# Global Optimization of Polynomial Functions and Applications 

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Abstract<br>Global Optimization of Polynomial Functions and Applications<br>by<br>Jiawang Nie<br>Doctor of Philosophy in Applied Mathematics<br>University of California, Berkeley<br>Professor James Demmel, Co-Chair<br>Professor Bernd Sturmfels, Co-Chair

This thesis discusses the global optimization problem whose objective function and constraints are all described by (multivariate) polynomials. The motivation is to find the global solution. For this problem, sum of squares (SOS) relaxations are able to get guaranteed lower bounds.

For unconstrained polynomial optimization problem, SOS relaxation generally only provides a lower bound. Sometimes this lower bound may be strictly smaller than the global minimum. In such situations, how can we do better? Much better lower bounds can be obtained if we apply SOS relaxation over the gradient ideal of the polynomial function. In fact, we can always get the exact lower bound, and have finite convergence, under some conditions that hold generically.

For constrained polynomial optimization, when the feasible set is compact, Lasserre's procedure is usually applied to to get a sequence of lower bounds. Under a
certain condition, these lower bounds will converge to the global minimum. However, no estimates of the speed of the convergence were available. For this purpose, we obtain the first upper bound on the convergence rate. When the feasible set is not compact, Lasserre's procedure may not converge. In such situations, better lower bounds can be obtained if we apply SOS relaxation over the Kuhn-Karush-Tucker (KKT) ideal. This new sequence of lower bounds has finite convergence under some generic conditions.

SOS relaxations can also be applied to minimize rational functions. The new features of SOS relaxations for this problem are studied in this thesis.

Polynomial optimization has wide applications. We studied the applications in shape optimization of transfer functions, finding minimum ellipsoid bounds for polynomial systems, solving the nearest GCD problem, maximum likelihood optimization, and sensor network localization.

James Demmel<br>Dissertation Committee Co-Chair

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## Chapter 1

## Introduction

Many problems in engineering and in science can be formulated as finding the optimal value of some objective function subject to some constraints on the decision variables. Finding the optimal decision variables is one main theme of the discipline of Mathematical Programming. There is a huge volume of work on the theory, algorithms and applications of Mathematical Programming.

When the objective and constraints are convex, the problem is called convex optimization. In this case, every local minimizer is also a global minimizer, the set of minimizers is convex, and specialized, very efficient algorithms are available. We refer to $[15,68,94,96]$ for the theory and methods for convex optimization.

When the objective and constraints are general nonlinear functions (often nonconvex), the optimization problem is called nonlinear programming. In such situations, a local minimizer might not be a global minimizer, and the set of minimizers may be nonconvex or even disconnected. There is much work on developing numerical methods to find local minimizers. We refer to $[3,7,5,58,65,75]$ for the theory
and methods for nonlinear programming.
Another important and active area of mathematical programming is global optimization - the theory and methods for finding global minimizers. Many global methods are based on branch-and-bound algorithms. They are often very expensive to implement. For computational efficiency, heuristic methods are developed. But the solutions returned by heuristic methods might not be globally optimal (or only globally optimal under some assumptions). We refer to [76, 77, 78] for the theory and methods of global optimization.

Usually, it is very difficult to find the global minimizer(s) of a general nonconvex nonlinear function. However, if the objective and constraints are described by multivariate polynomials (not necessarily convex), there are some certificates on the emptiness of polynomial systems from real algebra [13, 6, 27]. Therefore certificates of global solutions for polynomials can be computed and tractable algorithms can be developed.

This thesis concentrates on the special optimization problems whose objective and constraints are all polynomials. The main problem to be considered is of the form

$$
\begin{array}{rl}
f^{*}=\min _{x \in \mathbb{R}^{n}} & f(x) \\
\text { s.t. } & g_{1}(x) \geq 0, \cdots, g_{m}(x) \geq 0 \tag{1.0.2}
\end{array}
$$

where $f(x), g_{i}(x) \in \mathbb{R}[X]$, the ring of real multivariate polynomials in $X=\left(x_{1}, \cdots, x_{n}\right)$. Denote by $S$ be the feasible set defined by constraint (1.0.2). It is a basic closed semialgebraic set [13]. In this thesis, we do not have any convexity/concavity assumptions on $f(x)$ or $g_{i}(x)$. The goal is to find the global minimum $f^{*}$ and global minimizers (if
any). When (1.0.2) defines the whole space $\mathbb{R}^{n}$, the problem (1.0.1)-(1.0.2) becomes an unconstrained problem.

The formulation (1.0.1)-(1.0.2) contains quite a broad class of optimization problems, including some NP-hard problems.

- (Linear Programming (LP)) When $f(x)$ and all $g_{i}(x)$ are all affine functions, (1.0.1)-(1.0.2) becomes a linear programming of the form:

$$
\begin{aligned}
f^{*}=\min _{x \in \mathbb{R}^{n}} & c^{T} x \\
& \text { s.t. } \quad a_{i}^{T} x+b_{i} \geq 0, i=1, \cdots, m
\end{aligned}
$$

where $c$ and $a_{i}$ are all vectors in $\mathbb{R}^{n}$.

- (Nonconvex Quadratic Programming (QP)) When $f(x)$ and all $g_{i}(x)$ are all quadratic functions (not necessarily convex or concave), (1.0.1)-(1.0.2) becomes a QP:

$$
\begin{aligned}
f^{*}=\min _{x \in \mathbb{R}^{n}} & \frac{1}{2} x^{T} A_{0} x+a_{0}^{T} x \\
\text { s.t. } & \frac{1}{2} x^{T} A_{i} x+a_{i}^{T} x+c_{i} \geq 0, i=1, \cdots, m
\end{aligned}
$$

where $A_{i} \in \mathcal{S}^{n}$ are symmetric matrices and $a_{i} \in \mathbb{R}^{n}$ are vectors. To solve the nonconvex QP is NP-hard [64].

Many combinatorial problems can be formulated as polynomial optimization problems. Here are some examples.

- (Matrix Copositivity) A symmetric matrix $A \in \mathcal{S}^{n}$ is copositive if the quartic form $\sum_{i, j} A_{i j} x_{i}^{2} x_{j}^{2}$ is always nonnegative. This can be decided by finding
the global minimum of $\sum_{i, j} A_{i j} x_{i}^{2} x_{j}^{2}$ on the unit ball, which is a polynomial optimization problem. Testing the copositivity of a matrix is NP-hard [34].
- (Partition Problem) Given a vector $a \in \mathbb{Z}^{n}$, can it be partitioned into two parts with equal sums? This can be formulated as the polynomial optimization problem

$$
\min _{x \in \mathbb{R}^{n}}\left(a^{T} x\right)^{2}+\sum_{i}\left(x_{i}^{2}-1\right)^{2} .
$$

The global minimum is zero if and only if the vector $a$ can be partitioned into two parts with equal sums.

- (Maxcut problem) Given a graph $G=(V, E)$ with edge weights $W_{i j}$, how do we partition $G$ into two parts such that the edges connecting these two parts have maximum sum of weights? This can be formulated as

$$
\begin{aligned}
\max _{x \in \mathbb{R}^{n}} & \sum_{i, j} W_{i j}\left(x_{i}-x_{j}\right)^{2} \\
\text { s.t. } & x_{i}\left(x_{i}-1\right)=0, i=1, \cdots, n .
\end{aligned}
$$

### 1.1 Prior work

There has been a great deal of recent work in using Sum of Squares (SOS) relaxations to find global solutions to polynomial optimization problems. Here we give a very brief review of SOS methods.

### 1.1.1 Sum of squares (SOS) relaxations

The basic idea of relaxation is to approximate nonnegative polynomials by Sum of Squares (SOS) polynomials, i.e., those polynomials that can be written as
a summation of squares of other polynomials (e.g., $x_{1}^{2}-x_{1} x_{2}+x_{2}^{2}=\frac{1}{2} x_{1}^{2}+\frac{1}{2}\left(x_{1}-\right.$ $\left.x_{2}\right)^{2}+\frac{1}{2} x_{2}^{2}$ ). See Section 2.1 for an introduction to SOS polynomials.

To see the application of SOS relaxation in optimization, let us first consider problem (1.0.1) without any constraints. Assume the degree of $f(x)$ is even (otherwise $f(x)$ is unbounded from below). Obviously, the minimum $f^{*}$ equals the maximum lower bound of $f(x)$, i.e.,

$$
\begin{align*}
f^{*}=\max & \gamma  \tag{1.1.3}\\
\text { s.t. } & f(x)-\gamma \geq 0 \quad \forall x \in \mathbb{R}^{n} . \tag{1.1.4}
\end{align*}
$$

When $\operatorname{deg}(f) \geq 4$, it is NP-hard [67] to find $f^{*}$ and the minimizing values of the argument (if any). So in practice, one is interested in finding a lower bound of $f^{*}$ and extracting some approximate solutions. SOS relaxation is such a method, and it provides exact lower bounds in many cases. If we relax the nonnegativity condition (1.1.4) to an SOS condition, we get the convex optimization problem:

$$
\begin{align*}
f_{\text {sos }}^{*}=\max _{\gamma} & \gamma  \tag{1.1.5}\\
& \text { s.t. }  \tag{1.1.6}\\
& f(x)-\gamma \in \sum \mathbb{R}[X]^{2} .
\end{align*}
$$

Here $\sum \mathbb{R}[X]^{2}$ denotes the set of all polynomials that can be represented as sums of squares of polynomials. Notice that the decision variable above is $\gamma$ instead of $x \in \mathbb{R}^{n}$. The attractive property of (1.1.5)-(1.1.6) is that it can be transformed to Semidefinite Programming (SDP) problem, for which efficient algorithms exist (e.g., interior-point methods). Notice that $f(x)-\gamma$ being SOS implies that $f(x) \geq \gamma$ for any $x \in \mathbb{R}^{n}$. Thus $f_{\text {sos }}^{*}$ is a lower bound for $f(x)$, that is, $f_{\text {sos }}^{*} \leq f^{*}$. And $f_{\text {sos }}^{*}=f^{*}$ if and only if the polynomial $f(x)-f^{*}$ is SOS [52]. From Theorem 2.1.3 below, we
know in many occasions $f(x)-f^{*}$ may be nonnegative but not SOS. Thus the lower bound may not be exact.

Let $\ell=\lceil\operatorname{deg}(f) / 2\rceil$ and write $f(x)=\sum_{\alpha} f_{\alpha} x^{\alpha}$, where the indices $\alpha=$ $\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ are in $\mathbb{N}^{n}$ and $x^{\alpha}:=x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}$. Then the dual of problem (1.1.5)(1.1.6) is

$$
\begin{array}{ll}
\min _{y} & \sum_{\alpha} f_{\alpha} y_{\alpha} \\
\text { s.t. } & M_{\ell}(y) \succeq 0, \quad y_{(0, \cdots, 0)}=1 \tag{1.1.8}
\end{array}
$$

Here $y=\left(y_{\alpha}\right)$ is a monomial-indexed vector, i.e., indexed by integer vectors in $\mathbb{N}^{n}$, and $M_{\ell}(y)$ is the moment matrix generated by vector $y$ (see Definition 2.3.6 in Section 2.3). Here $A \succeq 0$ means the symmetric matrix $A$ is positive semidefinite.

