sin}[v[t]] == 0$$

$$g \text{Sin}[v[t]] L_2 m_2 - \text{Sin}[u[t] - v[t]] L_1 L_2 m_2 u'[t]^2 + \text{Cos}[u[t] - v[t]] L_1 L_2 m_2 u''[t] + L_2^2 m_2 v''[t] == 0$$

There is no general closed form solution for these equations, but if we assume small, slow oscillations, replacing $\text{Sin}[w]$ by w and $\text{Cos}[w]$ by 1 (for small w) and squares of velocities by 0, we obtain a simpler pair of equations with the form

$$\text{leq1} = \text{eq1} /. \{\text{Cos}[x_] \rightarrow 1, \text{Sin}[x_] \rightarrow x, v'[t]^2 \rightarrow 0\}$$

$$g L_1 (m_1 + m_2) u[t] + L_1^2 (m_1 + m_2) u''[t] + L_1 L_2 m_2 v''[t] == 0$$

$$\text{leq2} = \text{eq2} /. \{\text{Cos}[x_] \rightarrow 1, \text{Sin}[x_] \rightarrow x, u'[t] \rightarrow 0\}$$

$$g L_2 m_2 v[t] + L_1 L_2 m_2 u''[t] + L_2^2 m_2 v''[t] == 0$$

■ Question 1.

Find the explicit solutions to the equations above assuming units are in feet, seconds, taking

$$m_1 = 3; m_2 = 1; L_1 = L_2 = 16; g = 32;$$

and assuming the initial conditions

$$\text{init} = \{u[0] == 1, u'[0] == 0, v[0] == -1, v'[0] == 0\}$$

$$\{u[0] == 1, u'[0] == 0, v[0] == -1, v'[0] == 0\}$$

(The solutions may be given in complex form. Before using the solutions to define actual functions, use `ComplexExpand` and `Simplify` to get compact real forms.)

■ Question 2.

Give a ten second animation of the solutions obtained in Question 1, using the colors from the first picture and a total of 32 frames.

■ Question 3.

Now go back to the original equations `eq1` and `eq2`, and the same values of masses and lengths as above, but with oscillations that are definitely not small. The equations must now be solved numerically. Give a 10 second animation (same colors as above) of the numerical solution under the initial conditions

```
largeinit = {u[0] == 1, u'[0] == 2, v[0] == -1, v'[0] == 1}
```

```
{u[0] == 1, u'[0] == 2, v[0] == -1, v'[0] == 1}
```

(You will probably find it useful to first define a function of u and v which plots the individual frames.)

■ Question 4.

Use the solution in the preceding question to give a plot of the location of the second bob (the blue one) over the same 25 second period.