For example, consider minimizing the polynomial

$$
f(x)=4 x_{1}^{2}-\frac{21}{10} x_{1}^{4}+\frac{1}{3} x_{1}^{6}+x_{1} x_{2}-4 x_{2}^{2}+4 x_{2}^{4}
$$

The contour of $f(x)$ is shown in Figure 1.1. We can see that $f(x)$ is highly nonconvex and has several local minimizers. Now we apply SOS relaxation to find its global minimum and minimizers. SOS relaxation gives exact lower bound

$$
f_{\text {sos }}^{*} \approx-1.03
$$

and extracts two points


$$
x^{*} \approx \pm(0.09,-0.71)
$$

We may plug $x^{*}$ into $f(x)$, evaluate it and find that $f\left(x^{*}\right)=f_{\text {sos. }}^{*}$. In other words, we find a point where the value of the polynomial equals its lower bound $f_{\text {sos }}^{*}$. Therefore, this lower bound $f_{s o s}^{*}$ equals the minimum of $f(x)$ and $x^{*}$ is a global minimizer. (More precisely, this is only true up to roundoff. One may construct examples with several points at which the global minimum is nearly attained; in such cases roundoff may prevent us from identifying the correct minimum, or the minimum value very precisely. We will not consider these possibilities in this thesis.)

For the constrained problem (1.0.1)-(1.0.2), SOS relaxations can also be applied in a similar way. This is the frequently used Lasserre's procedure in polynomial optimization. For a fixed integer $N$, one lower bound of $f^{*}$ can be obtained by the SOS relaxation:

$$
\begin{align*}
f_{N}^{*}=\max & \gamma  \tag{1.1.9}\\
\text { s.t. } & f(x)-\gamma \equiv \sigma_{0}(x)+\sigma_{1}(x) g_{1}(x)+\cdots+\sigma_{m}(x) g_{m}(x) \tag{1.1.10}
\end{align*}
$$

where $\operatorname{deg}\left(\sigma_{i} g_{i}\right) \leq 2 N$ and $\sigma_{i}$ are all SOS polynomials. The integer $N$ is called the degree of the SOS relaxation. The dual problem of (1.1.9)-(1.1.10) is

$$
\begin{array}{ll}
\min _{y} & \sum_{\alpha} f_{\alpha} y_{\alpha} \\
\text { s.t. } & M_{N}(y) \succeq 0, \quad y_{0}=1 \\
& M_{N-w_{i}}\left(g_{i} * y\right) \succeq 0 \tag{1.1.13}
\end{array}
$$

where $w_{i}=\left\lceil\operatorname{deg}\left(g_{i}\right) / 2\right\rceil$. Here $g_{i} * y$ denotes another monomial-indexed vector defined as

$$
\left(g_{i} * y\right)_{\alpha}=\sum_{\beta} g_{i, \beta} y_{\alpha+\beta} \quad \text { where } g_{i}(x)=\sum_{\beta} g_{i, \beta} x^{\beta} .
$$

Obviously, if $\gamma$ is feasible in (1.1.10), then $f(x)-\gamma$ must be nonnegative for all feasible points $x$. Thus every feasible $\gamma$ and $f_{N}^{*}$ are lower bounds of minimum $f^{*}$. Furthermore, the lower bound $f_{N}^{*}$ is increasing as $N$ increases, since the feasible region of $\gamma$ defined by (1.1.10) is increasing. Under the constraint qualification condition, i.e., supposing there exists $M>0$ and SOS polynomials $s_{i}(x)$ such that the following identity holds

$$
\begin{equation*}
M-\|x\|^{2} \equiv s_{0}(x)+s_{1}(x) g_{1}(x)+\cdots+s_{m}(x) g_{m}(x) \tag{1.1.14}
\end{equation*}
$$

where $\|x\|^{2}=\sum_{i=1}^{n} x_{i}^{2}$, Lasserre [52] showed convergence $\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$. If this condition holds, we can see that the set of feasible points must be bounded. But the converse might not be true (see Section 2.3). When this constraint qualification condition fails, it might happen that $\lim _{N \rightarrow \infty} f_{N}^{*}<f^{*}$.

For an example, consider the following optimization problem

$$
\begin{array}{ll}
\min _{x_{1}, x_{2}} & -x_{1}-x_{2} \\
\text { s.t. } & x_{2} \leq 2 x_{1}^{4}-8 x_{1}^{3}+8 x_{1}^{2}+2 \\
& x_{2} \leq 4 x_{1}^{4}-32 x_{1}^{3}+88 x_{1}^{2}-96 x_{1}+36 \\
& 0 \leq x_{1} \leq 3,0 \leq x_{2} \leq 4
\end{array}
$$

Now we apply Lasserre's procedure to find the global solution. Since the highest degree of the polynomials is 3 and $2 N \geq \max _{i} \operatorname{deg}\left(g_{i}\right)=3$, the relaxation order $N$ should be at least 2. For $N=2,3,4$, we solve the relaxation (1.1.9)-(1.1.10) and get the Results

| $N$ | $f_{N}^{*}$ | minimizer |
| :---: | :---: | :---: |
| 2 | -7 | no sol. extracted |
| 3 | -6.667 | no sol. extracted |
| 4 | -5.5080 | $(2.3295,3.1785)$ |

Figure 1.2: An example of Lasserre's procedure
in Figure 1.2. When $N=2$ or 3 , only a
lower bound is returned and no minimizer can be extracted. When $N=4$, one lower bound $f_{3}^{*} \approx-5.5080$ is returned and a feasible point $(2.3295,3.1785)$ is extracted from the dual solutions (see [41]). We plug this point into the objective polynomial, evaluate it and find that the value equals the lower bound $f_{3}^{*}$. This implies that the global minimum is $f^{*} \approx-5.5080$ and one global minimizer is $(2.3295,3.1785)$.

SOS relaxations are very attractive for solving some hard global optimization problems. There has been a lot of work in this exciting area. We refer to Kojima [47, 50, 114], Laurent [55, 45, 26, 56], Henrion [40, 41], Lasserre [52, 53, 54], Parrilo [79, 80, 81, 26], Schweighofer [101, 103], Sturmfels [80] and many others.

### 1.1.2 SOS methods versus symbolic methods

The optimization problem (1.0.1)-(1.0.2) can be formulated as a solvability problem of a particular system of polynomial equalities and inequalities, and therefore can be solved using a special case of symbolic methods like Quantifier Elimination (QE). Geometric algorithms for QE exist. We refer the reader to [6, 93]. (In fact, describing an algorithm for the global optimization problem using QE is given as Exercise 14.23 in [6].) The complexity of these methods (e.g., the QE algorithms in $[6,93])$ is

$$
\left((1+m) \max \left\{\operatorname{deg}(f), \operatorname{deg}\left(g_{1}\right), \cdots, \operatorname{deg}\left(g_{m}\right)\right\}\right)^{O(n)}
$$

This exponential complexity is consistent with the NP-hardness of general polynomial optimization problems. On the other hand, very few of these QE algorithms have been implemented and, to our best knowledge, they are rarely applied to polynomial optimization. Therefore it is of interest to find approximation methods for
polynomial optimization. SOS relaxation is a special approximation method. It has the advantage that it is easy to implement and can be solved numerically, since the relaxations (1.1.5)-(1.1.6) or (1.1.9)-(1.1.10) are SDPs.

### 1.2 Contributions of this thesis

The main contributions of this thesis are as follows:

1. When the feasible set $S$ is compact and the constraint qualification condition (1.1.14) holds, Lasserre's procedure converges, that is, the lower bounds $\left\{f_{N}^{*}\right\}$ from (1.1.9)-(1.1.10) converge to the minimum $f^{*}$. However, no estimates of the speed of the convergence were available. The author obtained the first upper bound on the convergence rate, in cooperation with M. Schweighofer. The convergence rate analysis is based on the degree bounds in Putinar's Theorem. This will be presented in Chapter 3.
2. To solve the unconstrained optimization (1.0.1), SOS relaxation (1.1.5)-(1.1.6) generally only provides a lower bound $f_{\text {sos }}^{*}$. But sometimes it happens that $f_{\text {sos }}^{*}<f^{*}$. In such situations, how can we get better lower bounds? A very good lower bound can be obtained if we apply SOS relaxation over the gradient ideal of the polynomial $f(x)$. In fact, we can always get the exact lower bound, and have finite convergence, under some conditions that hold generically. This will be presented in Chapter 4.
3. In the constrained optimization problem (1.0.1)-(1.0.2), we may not have convergence $\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$ if the semialgebraic set $S$ is not compact. How can we get better lower bounds in such situations? Similarly to the unconstrained
case, a very good lower bound can be obtained if we apply SOS relaxation over the Kuhn-Karush-Tucker (KKT) ideal. This lower bound is exact, and we have finite convergence, under some conditions that hold generically. This will be presented in Chapter 5.
4. There is a broader class of optimization problems which are described by rational functions. To our best knowledge, there is little work on the global optimization of rational function using SOS methods, even though there is a direct way to do so. We studied SOS methods for minimizing rational functions exploiting their special features. This will be presented in Chapter 6.
5. Polynomial optimization problems have wide applications. We studied applications in shape optimization of transfer functions, finding minimum ellipsoid bounds for polynomial systems, solving the nearest GCD problem, maximum likelihood optimization, and sensor network localization. These will be presented in Chapter 7.

## Chapter 2

## Some Basic Tools for

## Polynomial Optimization

The basic idea in polynomial optimization is to use sum of squares (SOS) representations of polynomials which are positive on some feasible sets defined by polynomial equalities and/or inequalities. The attractive property of the SOS representation is that it can be reduced to some particular semidefinite program (SDP). To study the SOS representation, we need some basic tools from algebraic geometry, real algebra and the theory of positive polynomials. They are the fundamentals of polynomial optimization.

### 2.1 SOS and nonnegative polynomials

A polynomial $p(x) \in \mathbb{R}[X]$ is nonnegative if $p(x) \geq 0$ for all $x \in \mathbb{R}^{n}$. A polynomial $p(x) \in \mathbb{R}[X]$ is a Sum Of Squares (SOS) if $p(x)=\sum_{i} q_{i}^{2}(x)$ for some finite number of $q_{i}(x) \in \mathbb{R}[X]$. Denote by $\sum \mathbb{R}[X]^{2}$ the set of all SOS polynomials. Obviously, if $p(x)$ is SOS, then $p(x)$ is always nonnegative.

Example 2.1.1. The following identity

$$
\begin{aligned}
& 3 \cdot\left(x_{1}^{4}+x_{2}^{4}+x_{3}^{4}+x_{4}^{4}-4 x_{1} x_{2} x_{3} x_{4}\right) \\
= & \left(x_{1}^{2}-x_{2}^{2}-x_{4}^{2}+x_{3}^{2}\right)^{2}+\left(x_{1}^{2}+x_{2}^{2}-x_{4}^{2}-x_{3}^{2}\right)^{2}+\left(x_{1}^{2}-x_{2}^{2}-x_{3}^{2}+x_{4}^{2}\right)^{2}+ \\
& \quad 2\left(x_{1} x_{4}-x_{2} x_{3}\right)^{2}+2\left(x_{1} x_{2}-x_{3} x_{4}\right)^{2}+2\left(x_{1} x_{3}-x_{2} x_{4}\right)^{2}
\end{aligned}
$$

shows that the polynomial $x_{1}^{4}+x_{2}^{4}+x_{3}^{4}+x_{4}^{4}-4 x_{1} x_{2} x_{3} x_{4}$ is SOS and hence nonnegative.

Example 2.1.2. The following polynomial

$$
2 x_{1}^{4}+2 x_{1}^{3} x_{2}-x_{1}^{2} x_{2}^{2}+5 x_{2}^{4}=\frac{1}{2}\left[\left(2 x_{1}^{2}-3 x_{2}^{2}+x_{1} x_{2}\right)^{2}+\left(x_{2}^{2}+3 x_{1} x_{2}\right)^{2}\right]
$$

is SOS and hence nonnegative.
The set $\sum \mathbb{R}[X]^{2}$ is a cone within the polynomial ring $\mathbb{R}[X]$, since the following three properties hold: (i) if $f, g \in \sum \mathbb{R}[X]^{2}$, then $f+g \in \sum \mathbb{R}[X]^{2}$; (ii) if $f, g \in \sum \mathbb{R}[X]^{2}$, then $f \cdot g \in \sum \mathbb{R}[X]^{2} ;$ (iii) for any $f \in \mathbb{R}[X], f^{2} \in \sum \mathbb{R}[X]^{2}$.

As we have seen, $p(x)$ being SOS implies that $p(x)$ is nonnegative. However, the converse may not be true. For instance, the Motzkin polynomial

$$
M(x):=x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+x_{3}^{6}-3 x_{1}^{2} x_{2}^{2} x_{3}^{2}
$$

is nonnegative, but not SOS [95]. The following theorem characterizes the relationship between nonnegative and SOS polynomials:

Theorem 2.1.3 (Hilbert, 1888). Let $\mathcal{P}_{n, d}$ be the set of all nonnegative polynomials in $n$ variables with degree at most d, and $\Sigma_{n, d}$ be the set of all SOS polynomials in $n$ variables with degree at most $d$. Then $\mathcal{P}_{n, d}=\Sigma_{n, d}$ if and only if $n=1$, or $d=2$, or $(n, d)=(2,4)$.

Denote by $m(x)$ the column vector of monomials up to degree $d$

$$
m(x)^{T}=\left[1, x_{1}, \cdots, x_{n}, x_{1}^{2}, x_{1} x_{2}, \cdots, x_{n}^{2}, x_{1}^{3}, \cdots, x_{n}^{d}\right] .
$$

Notice that the length of vector $m(x)$ is $\binom{n+d}{d}$. Let $p(x)$ be a polynomial with degree $2 d$. Then $p(x)$ is SOS if and only if $[81,105]$ there exists a real symmetric matrix $W \succeq 0$ of dimension $\binom{n+d}{d}$ such that the identity holds:

$$
p(x) \equiv m(x)^{T} W m(x) .
$$

Now we write $p(x)$ as $\sum_{\alpha \in P} p_{\alpha} x^{\alpha}$, where $P$ is a finite subset of $\mathbb{N}^{n}$, i.e., $P$ is the support of polynomial $p(x)$. Let $B_{\alpha}$ be the $\binom{n+d}{d}$-dimensional coefficient matrix of $x^{\alpha}$ in $m(x) \cdot m(x)^{T}$, i.e,.

$$
m(x) \cdot m(x)^{T}=\sum_{|\alpha| \leq 2 d} B_{\alpha} x^{\alpha} .
$$

Then we can see that $p(x)$ is SOS if and only if there exists a symmetric matrix $W$ such that

$$
W \succeq 0, \quad<W, B_{\alpha}>=p_{\alpha}, \quad \forall \alpha \in P .
$$

The inner product $\langle\cdot \cdot \cdot\rangle$ above is defined as $\langle A, B\rangle=\operatorname{Trace}(A B)$ for any two symmetric matrices $A, B$. Testing whether a polynomial is SOS or not can be done by solving a SDP feasibility problem. The condition that a polynomial is SOS poses an Linear Matrix Inequality (LMI) constraint on the coefficients of the polynomial. See [81] for more detailed descriptions of connections between SOS polynomials and SDP.

### 2.2 Elementary algebraic geometry

This section will introduce some basic tools in algebraic geometry. Readers may consult [13, 21, 22, 29] for more details.

A subset $I$ of $\mathbb{R}[X]$ is an ideal if $p \cdot q \in I$ for any $p \in I$ and $q \in \mathbb{R}[X]$. For $p_{1}, \ldots, p_{r} \in \mathbb{R}[X],\left\langle p_{1}, \cdots, p_{r}\right\rangle$ denotes the smallest ideal containing the $p_{i}$. Equivalently, $\left\langle p_{1}, \cdots, p_{r}\right\rangle$ is the set of all polynomials that are polynomial linear combinations of the $p_{i}$. Every ideal arises in this way:

Theorem 2.2.1 (Hilbert Basis Theorem). Every ideal $I \subset \mathbb{R}[x]$ has a finite generating set, i.e., $I=\left\langle p_{1}, \cdots, p_{\ell}\right\rangle$ for some $p_{1}, \cdots, p_{\ell} \in I$.

The variety of an ideal $I$ is the set of all common complex zeros of the polynomials in $I$ :

$$
V(I)=\left\{x \in \mathbb{C}^{n}: p(x)=0 \text { for all } p \in I\right\} .
$$

The subset of all real points in $V(I)$ is the real variety of $I$. It is denoted

$$
V^{\mathbb{R}}(I)=\left\{x \in \mathbb{R}^{n}: p(x)=0 \text { for all } p \in I\right\} .
$$

If $I=\left\langle p_{1}, \ldots, p_{r}\right\rangle$ then $V(I)=V\left(p_{1}, \ldots, p_{r}\right)=\left\{x \in \mathbb{C}^{n}: p_{1}(x)=\cdots=p_{r}(x)=\right.$ $0\}$. An ideal $I \subseteq \mathbb{R}[X]$ is zero-dimensional if its variety $V(I)$ is a finite set. This condition is much stronger than requiring that the real variety $V^{\mathbb{R}}(I)$ be a finite set. For example, $I=\left\langle x_{1}^{2}+x_{2}^{2}\right\rangle$ is not zero-dimensional, however the real variety $V^{\mathbb{R}}(I)=\{(0,0)\}$ consists of one point of the curve $V(I)$.

A variety $V \subseteq \mathbb{C}^{n}$ is irreducible if there do not exist two proper subvarieties $V_{1}, V_{2} \subseteq V$ such that $V=V_{1} \cup V_{2}$. Here "irreducible" means that the set of complex
zeros cannot be written as a proper union of subvarieties defined by real polynomials. Given a variety $V \subseteq \mathbb{C}^{m}$, the set of all polynomials that vanish on $V$ is an ideal

$$
I(V)=\{p \in \mathbb{R}[X]: p(u)=0 \text { for all } u \in V\}
$$

Given any ideal $I$ of $\mathbb{R}[X]$, its radical is the ideal

$$
\sqrt{I}=\left\{q \in \mathbb{R}[X]: q^{\ell} \in I \text { for some } \ell \in \mathbb{N}\right\} .
$$

Note that $I \subseteq \sqrt{I}$. We say that $I$ is a radical ideal if $\sqrt{I}=I$. Clearly, the ideal $I(V)$ defined by a variety $V$ is a radical ideal. The following theorems offer a converse to this observation:

## Theorem 2.2.2 (Hilbert's Weak Nullstellensatz).

If $I$ is an ideal in $\mathbb{R}[X]$ such that $V(I)=\emptyset$ then $1 \in I$.

Theorem 2.2.3 (Hilbert's Strong Nullstellensatz).
If $I$ is an ideal in $\mathbb{R}[X]$ then $I(V(I))=\sqrt{I}$.

### 2.3 Positive polynomials on semialgebraic sets

In polynomial optimization problems, we are often interested in a feasible set $S$ of the form

$$
S=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \cdots, g_{m}(x) \geq 0\right\}
$$

where $g_{i}(x) \in \mathbb{R}[X]$. Such an $S$ is called a basic closed semialgebraic set, and plays an important role in real algebraic geometry [13]. Notice that different polynomial tuples $\left(g_{1}(x), \cdots, g_{m}(x)\right)$ may define the same semialgebraic set $S$ in $\mathbb{R}^{n}$, but these tuples might have different algebraic properties (e.g., archimedeanness as defined
below). So when we refer to a semialgebraic set $S$, we assume that a polynomial tuple $\left(g_{1}(x), \cdots, g_{m}(x)\right)$ (which is often clear from the text) is associated with it. Given $S$ with polynomial tuple $\left(g_{1}(x), \cdots, g_{m}(x)\right)$, the preorder and linear cones associated with $S$ are defined as

$$
\begin{aligned}
\mathcal{P}(S) & =\left\{\sum_{\theta \in\{0,1\}^{m}} s_{\theta}(x) g_{1}^{\theta_{1}}(x) \cdots g_{\ell}^{\theta_{\ell}}(x) \mid s_{\theta} \in \Sigma \mathbb{R}[X]^{2}\right\} \\
\mathcal{M}(S) & =\left\{\sigma_{0}(x)+\sum_{j=1}^{\ell} g_{j}(x) \sigma_{j}(x) \mid \sigma_{i} \in \Sigma \mathbb{R}[X]^{2}\right\}
\end{aligned}
$$

$\mathcal{M}(S)$ is also called the quadratic module generated by $S$. We also denote by $\mathcal{P}(S)_{N}$ (and $\mathcal{M}(S)_{N}$ respectively) the subset of $\mathcal{P}(S)$ (and $\mathcal{M}(S)$ respectively) such that the degree in each summand is no greater than $N$.

A subset $M \subseteq \mathbb{R}[X]$ is called a quadratic module if it contains 1 and it is closed under addition and under multiplication with squares, i.e.,

$$
1 \in M, \quad M+M \subseteq M \quad \text { and } \quad \mathbb{R}[X]^{2} M \subseteq M
$$

A subset $T \subseteq \mathbb{R}[x]$ is called a preordering if it contains all squares in $\mathbb{R}[X]$ and it is closed under addition and multiplication, i.e.,

$$
\mathbb{R}[X]^{2} \subseteq T, \quad T+T \subseteq T \quad \text { and } \quad T T \subseteq T
$$

In other words, the preorderings are exactly the multiplicatively closed quadratic modules. In 1991, Schmüdgen [98] proved the following "Positivstellensatz" (a commonly used German term explained by the analogy with Hilbert's Nullstellensatz).

Theorem 2.3.1 (Schmüdgen's Positivstellensatz, [98]). Suppose the set $S$ is compact. Then every polynomial $p(x)$ which is positive on $S$ belongs to $\mathcal{P}(S)$.

The quadratic module $\mathcal{M}(S)$ is archimedean if there exists $\rho(x) \in \mathcal{M}(S)$ such that the set $\left\{x \in \mathbb{R}^{m}: \rho(x) \geq 0\right\}$ is compact, equivalently, if there exists $N \in \mathbb{N}$ such that $N-\sum_{i=1}^{n} x_{i}^{2} \in \mathcal{M}(S)$. The condition that $\mathcal{M}(\mathcal{S})$ is archimedean is also called Putinar's constraint qualification [91], or constraint qualification condition [52].

In particular, we see that $S$ is compact if and only if $\mathcal{P}(S)$ is archimedean. Unfortunately, $S$ might be compact without $\mathcal{M}(S)$ being archimedean (see [27, Example 6.3.1]). What has to be added to compactness of $S$ in order to ensure that $\mathcal{M}(S)$ is archimedean has been extensively investigated by Jacobi and Prestel [44, 27]. Now we can state the Positivstellensatz proved by Putinar [91] in 1993.

Theorem 2.3.2 (Putinar's Positivstellensatz, [91]). Suppose $\mathcal{M}(S)$ is archimedean. Then every polynomial $p(x)$ which is positive on $S$ belongs to $\mathcal{M}(S)$.

Remark 2.3.3. There are examples of compact $S$ for which $\mathcal{M}(S)$ is not archimedean and the conclusion of Putinar's Theorem does not hold. For instance, for $S=$ $\left\{\left(x_{1}, x_{2}\right): 2 x_{1}-1 \geq 0,2 x_{2}-1 \geq 0,1-x_{1} x_{2} \geq 0\right\}, M-x_{1}^{2}-x_{2}^{2} \notin \mathcal{M}(S)$ for any $M>0$. Otherwise, suppose $M-x_{1}^{2}-x_{2}^{2}=s_{0}+s_{1}\left(2 x_{1}-1\right)+s_{2}\left(2 x_{2}-1\right)+s_{3}\left(1-x_{1} x_{2}\right)$ for some SOS polynomials $s_{0}, s_{1}, s_{2}, s_{3}$. Since the highest degree on the left hand side is 2 , the highest degree in the right hand side must come from $s_{0}+s_{3}\left(1-x_{1} x_{2}\right)$. If $s_{3} \equiv 0$, the leading coefficient on the right hand side is nonnegative, which is a contradiction. If $s_{3} \neq 0$, since the leading coefficient of $s_{0}$ is nonnegative, the leading term must come from $s_{3}\left(1-x_{1} x_{2}\right)$. But the leading term of $s_{3}\left(1-x_{1} x_{2}\right)$ is of the form $x_{1}^{2 k+1} x_{2}^{2 \ell+1}$, which is a contradiction by comparison with the left hand side. In practice, if we know some integer $N$ such that $S$ is contained in the sphere $\left\{x \in \mathbb{R}^{n}: N-\sum_{i=1}^{n} x_{i}^{2} \geq 0\right\}$, we can add the redundant constraint $N-\sum_{i=1}^{n} x_{i}^{2} \geq 0$
to force $\mathcal{M}(S)$ to be archimedean.
Theorem 2.3.4 (Parrilo, [89]). Suppose $\mathcal{M}(S)$ contains an ideal $J=\left\langle p_{1}, \cdots, p_{r}\right\rangle$. If $J$ is a zero-dimensional radical ideal in $\mathbb{R}[X]$, then a polynomial $w(X) \in \mathbb{R}[X]$ is nonnegative on $S$ if and only if $w(X) \in \mathcal{M}(S)$.

Theorem 2.3.5 (Stengle's Positivstellensatz,[108]). Suppose $S$ and $\mathcal{P}(S)$ are defined as above. Then $S=\emptyset$ if and only if $-1 \in \mathcal{P}(S)$.

Definition 2.3.6 (Moment Matrix). Let $y=\left(y_{\alpha}\right)_{\alpha \in \mathbb{N}^{n}}$ be a sequence indexed by $\alpha=\left(\alpha_{1}, \cdots, \alpha_{n}\right)$, i.e., it is multi-indexed or monomial-indexed. Then the moment matrix induced by the vector $y$ is $M(y)=\left(y_{\alpha+\beta}\right)$, i.e., the $(\alpha, \beta)$-th entry of $M(y)$ is $y_{\alpha+\beta}$.

The $N$-th truncation matrix $M_{N}(y)$ of $M(y)$ is the leading submatrix such that $M_{N}(y)=\left(y_{\alpha+\beta}\right)_{|\alpha| \leq N,|\beta| \leq N}$. For instance, when $n=1$,

$$
M_{3}(y)=\left[\begin{array}{llll}
y_{0} & y_{1} & y_{2} & y_{3} \\
y_{1} & y_{2} & y_{3} & y_{4} \\
y_{2} & y_{3} & y_{4} & y_{5} \\
y_{3} & y_{4} & y_{5} & y_{6}
\end{array}\right]
$$

For a polynomial $h=\sum_{\beta} h_{\beta} x^{\beta}$, define the convolution of $h$ and vector $y$ as the new multi-indexed vector $(h * y)_{\alpha}=\sum_{\beta} h_{\beta} y_{\alpha+\beta}$. The multi-indexed vector $y$ is a moment sequence if there exists a measure $\mu$ on $\mathbb{R}^{n}$ such that $y_{\alpha}=\int_{\mathbb{R}^{n}} x^{\alpha} \mu(d x)$. See [24] for more details about moment theories.

When $n=1$, i.e., in case of univariate polynomials, there are some characterizations of polynomials which are nonnegative on some interval. We refer to [85]. We will use these results in Section 7.1.

Theorem 2.3.7 (Markov, Lukacs [59, 61, 84]). Let $q(t) \in \mathbb{R}[t]$ be a real polynomial of degree $n$. Let $n_{1}=\left\lfloor\frac{n}{2}\right\rfloor$ and $n_{2}=\left\lfloor\frac{n-1}{2}\right\rfloor$. If $q(t) \geq 0$ for all $t \geq 0$, then $q(t)=q_{1}(t)^{2}+t q_{2}(t)^{2}$ where $\operatorname{deg}\left(q_{1}\right) \leq n_{1}$ and $\operatorname{deg}\left(q_{2}\right) \leq n_{2}$.

Theorem 2.3.8 (Markov, Lukacs [59, 61, 84]). Let $q(t) \in \mathbb{R}[t]$ be a real polynomial. Suppose $q(t) \geq 0$ for all $t \in[a, b]$, then one of the following holds.

1. If $\operatorname{deg}(q)=n=2 m$ is even, then $q(t)=q_{1}(t)^{2}+(t-a)(b-t) q_{2}(t)^{2}$ where $\operatorname{deg}\left(q_{1}\right) \leq m$ and $\operatorname{deg}\left(q_{2}\right) \leq m-1$.
2. If $\operatorname{deg}(q)=n=2 m+1$ is odd, then $q(t)=(t-a) q_{1}(t)^{2}+(b-t) q_{2}(t)^{2}$ where $\operatorname{deg}\left(q_{1}\right) \leq m$ and $\operatorname{deg}\left(q_{2}\right) \leq m$.

## Chapter 3

## On the Convergence Rate of

## Lasserre's Procedure

Consider the constrained polynomial optimization problem

$$
f^{*}=\min _{x \in S} f(x)
$$

where $S=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \cdots, g_{m}(x) \geq 0\right\}$. Lasserre [52] proposed the SOS relaxation ( $k$ is the order)

$$
f_{k}^{*}=\sup \quad \gamma \quad \text { s.t. } \quad f(x)-\gamma \in \mathcal{M}_{2 k}(S) .
$$

Obviously each $f_{k}^{*}$ is a lower bound of $f^{*}$. Under condition (1.1.14), Lasserre [52] showed convergence $\lim _{N \rightarrow \infty} f_{k}^{*}=f^{*}$. A naturally arising question is how fast does $f_{k}^{*}$ converge to $f^{*}$ ? This chapter will give the first estimate on the convergence rate of $f_{k}^{*} \rightarrow f^{*}$ as $k$ goes to infinity. This is joint work with Markus Schweighofer [72].

### 3.1 Convergence rate of Lasserre's procedure

Let $S=S(\bar{g}):=\left\{x \in \mathbb{R}^{n}: g_{1}(x) \geq 0, \cdots, g_{m}(x) \geq 0\right\}$ be the feasible set, where $\bar{g}=\left(g_{1}, \cdots, g_{m}\right)$ is the tuple of polynomials defining the constraints. For convenience, set $g_{0}(x) \equiv 1$. Denote by $\mathcal{M}(S)\left(\mathcal{M}(S)_{N}\right)$ the (truncated) quadratic module generated by the tuple $\bar{g}$, i.e.,

$$
\begin{aligned}
\mathcal{M}(S) & =\left\{\sum_{j=0}^{m} \sigma_{j}(x) g_{j}(x) \mid \sigma_{j} \in \Sigma \mathbb{R}[X]^{2}\right\} \\
\mathcal{M}(S)_{N} & =\left\{\sum_{j=0}^{m} \sigma_{j}(x) g_{j}(x) \mid \sigma_{j} \in \Sigma \mathbb{R}[X]^{2}, \operatorname{deg}\left(\sigma_{j} g_{j}\right) \leq N\right\}
\end{aligned}
$$

For each integer $k$, we can see that

$$
\begin{equation*}
f_{k}^{*}:=\sup \left\{\gamma \in \mathbb{R} \mid f-\gamma \in \mathcal{M}(S)_{2 k}\right\} \in \mathbb{R} \cup\{-\infty\} \tag{3.1.1}
\end{equation*}
$$

The problem of finding $f_{k}^{*}$ is essentially a semidefinite program (SDP) whose size gets bigger as $k$ grows (see $[52,80,81]$ ). One can now solve a sequence of larger and larger semidefinite programs in order to get tighter and tighter lower bounds for $f^{*}$. Lasserre [52] showed convergence by applying Putinar's Positivstellensatz.

Indeed, it is easy to see that Putinar's theorem just says that the ascending sequence $\left(f_{k}^{*}\right)_{k \in \mathbb{N}}$ converges to $f^{*}$ under the condition that $\mathcal{M}(S)$ be archimedean (see Section 2.3). In this section, we will interpret our bound for Putinar's Positivstellensatz as a result about the speed of convergence of this sequence.

To get the bound for Putinar's Positivstellensatz, we will need a convenient measure of the size of the coefficients of a polynomial. For $\alpha \in \mathbb{N}^{n}$, we introduce the notation

$$
|\alpha|:=\alpha_{1}+\cdots+\alpha_{n} \quad \text { and } \quad x^{\alpha}:=x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}
$$

as well as the multinomial coefficient

$$
\binom{|\alpha|}{\alpha}:=\frac{|\alpha|!}{\alpha_{1}!\ldots \alpha_{n}!}
$$

For a polynomial $f(x)=\sum_{\alpha} a_{\alpha} x^{\alpha} \in \mathbb{R}[X]$ with coefficients $a_{\alpha} \in \mathbb{R}$, we set

$$
\|f\|:=\max _{\alpha} \frac{\left|a_{\alpha}\right|}{\binom{|\alpha|}{\alpha}}
$$

This defines a norm on the real vector space $\mathbb{R}[X]$ with convenient properties illustrated by Proposition 3.3.1 in Section 3.3. The following technical lemma estimates the value of a polynomial in term of its norm on the unit box, which will be needed in Section 3.3.

Lemma 3.1.1. For any polynomial $f \in \mathbb{R}[X]$ of degree $d \geq 1$ and all $x \in[-1,1]^{n}$,

$$
|f(x)| \leq 2 d n^{d}\|f\|
$$

Proof. Writing $f=\sum_{\alpha} a_{\alpha}\binom{|\alpha|}{\alpha} \bar{X}^{\alpha}\left(a_{\alpha} \in \mathbb{R}\right)$, we have $\|f\|=\max _{\alpha}\left|a_{\alpha}\right|$ and

$$
|f(x)|=\left|\sum_{\alpha} a_{\alpha}\binom{|\alpha|}{\alpha} x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}}\right| \leq \sum_{\alpha}\left|a_{\alpha}\right|\binom{|\alpha|}{\alpha}\left|x_{1}\right|^{\alpha_{1}} \cdots\left|x_{n}\right|^{\alpha_{n}}
$$

for all $x \in[-1,1]^{n}$. Using that $\left|a_{\alpha}\right| \leq\|f\|$ and $\left|x_{i}\right| \leq 1$, the multinomial identity now shows that $|f(x)| \leq\|f\| \sum_{k=0}^{d} n^{k} \leq(d+1) n^{d}\|f\| \leq 2 d n^{d}\|f\|$.

Now we are ready to prove the main theorem of this section, which is based on the complexity result in Theorem 3.2.4.

Theorem 3.1.1. For every polynomial tuple $\bar{g}$ defining an archimedean quadratic module $\mathcal{M}(S)$ and a set $\emptyset \neq S=S(\bar{g}) \subseteq(-1,1)^{n}$, there is some $c>0$ (depending on
$\bar{g})$ such that for all $f \in \mathbb{R}[X]$ of degree $d$ with minimum $f^{*}$ on $S$ and for all integers $k>\frac{1}{2} c \exp \left(\left(2 d^{2} n^{d}\right)^{c}\right)$, we have

$$
\left(f-f^{*}\right)+\frac{6 d^{3} n^{2 d}\|f\|}{\sqrt[c]{\log \frac{2 k}{c}}} \in \mathcal{M}(S)_{2 k}
$$

and hence

$$
0 \leq f^{*}-f_{k}^{*} \leq \frac{6 d^{3} n^{2 d}\|f\|}{\sqrt[c]{\log \frac{2 k}{c}}}
$$

where $f_{k}^{*}$ is defined as in (3.1.1).
Proof. Given $\bar{g}$, we choose $c>0$ as in Theorem 3.2.4. Now let $f \in \mathbb{R}[X]$ be of degree $d$ with minimum $f^{*}$ on $S$ and let

$$
\begin{equation*}
k>\frac{1}{2} c \exp \left(\left(2 d^{2} n^{d}\right)^{c}\right) \tag{3.1.2}
\end{equation*}
$$

be an integer. The case $d=0$ is trivial. We assume therefore $d \geq 1$. Note that $k>\frac{c}{2}$ and hence $\log (2 k / c)>0$. Setting

$$
\begin{equation*}
a:=\frac{6 d^{3} n^{2 d}\|f\|}{\sqrt[c]{\log \frac{2 k}{c}}}, \tag{3.1.3}
\end{equation*}
$$

all we have to prove is $h:=f-f^{*}+a \in \mathcal{M}(S)_{2 k}$ because the second claim follows from this. By our choice of $c$ and the observation $\operatorname{deg} h=\operatorname{deg} f=d$, it is enough to show that

$$
c \exp \left(\left(d^{2} n^{d} \frac{\|h\|}{a}\right)^{c}\right) \leq 2 k,
$$

or equivalently

$$
d^{2} n^{d}\|h\| \leq a \sqrt[c]{\log \frac{2 k}{c}}=6 d^{3} n^{2 d}\|f\|
$$

Observing that $\|h\| \leq\|f\|+\left|f^{*}\right|+a$, it suffices to show that

$$
\|f\|+\left|f^{*}\right|+a \leq 6 d n^{d}\|f\| .
$$

Lemma 3.1.1 tells us that $\left|f^{*}\right| \leq 2 d n^{d}\|f\|$ and we are thus reduced to verifying that

$$
a \leq\left(4 d n^{d}-1\right)\|f\|
$$

which is by (3.1.3) equivalent to

$$
6 d^{3} n^{2 d} \leq\left(4 d n^{d}-1\right) \sqrt[c]{\log \frac{2 k}{c}}
$$

By (3.1.2), it is finally enough to check that $6 d^{3} n^{2 d} \leq\left(4 d n^{d}-1\right)\left(2 d^{2} n^{d}\right)$.

The hypothesis that $S(\bar{g})$ is contained in the open unit hypercube is just a technical assumption, which helps avoid a more complicated bound (see Remark 3.2.1). If one does not insist on all the information given in Theorem 3.1.1, one can get a corollary which is easy to remember and still gives the most important part of the information.

Corollary 3.1.2. Suppose $\mathcal{M}(S)$ is archimedean, $S(\bar{g}) \neq \emptyset$ and $f \in \mathbb{R}[X]$. There is

- a constant $c>0$ depending only on $\bar{g}$ and
- $a$ constant $c^{\prime}>0$ depending on $\bar{g}$ and $f$
such that for $f^{*}$ and $f_{k}^{*}$ as defined in (3.1.1)),

$$
0 \leq f^{*}-f_{k}^{*} \leq \frac{c^{\prime}}{\sqrt[c]{\log \frac{2 k}{c}}} \quad \text { for all large } k \in \mathbb{N} .
$$

Proof. Without loss of generality, assume $f \neq 0$. Set $d:=\operatorname{deg} f$. Since $\mathcal{M}(S)$ is archimedean, $S$ is compact. We can hence choose a rescaling factor $r>0$ depending only on $\bar{g}$ such that $S(\bar{g}(r x)) \subseteq(-1,1)^{n}$. Here $\bar{g}(r x)$ denotes the tuple of rescaled polynomials $g_{i}(r \bar{X})$. Now Theorem 3.1.1 applied to $g(r x)$ instead of $\bar{g}$ yields $c>0$ that will together with $c^{\prime}:=6 d^{3} n^{2 d}\|f(r x)\|$ have the desired properties by simple scaling arguments.

Remark 3.1.3. The bound on the difference $f^{*}-f_{k}^{*}$ presented in this section is much worse than the corresponding one presented in [100, Section 2] which is based on preordering representations (i.e., where $f_{k}^{*}$ would be defined using $\mathcal{P}(S)$ instead of $\mathcal{M}(S)$. This raises the question whether it is after all not such a bad thing to use preordering (instead of quadratic module) representations for optimization though they involve the $2^{m}$ products $\bar{g}^{\delta}$ thereby letting the semidefinite programs get huge when $m$ grows. However, it is not known if Theorem 3.1.1 holds perhaps even with the bound from [100, Theorem 4]. Compare also [100, Remark 5].

### 3.2 On the complexity of Putinar's positivstellensatz

Denote by $\bar{g}:=\left(g_{1}, \ldots, g_{m}\right)$ the tuple of polynomials defining the constraints, and set $g_{0}:=1 \in \mathbb{R}[X]$ for convenience. The quadratic module $\mathcal{M}(S)$ generated by $\bar{g}$ is

$$
\begin{equation*}
\mathcal{M}(S):=\left\{\sum_{i=0}^{m} \sigma_{i} g_{i} \mid \sigma_{i} \in \sum \mathbb{R}[X]^{2}\right\} . \tag{3.2.4}
\end{equation*}
$$

Using the notation

$$
\bar{g}^{\delta}:=g_{1}^{\delta_{1}} \ldots g_{m}^{\delta_{m}}
$$

the preordering $\mathcal{P}(S)$ generated by $\bar{g}$ can be written as

$$
\begin{equation*}
\mathcal{P}(S):=\left\{\sum_{\delta \in\{0,1\}^{m}} \sigma_{\delta} \bar{g}^{\delta} \mid \sigma_{\delta} \in \sum \mathbb{R}[X]^{2}\right\}, \tag{3.2.5}
\end{equation*}
$$

i.e., $\mathcal{P}(S)$ is the quadratic module generated by the $2^{m}$ products of $g_{i}$. It is obvious that all polynomials lying in $\mathcal{P}(S) \supseteq \mathcal{M}(S)$ are nonnegative on the feasible set

$$
S:=\left\{x \in \mathbb{R}^{n} \mid g_{1}(x) \geq 0, \ldots, g_{m}(x) \geq 0\right\} .
$$

Sets of this form are called basic closed semialgebraic sets (see [13]). In 1991, Schmüdgen [98] proved the following "Positivstellensatz" (a commonly used German term explained by the analogy with Hilbert's Nullstellensatz).

Theorem 3.2.1 (Schmüdgen). Suppose the feasible set $S$ is compact. Then for every $f \in \mathbb{R}[X]$,

$$
f>0 \text { on } S \Longrightarrow f \in \mathcal{P}(S) .
$$

Under a certain extra property that $\mathcal{M}(S)$ is archimedean, i.e., $N-\|x\|^{2} \in$ $\mathcal{M}(S)$ for some $N>0$ (see Section 2.3), the Theorem 3.2.1 remains true if $\mathcal{P}(S)$ is replaced by $\mathcal{M}(S)$.

In particular, we see that $S=S(\bar{g})$ is compact if and only if $\mathcal{P}(S)$ is archimedean. Unfortunately, $S$ might be compact without $\mathcal{M}(S)$ being archimedean (see [27, Example 6.3.1]). What has to be added to compactness of $S$ in order to ensure that $\mathcal{M}(S)$ is archimedean has been extensively investigated by Jacobi and Prestel [44, 27]. Now we can state the Positivstellensatz proved by Putinar [91] in 1993.

Theorem 3.2.2 (Putinar). Suppose $\mathcal{M}(S)$ is archimedean. Then for every $f \in$ $\mathbb{R}[X]$,

$$
f>0 \text { on } S(\bar{g}) \Longrightarrow f \in \mathcal{M}(S)
$$

Both the proofs of Schmüdgen and Putinar use functional analysis and real algebraic geometry. They do not give information how to construct a representation of $f$ showing that $f$ lies in the preordering (an expression like (3.2.5) involving $2^{m}$ sums of squares) or the quadratic module (a representation like (3.2.4) with $m+1$ sums of squares).

Based on an old theorem of Pólya [83], new proofs of both Schmüdgen's and Putinar's Positivstellensatz have been given in [99, 101] which are to some extent constructive. By carefully analyzing a tame version of [101] and using an effective version of Pólya's theorem [87], upper bounds on the degrees of the sums of squares appearing in Schmüdgen's preordering representation have been obtained in [100]. The aim of this section is to prove bounds on Putinar's Theorem. They will depend on the same data but will be worse than the ones known for Schmüdgen's theorem.

For any $k \in \mathbb{N}$, define the truncated convex cones $\mathcal{P}(S)_{k}$ and $\mathcal{M}(S)_{k}$ in the finite-dimensional vector space $\mathbb{R}[X]_{\leq k}$ of polynomials of degree at most $k$ by setting

$$
\begin{aligned}
& \mathcal{P}(S)_{k}=\left\{\sum_{\delta \in\{0,1\}^{m}} \sigma_{\delta} \bar{g}^{\delta} \mid \sigma_{\delta} \in \sum \mathbb{R}[X]^{2}, \operatorname{deg}\left(\sigma_{\delta} \bar{g}^{\delta}\right) \leq k\right\} \subseteq \mathcal{P}(S) \cap \mathbb{R}[X]_{\leq k}, \\
& \mathcal{M}(S)_{k}=\left\{\sum_{i=0}^{m} \sigma_{\delta} \bar{g}^{\delta} \mid \sigma_{\delta} \in \sum \mathbb{R}[X]^{2}, \operatorname{deg}\left(\sigma_{\delta} \bar{g}^{\delta}\right) \leq k\right\} \quad \subseteq M(s) \cap \mathbb{R}[X]_{\leq k}
\end{aligned}
$$

We now recall the previously proved bound for Schmüdgen's theorem.
Theorem 3.2.3 ([100]). For all $\bar{g}$ defining a basic closed semialgebraic set $S(\bar{g})$ which is non-empty and contained in the open hypercube $(-1,1)^{n}$, there is some $c \geq 1$ (depending on $\bar{g}$ ) such that for all $f \in \mathbb{R}[X]$ of degree $d$ with

$$
f^{*}:=\min \{f(x) \mid x \in S(\bar{g})\}>0,
$$

we have $f \in \mathcal{P}(S)_{N}$ with

$$
N=c d^{2}\left(1+\left(d^{2} n^{d} \frac{\|f\|}{f^{*}}\right)^{c}\right) .
$$

In this chapter, we will prove the following bound for Putinar's theorem.

Theorem 3.2.4. For all $\bar{g}$ defining an archimedean quadratic module $\mathcal{M}(S)$ and a set $\emptyset \neq S(\bar{g}) \subseteq(-1,1)^{n}$, there is a constant $c>0$ (depending on $\bar{g}$ ) such that for all $f \in \mathbb{R}[X]$ of degree $d$ with

$$
f^{*}:=\min \{f(x) \mid x \in S(\bar{g})\}>0,
$$

we have $f \in \mathcal{M}(S)_{N}$ with

$$
N=c \exp \left(\left(d^{2} n^{d} \frac{\|f\|}{f^{*}}\right)^{c}\right) .
$$

Remark 3.2.1. In both theorems above, there have been additional assumptions made compared to the original theorems. But these are not very serious and have only been made to simplify the statements. For example, if $S=\emptyset$, then $-1 \in \mathcal{P}(S)_{k}$ for some $k \in \mathbb{N}$ by Schmüdgen's theorem. Therefore $4 f=(f+1)^{2}+(f-1)^{2}(-1) \in$ $\mathcal{P}(S)_{2 d+k}$ for each $f \in \mathbb{R}[X]$ of degree $d \geq 0$. The other hypothesis that $S(\bar{g})$ be contained in the open hypercube $(-1,1)^{n}$ is only a matter of rescaling by a linear (or affine linear) transformation on $\mathbb{R}^{n}$. For example, if $r>0$ is such that $S \subseteq(-r, r)^{n}$, then Theorem 3.2.3 remains true with $\|f\|$ replaced by $\|f(r x)\|$. Here it is important to note that the property that $\mathcal{M}(S)$ be archimedean is preserved under affine linear coordinate changes.

In both Theorem 3.2.3 and 3.2.4, the bound depends on three parameters:

- The description $\bar{g}$ of the basic closed semialgebraic set,
- the degree $d$ of $f$ and
- a measure of how close $f$ comes to have a zero on $S(\bar{g})$, namely $\|f\| / f^{*}$.

The main difference between the two bounds is the exponential function appearing in the degree bound for the quadratic module representation. It is an open problem whether this exponential function can be avoided. It could even be possible that the same bound for Schmüdgen's theorem holds also for Putinar's theorem. In view of the impact on the convergence rate of Lasserre's optimization procedure (see Section 3.1), this question seems very interesting for applications. Whereas the bound for the preordering representation cannot be improved significantly (see [109]), we are not sure whether this is possible for the quadratic module representation.

The dependence on the third parameter $\|f\| / f^{*}$ is consistent with the fact that the condition $f^{*}>0$ cannot be weakened to $f^{*} \geq 0$ in either Schmüdgen's nor Putinar's theorem. Under certain conditions (e.g., on the derivatives of $f$ ), both theorems can however be extended to nonnegative polynomials (see [97, 63]). With the partially constructive approach from [102] applied to representations of nonnegative polynomials with zeros, one might perhaps in the future gain bounds even for the case of nonnegative polynomials which depend on further data (for example the norm of the Hessian at the zeros).

In contrast to this, our more constructive approach yields information in what way the above bound depends on the two parameters $d$ and $\|f\| / f^{*}$. The constant $c$ depends on the description $\bar{g}$ of the semialgebraic set, but no explicit formula is given. For a concretely given $\bar{g}$, one could possibly determine a constant $c$ in Theorems 3.2.3 and 3.2.4 by a very tedious analysis of the proofs (cf. [100, Remark 10]).

### 3.3 The proof of Theorem 3.2.4

In this section, we give the proof of Theorem 3.2.4. The three main ingredients in the proof are

- the bound for Schmüdgen's theorem presented in Theorem 3.2.3 above,
- ideas from the (to some extent constructive) proof of Putinar's theorem in [101, Section 2] and
- the Lojasiewicz inequality from semialgebraic geometry.

We start with some simple facts from calculus.
Lemma 3.3.1. If $0 \neq f \in \mathbb{R}[X]$ has degree $d$, then

$$
|f(x)-f(y)| \leq\|x-y\|_{2} d^{2} n^{d-1} \sqrt{n}\|f\|
$$

for all $x, y \in[-1,1]^{n}$.
Proof. Denoting by $D f$ the derivative of $f$, by the mean value theorem, it is enough to show that

$$
\begin{equation*}
|D f(x)(e)| \leq d^{2} n^{d-1} \sqrt{n}\|f\| \tag{3.3.6}
\end{equation*}
$$

for all $x \in[-1,1]^{n}$ and $e \in \mathbb{R}^{n}$ with $\|e\|_{2}=1$. A small computation (compare the proof of Lemma 3.1.1) shows that

$$
\left|\frac{\partial f(x)}{\partial x_{i}}\right| \leq\|f\| \sum_{k=1}^{d} k\left(\left|x_{1}\right|+\cdots+\left|x_{n}\right|\right)^{k-1} \leq\|f\| \sum_{k=1}^{d} k n^{k-1} \leq\|f\| d^{2} n^{d-1}
$$

from which we conclude for all $x \in[-1,1]^{n}$ and $e \in \mathbb{R}^{n}$ with $\|e\|=1$,

$$
|D f(x)(e)|=\left|\sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_{i}} e_{i}\right| \leq \sum_{i=1}^{n}\left|\frac{\partial f(x)}{\partial x_{i}}\right| \cdot\left|e_{i}\right| \leq\|f\| d^{2} n^{d-1} \sum_{i=1}^{n}\left|e_{i}\right| .
$$

Because for a vector $e$ on the unit sphere in $\mathbb{R}^{n}, \sum_{i=1}^{n}\left|e_{i}\right|$ can reach at most $\sqrt{n}$, this implies (3.3.6).

Remark 3.3.2. For all $k \in \mathbb{N}$ and $y \in[0,1],(y-1)^{2 k} y \leq \frac{1}{2 k+1}$.
In [101, Lemma 2.3], it is shown that, if $C \subseteq \mathbb{R}^{n}$ is any compact set, $g_{i} \leq 1$ on $C$ for all $i$ and $f \in \mathbb{R}[X]$ is a polynomial with $f>0$ on $S(\bar{g})$, then there exists $\lambda \geq 0$ such that for all sufficiently large $k \in \mathbb{N}$,

$$
\begin{equation*}
f-\lambda \sum_{i=1}^{m}\left(g_{i}-1\right)^{2 k} g_{i}>0 \quad \text { on } C \text {. } \tag{3.3.7}
\end{equation*}
$$

The idea is that, to show $f \in \mathcal{M}(S)$, you first subtract another polynomial from $f$ which lies obviously in $\mathcal{M}(S)$ such that the difference can be proved to lie in $\mathcal{M}(S)$ as well. This other polynomial must necessarily be nonnegative on $S(\bar{g})$ but it should take on only very small values on $S(\bar{g})$ so that the difference is still positive on $S(\bar{g})$. On the region where it is outside but not too far away from $S(\bar{g})$, the polynomial you subtract should take large negative values so that the difference gets positive on this region outside of $S(\bar{g})$ (where $f$ itself might be negative). The hope is that the difference satisfies an improved positivity condition which will help us to show that it lies in $\mathcal{M}(S)$. To understand the lemma, it is helpful to observe that the pointwise limit for $k \rightarrow \infty$ of this difference, which is the left hand side of (3.3.10), is $f$ on $S(\bar{g})$ and $\infty$ outside of $S(\bar{g})$. This is the motivation of the following lemma:

Lemma 3.3.3. For all $\bar{g}$ such that $S:=S(\bar{g}) \cap[-1,1]^{n} \neq \emptyset$ and $g_{i} \leq 1$ on $[-1,1]^{n}$, there are $c_{0}, c_{1}, c_{2}>0$ with the following property:

For all polynomials $f \in \mathbb{R}[X]$ of degree $d$ with minimum $f^{*}>0$ on $S$, if we set

$$
\begin{equation*}
L:=d^{2} n^{d-1} \frac{\|f\|}{f^{*}}, \quad \lambda:=c_{1} d^{2} n^{d-1}\|f\| L^{c_{2}} \tag{3.3.8}
\end{equation*}
$$

and if $k \in \mathbb{N}$ satisfies

$$
\begin{equation*}
2 k+1 \geq c_{0}\left(1+L^{c_{0}}\right) \tag{3.3.9}
\end{equation*}
$$

then the inequality

$$
\begin{equation*}
f-\lambda \sum_{i=1}^{m}\left(g_{i}-1\right)^{2 k} g_{i} \geq \frac{f^{*}}{2} \tag{3.3.10}
\end{equation*}
$$

holds on $[-1,1]^{n}$.

Proof. By the Łojasiewicz inequality for semialgebraic functions (Corollary 2.6.7 in [13]), we can choose $c_{2}, c_{3}>0$ such that

$$
\begin{equation*}
\operatorname{dist}(x, S)^{c_{2}} \leq-c_{3} \min \left\{g_{1}(x), \ldots, g_{m}(x), 0\right\} \tag{3.3.11}
\end{equation*}
$$

for all $x \in[-1,1]^{n}$ where $\operatorname{dist}(x, S)$ denotes the Euclidean distance of $x$ to $S$. Set

$$
\begin{align*}
& c_{4}:=c_{3}(4 n)^{c_{2}}  \tag{3.3.12}\\
& c_{1}:=4 n c_{4} \tag{3.3.13}
\end{align*}
$$

and choose $c_{0} \in \mathbb{N}$ big enough to guarantee that

$$
\begin{align*}
& c_{0}\left(1+r^{c_{0}}\right) \geq 2(m-1) c_{4} r^{c_{2}} \quad \text { and }  \tag{3.3.14}\\
& c_{0}\left(1+r^{c_{0}}\right) \geq 4 m c_{1} r^{c_{2}+1} \tag{3.3.15}
\end{align*}
$$

for all $r \geq 0$. Now suppose $f \in \mathbb{R}[X]$ is of degree $d$ with minimum $f^{*}>0$ on $S$ and consider the set

$$
A:=\left\{x \in[-1,1]^{n} \left\lvert\, f(x) \leq \frac{3}{4} f^{*}\right.\right\}
$$

By Lemma 3.3.1, we get for all $x \in A$ and $y \in S$

$$
\frac{f^{*}}{4} \leq f(y)-f(x) \leq\|x-y\| d^{2} n^{d-1} \sqrt{n}\|f\| \leq\|x-y\| d^{2} n^{d}\|f\|
$$

Since this is valid for arbitrary $y \in S$, it holds that

$$
\frac{f^{*}}{4 d^{2} n^{d}\|f\|} \leq \operatorname{dist}(x, S)
$$

for all $x \in A$. We combine this now with (3.3.11) and get

$$
\min \left\{g_{1}(x), \ldots, g_{m}(x)\right\} \leq-\frac{1}{c_{3}}\left(\frac{f^{*}}{4 d^{2} n^{d}\|f\|}\right)^{c_{2}}
$$

for $x \in A$. We have omitted the argument 0 in the minimum which is here redundant because of $A \cap S=\emptyset$. By setting

$$
\begin{equation*}
\delta:=\frac{1}{c_{4} L^{c_{2}}}>0 \tag{3.3.16}
\end{equation*}
$$

where we define $L$ like in (3.3.8), and having a look at (3.3.12), we can rewrite this as

$$
\begin{equation*}
\min \left\{g_{1}(x), \ldots, g_{m}(x)\right\} \leq-\delta \tag{3.3.17}
\end{equation*}
$$

Define $\lambda$ and $k$ like in (3.3.8) and (3.3.9). For later use, we note

$$
\begin{equation*}
\lambda=c_{1} L^{c_{2}+1} f^{*} \tag{3.3.18}
\end{equation*}
$$

We claim now that

$$
\begin{align*}
f+\frac{\lambda \delta}{2} & \geq \frac{f^{*}}{2} \text { on }[-1,1]^{n}  \tag{3.3.19}\\
\frac{\delta}{2} & \geq \frac{m-1}{2 k+1} \quad \text { and }  \tag{3.3.20}\\
\frac{f^{*}}{4} & \geq \frac{\lambda m}{2 k+1} \tag{3.3.21}
\end{align*}
$$

Let us prove these claims. If we choose in Lemma 3.3.1 for $y$ a minimizer of $f$ on $S$, we obtain

$$
\left|f(x)-f^{*}\right| \leq \operatorname{diam}\left([-1,1]^{n}\right) d^{2} n^{d-1} \sqrt{n}\|f\|=2 \sqrt{n} d^{2} n^{d-1} \sqrt{n}\|f\|=2 d^{2} n^{d}\|f\|
$$

for all $x \in[-1,1]^{n}$, noting that the diameter of $[-1,1]^{n}$ is $2 \sqrt{n}$. In particular, we observe

$$
f \geq f^{*}-2 d^{2} n^{d}\|f\| \geq \frac{f^{*}}{2}-2 d^{2} n^{d}\|f\| \quad \text { on }[-1,1]^{n} .
$$

Together with the equation

$$
\frac{\lambda \delta}{2}=2 d^{2} n^{d}\|f\|,
$$

which is clear from (3.3.8), (3.3.13) and (3.3.16), this yields (3.3.19). Using (3.3.9), (3.3.14) and (3.3.16), we see that

$$
(2 k+1) \delta \geq c_{0}\left(1+L^{c_{0}}\right) \delta \geq 2(m-1) c_{4} L^{c_{2}} \delta=2(m-1)
$$

which is nothing else than (3.3.20). Finally, we exploit (3.3.9), (3.3.15) and (3.3.18), to see that

$$
(2 k+1) f^{*} \geq c_{0}\left(1+L^{c_{0}}\right) f^{*} \geq 4 m c_{1} L^{c_{2}+1} f^{*}=4 m \lambda,
$$

i.e., (3.3.21) holds. Now (3.3.19), (3.3.20) and (3.3.21) will enable us to show our claim (3.3.10). If $x \in A$, then in the sum

$$
\begin{equation*}
\sum_{i=1}^{m}\left(g_{i}(x)-1\right)^{2 k} g_{i}(x) \tag{3.3.22}
\end{equation*}
$$

at most $m-1$ summands are nonnegative. By Remark 3.3.2, these nonnegative summands add up to at most $(m-1) /(2 k+1)$. At least one summand is negative, in fact $\leq-\delta$ by (3.3.17). All in all, if we evaluate the left hand side of inequality (3.3.10) at a point $x \in A$, then we get
$f-\lambda \sum_{i=1}^{m}\left(g_{i}-1\right)^{2 k} g_{i} \geq f(x)-\lambda \frac{m-1}{2 k+1}+\lambda \delta \geq \underbrace{f(x)+\frac{\lambda \delta}{2}}_{\geq \frac{f^{*}}{2} \text { by }(3.3 .19)}+\lambda \underbrace{\left(\frac{\delta}{2}-\frac{m-1}{2 k+1}\right)}_{\geq 0 \text { by (3.3.20) }} \geq \frac{f^{*}}{2}$.
When we evaluate it in a point $x \in[-1,1]^{n} \backslash A$, all summands of the sum (3.3.22) might happen to be nonnegative. Again by Remark 3.3.2, they add up to at most
$m /(2 k+1)$. But at the same time, the definition of $A$ gives us a good lower bound on $f(x)$ so that the result is

$$
\geq \frac{3}{4} f^{*}-\lambda \frac{m}{2 k+1} \geq \frac{f^{*}}{2}+\underbrace{\frac{f^{*}}{4}-\frac{\lambda m}{2 k+1}}_{\geq 0 \text { by }(3.3 .21)} \geq \frac{f^{*}}{2} .
$$

Proposition 3.3.1. If $p, q \in \mathbb{R}[X]$ are both homogeneous (i.e., all of their respective monomials have the same degree), then $\|p q\| \leq\|p\|\|q\|$. For arbitrary $s \in \mathbb{N}$ and polynomials $p_{1}, \ldots, p_{s} \in \mathbb{R}[X]$, we have

$$
\left\|p_{1} \cdots p_{s}\right\| \leq\left(1+\operatorname{deg} p_{1}\right) \cdots\left(1+\operatorname{deg} p_{s}\right)\left\|p_{1}\right\| \cdots\left\|p_{s}\right\| .
$$

Proof. The statement for homogeneous $p$ and $q$ can be found in [100, Lemma 8]. The second claim follows from this by writing each $p_{i}$ as a sum $p_{i}=\sum_{k} p_{i k}$ of homogeneous degree $k$ polynomials $p_{i k}$. Multiply the $p_{i}$ by distributing out all such sums and apply the triangle inequality to the sum which arises in this way. Then use

$$
\left\|p_{1 k_{1}} \cdots p_{s k_{s}}\right\| \leq\left\|p_{1 k_{1}}\right\| \cdots\left\|p_{s k_{s}}\right\| \leq\left\|p_{1}\right\| \cdots\left\|p_{s}\right\| .
$$

Now factor out $\left\|p_{1}\right\| \cdots\left\|p_{s}\right\|$ and recombine the terms of the sum which now are all equal to 1 .

Lemma 3.3.4. For all $c_{1}, c_{2}, c_{3}>0$, there is $c>0$ such that

$$
c_{1} \exp \left(c_{2} r^{c_{3}}\right) \leq c \exp \left(r^{c}\right) \quad \text { for all } r \geq 0 .
$$

Proof. Choose any $c \geq c_{1} \exp \left(c_{2} 2^{c_{3}}\right)$ such that $c_{3} \leq c / 2$ and $c_{2} \leq 2^{c / 2}$. Then for $r \in[0,2]$,

$$
c_{1} \exp \left(c_{2} r^{c_{3}}\right) \leq c_{1} \exp \left(c_{2} 2^{c_{3}}\right) \leq c \leq c \exp \left(r^{c}\right)
$$

and for $r \geq 2$ (observing that $\left.c_{1} \leq c\right), c_{1} \exp \left(c_{2} r^{c_{3}}\right) \leq c \exp \left(2^{c / 2} r^{c / 2}\right) \leq c \exp \left(r^{c}\right)$.
We resume the discussion before Lemma 3.3.3. With regard to (3.3.10), we can for the moment concentrate on polynomials positive on the hypercube $[-1,1]^{n}$. If this hypercube could be described by a single polynomial inequality, i.e., if we had $[-1,1]^{n}=S(p)$ for some $p \in \mathbb{R}[X]$, then the idea would be to apply the bound for Schmüdgen's Positivstellensatz now. The clue is here that $p$ is a single polynomial and hence preordering and quadratic module representations are the same, i.e., $\mathcal{P}(p)=\mathcal{M}(p)$. The following lemma works around the fact that $[-1,1]^{n}=S(p)$ can only happen when $n=1$. We round the edges of the hypercube.

Lemma 3.3.5. Let $S \subseteq(-1,1)^{n}$ be compact. Then $1-\frac{1}{d}-\left(X_{1}^{2 d}+\ldots X_{n}^{2 d}\right)>0$ on $S$ for all sufficiently large $d \in \mathbb{N}$.

Proof. Consider for each $1 \leq d \in \mathbb{N}$ the set

$$
A_{d}:=\left\{x \in S \left\lvert\, x_{1}^{2 d}+\cdots+x_{n}^{2 d} \geq 1-\frac{1}{d}\right.\right\} .
$$

This gives a decreasing sequence $A_{1} \supseteq A_{2} \supseteq A_{3} \supseteq \ldots$ of compact sets whose intersection $\cap_{d=1}^{\infty} A_{d}$ is empty. By compactness, a finite subintersection is empty, i.e., $A_{d}=\emptyset$ for all large $d \in \mathbb{N}$.

Finally, we are ready to give the proof of Theorem 3.2.4.

Proof of Theorem 3.2.4. By a simple scaling argument, we may assume that $\left\|g_{i}\right\| \leq 1$ and $g_{i} \leq 1$ on $[-1,1]$ for all $i$. According to Lemma 3.3.5, we can choose $d_{0} \in \mathbb{N}$ such that

$$
p:=1-\frac{1}{d_{0}}-\left(X_{1}^{2 d}+\cdots+X_{n}^{2 d}\right)>0 \text { on } S(\bar{g}) .
$$

By Putinar's Theorem 3.2.2, we have $p \in \mathcal{M}(S)$ and therefore

$$
\begin{equation*}
p \in \mathcal{M}(S)_{d_{1}} \tag{3.3.23}
\end{equation*}
$$

for some $d_{1} \in \mathbb{N}$. Choose $d_{2} \in \mathbb{N}$ such that

$$
\begin{equation*}
1+\operatorname{deg} g_{i} \leq d_{2} \quad \text { for all } i \in\{1, \ldots, m\} \tag{3.3.24}
\end{equation*}
$$

Now we choose $c_{0}, c_{1}, c_{2}$ as in Lemma 3.3.3, define $L$ and $\lambda$ as in (3.3.8) and choose the smallest $k \in \mathbb{N}$ satisfying (3.3.9). Then

$$
\begin{equation*}
2 k+1 \leq c_{0}\left(1+L^{c_{0}}\right)+2 \tag{3.3.25}
\end{equation*}
$$

Let $c_{3} \geq 1$ denote the constant existing by Theorem 3.2.3 (which is called $c$ there and gives the bound for preordering representations of polynomials positive on $S(\bar{g})$ ). Using Lemma 3.3.4, it is easy to see that we can choose $c_{4}, c_{5}, c_{6}, c_{7}, c \geq 0$ satisfying

$$
\begin{align*}
c_{3} 2^{c_{3}} r^{2+2 c_{3}} n^{c_{3} r} & \leq c_{4}\left(\exp \left(c_{4} r\right)\right)  \tag{3.3.26}\\
2 r+2 c_{1} r^{c_{2}+1} d_{2}^{r\left(1+r^{c_{0}}\right)+1} & \leq c_{5} \exp \left(r^{c_{5}}\right)  \tag{3.3.27}\\
c_{4} \exp \left(2 c_{4} d_{2} r\left(1+r^{c_{0}}+3\right)\right) & \leq c_{6} \exp \left(r^{c_{6}}\right)  \tag{3.3.28}\\
c_{5}^{c_{3}} c_{6} \exp \left(c_{3} r^{c_{5}}+r^{c_{6}}\right) & \leq c_{7} \exp \left(r^{c_{7}}\right)  \tag{3.3.29}\\
c_{7} \exp \left(r^{c_{7}}\right)+d_{1} & \leq c \exp \left(r^{c}\right) \tag{3.3.30}
\end{align*}
$$

for all $r \geq 0$. Now let $f \in \mathbb{R}[X]$ be a polynomial of degree $d \geq 1$ with

$$
f^{*}:=\min \{f(x) \mid x \in S(\bar{g})\}>0
$$

We are going to apply Theorem 3.2.3 to

$$
h:=f-\lambda \sum_{i=1}^{m}\left(g_{i}-1\right)^{2 k} g_{i}
$$

By Lemma 3.3.3, (3.3.10) holds for this polynomial, in particular

$$
\begin{equation*}
h^{*}:=\min \{h(x) \mid x \in S(p)\} \geq \frac{f^{*}}{2} \tag{3.3.31}
\end{equation*}
$$

By Proposition 3.3 .1 and the definition of $d_{2}$ in (3.3.24),

$$
\begin{align*}
\|h\| & \leq\|f\|+\lambda d_{2}^{2 k+1}  \tag{3.3.32}\\
\operatorname{deg} h & \leq \max \left\{d,(2 k+1) d_{2}, 1\right\}=: d_{h} \tag{3.3.33}
\end{align*}
$$

By Theorem 3.2.3 (respectively the above choice of $c_{3} \geq 1$ ), we get

$$
\begin{equation*}
h \in \mathcal{P}(\{p\})_{k_{h}} \quad \text { where } k_{h}:=c_{3} d_{h}^{2}\left(1+d_{h}^{2} n^{d_{h}} \frac{\|h\|}{h^{*}}\right)^{c_{3}} \tag{3.3.34}
\end{equation*}
$$

Note that $\|h\| / h^{*} \geq 1$ since $0<h^{*} \leq h(0) \leq\|h\|$. We use this to simplify the degree bound in (3.3.34). Obviously

$$
\begin{equation*}
k_{h} \leq c_{3} d_{h}^{2}\left(2 d_{h}^{2} n^{d_{h}} \frac{\|h\|}{h^{*}}\right)^{c_{3}} \leq c_{3} 2^{c_{3}} d_{h}^{2+2 c_{3}} n^{c_{3} d_{h}}\left(\frac{\|h\|}{h^{*}}\right)^{c_{3}} \leq c_{4} \exp \left(c_{4} d_{h}\right)\left(\frac{\|h\|}{h^{*}}\right)^{c_{3}} \tag{3.3.35}
\end{equation*}
$$

by choice of $c_{4}$ in (3.3.26). Moreover, we have

$$
\begin{align*}
\frac{\|h\|}{h^{*}} \leq & \frac{2}{f^{*}}\left(\|f\|+\lambda d_{2}^{2 k+1}\right)=2 \frac{\|f\|}{f^{*}}+2 c_{1} d_{2}^{2 k+1} L^{c_{2}+1} \\
& \leq 2 L+2 c_{1} d_{2}^{2 k+1} L^{c_{2}+1}=2 L+2 c_{1} L^{c_{2}+1} d_{2}^{c_{0}\left(1+L^{c_{0}}\right)+1} \leq c_{5} \exp \left(L^{c_{5}}\right) \tag{3.3.36}
\end{align*}
$$

by $(3.3 .32),(3.3 .31),(3.3 .25),(3.3 .18)$ and by the choice of $c_{5}$ in (3.3.27). It follows
that

$$
\begin{align*}
d_{h} & \leq d(2 k+2) d_{2}  \tag{3.3.33}\\
& \leq d\left(c_{0}\left(1+L^{c_{0}}\right)+3\right) d_{2}  \tag{3.3.25}\\
& \leq 2 d_{2} d^{2} n^{d} \frac{\|f\|}{2 d n^{d}\|f\|}\left(c_{0}\left(1+L^{c_{0}}\right)+3\right) \\
& \leq 2 d_{2} d^{2} n^{d} \frac{\|f\|}{f^{*}}\left(c_{0}\left(1+L^{c_{0}}\right)+3\right)  \tag{byLemma3.1.1}\\
& \leq 2 d_{2} n L\left(c_{0}\left(1+(n L)^{c_{0}}+3\right)\right) \tag{3.3.8}
\end{align*}
$$

and therefore

$$
\begin{equation*}
c_{4} \exp \left(c_{4} d_{h}\right) \leq c_{6} \exp \left((n L)^{c_{6}}\right) \tag{3.3.37}
\end{equation*}
$$

for the constant $c_{6}$ chosen in (3.3.28). We now get

$$
\begin{array}{rlr}
k_{h} & \leq c_{4} \exp \left(c_{4} d_{h}\right)\left(\frac{\|h\|}{h^{*}}\right)^{c_{3}} & \quad \text { (by (3.3.35)) } \\
& \leq c_{6} \exp \left((n L)^{c_{6}}\right)\left(c_{5} \exp \left(L^{c_{5}}\right)\right)^{c_{3}} & \quad \text { (by (3.3.37) and (3.3.36)) } \\
& =c_{5}^{c_{3}} c_{6} \exp \left(c_{3}(n L)^{c_{5}}+(n L)^{c_{6}}\right) & \\
& \leq c_{7} \exp \left((n L)^{c_{7}}\right) & \text { (by choice of } c_{7} \text { in (3.3.29)). }
\end{array}
$$

Combining this with (3.3.34) and (3.3.23), i.e.,

$$
h \in \mathcal{P}(\{p\})_{c_{7} \exp \left((n L)^{c_{7}}\right)} \quad \text { and } \quad p \in \mathcal{M}(S)_{d_{1}}
$$

yields (by composing corresponding representations)

$$
h \in \mathcal{M}(S)_{c \exp \left((n L)^{c}\right)}
$$

according to the choice of $c$ in (3.3.30). Finally, we have that

$$
f=h+\lambda \sum_{i=1}^{m}\left(g_{i}-1\right)^{2 k} g_{i} \in \mathcal{M}(S)_{c \exp \left((n L)^{c}\right)}
$$

since

$$
\operatorname{deg}\left(\left(g_{i}-1\right)^{2 k} g_{i}\right) \leq d_{h} \leq k_{h} \leq c_{7} \exp \left((n L)^{c_{7}}\right) \leq c \exp \left((n L)^{c}\right)
$$

by choice of $d_{2}$ in (3.3.24), $d_{h}$ in (3.3.33), $k_{h}$ in (3.3.34) and c in (3.3.30).

## Chapter 4

## SOS Methods Based on the

## Gradient Ideal

As we have seen in Chapter 1, a very good lower bound $f_{\text {sos }}^{*}$ of the polynomial $f(x)$ can be found by applying SOS relaxations. But sometimes the SOS relaxation may not be exact, i.e., $f_{s o s}^{*}<f^{*}=\min _{x \in \mathbb{R}^{n}} f(x)$. In such situations, how can we improve the quality of the lower bound by applying some appropriately modified SOS relaxations?

This chapter will introduce a new method to get a sequence of better lower bounds $\left\{f_{N, \text { grad }}^{*}\right\}_{N=1}^{\infty}$. Every lower bound $f_{N, \text { grad }}^{*}$ is better than $f_{\text {sos }}^{*}$. The method combines the SOS relaxation and gradient of $f(x)$. It has the nice property that $\lim _{N \rightarrow \infty} f_{N, \text { grad }}^{*}=f^{*}$ whenever the minimum $f^{*}$ is attainable. Furthermore, the method also has finite convergence under some generic conditions, i.e., with probability one. A full version of this chapter is in [71].

### 4.1 Introduction

In this chapter, we consider the unconstrained polynomial optimization problem

$$
\begin{equation*}
f^{*}=\min _{x \in \mathbb{R}^{n}} f(x) \tag{4.1.1}
\end{equation*}
$$

where $x \in \mathbb{R}^{n}$ and $f(x)$ is a real multivariate polynomial of degree $d$. As is wellknown, the optimization problem (4.1.1) is NP-hard even when $d$ is fixed to be four [67]. A lower bound can be computed efficiently using the Sum Of Squares (SOS) relaxation

$$
\begin{equation*}
f_{\text {sos }}^{*}=\text { maximize } \gamma \text { subject to } f(x)-\gamma \succeq_{\text {sos }} 0, \tag{4.1.2}
\end{equation*}
$$

where the inequality $g \succeq_{\text {sos }} 0$ means that the polynomial $g$ is SOS, i.e. a sum of squares of other polynomials. See Section 2.1 for an elementary introduction to SOS polynomials. The relationship between (4.1.1) and (4.1.2) is as follows: $f_{\text {sos }}^{*} \leq f^{*}$ and the equality holds if and only if $f(x)-f^{*}$ is SOS.

Blekherman [12] recently showed that, for fixed even degree $d \geq 4$, the ratio between the volume of all nonnegative polynomials and the volume of all SOS polynomials tends to infinity when $n$ goes to infinity. In other words, for large $n$, there are many more nonnegative polynomials than SOS polynomials. For dealing with the challenging case when $f_{\text {sos }}^{*}<f^{*}$, Lasserre [52] proposed finding a sequence of lower bounds for $f(x)$ in some large ball $\left\{x \in \mathbb{R}^{n}:\|x\|_{2} \leq R\right\}$. His approach is based on the result [4] that SOS polynomials of all possible degrees are dense among polynomials which are nonnegative on some compact set. This sequence converges to $f^{*}$ when the degrees of the polynomials introduced in the algorithm go to infinity. But it may not converge in finitely many steps, and the degrees of the required auxiliary polynomials can be very large.

In this chapter, we introduce a method which can find the global minimum and terminate in finitely many steps, under some weak assumptions. Our point of departure is the observation that all local minima and global minima of (4.1.1) occur at points in the real gradient variety

$$
\begin{equation*}
V_{g r a d}^{\mathbb{R}}(f)=\left\{u \in \mathbb{R}^{n}:(\nabla f)(u)=0\right\} \tag{4.1.3}
\end{equation*}
$$

The gradient ideal of $f$ is the ideal in $\mathbb{R}\left[x_{1}, \ldots, x_{n}\right]$ generated by all partial derivatives of $f$ :

$$
\begin{equation*}
\mathcal{I}_{\text {grad }}(f)=\langle\nabla f(x)\rangle=\left\langle\frac{\partial f}{\partial x_{1}}, \frac{\partial f}{\partial x_{2}}, \cdots, \frac{\partial f}{\partial x_{n}}\right\rangle . \tag{4.1.4}
\end{equation*}
$$

There are several recent references on minimizing polynomials by way of the gradients. Hanzon and Jibetean [39] suggest applying perturbations to $f$ to produce a sequence of polynomials $f_{\lambda}$ (for small $\lambda$ ) with the property that the gradient variety of $f_{\lambda}$ is finite and the minima $f_{\lambda}^{*}$ converge to $f^{*}$ as $\lambda$ goes to 0 . Laurent [55] and Parrilo [89] discuss the more general problem of minimizing a polynomial subject to polynomial equality constraints (not necessarily partial derivatives). Under the assumption that the variety defined by the equations is finite, the matrix method proposed in [55] has finite convergence even if the ideal generated by the constraints is not radical. Building on [39, 55], Jibetean and Laurent [45] propose to compute $f^{*}$ by solving a single SDP, provided the gradient variety is finite (radicalness is not necessary).

The approach of this chapter is to find a lower bound $f_{g r a d}^{*}$ for (4.1.1) by requiring $f-f_{\text {grad }}^{*}$ to be SOS in the quotient ring $\mathbb{R}[X] / \mathcal{I}_{\text {grad }}(f)$ instead of in $\mathbb{R}[X]$. Let $\mathbb{R}[X]_{m}$ denote the vector space of polynomials with degree up to $m$. We consider
the SOS relaxation

$$
\begin{array}{ll}
f_{N, \text { grad }}^{*}=\sup & \gamma \\
\text { s.t. } & f(x)-\gamma-\sum_{j=1}^{n} \phi_{j}(x) \frac{\partial f}{\partial x_{j}} \in \sum \mathbb{R}[X]^{2} \\
& \phi_{j}(x) \in \mathbb{R}[X]_{2 N-d+1} . \tag{4.1.7}
\end{array}
$$

Here $d=\operatorname{deg}(f), N$ is an fixed integer, and $\sum \mathbb{R}[X]^{2}$ denotes the cone of SOS polynomials. Obviously $f_{N, g r a d}^{*} \leq f^{*}$ for all $2 N \geq d$, provided $f^{*}$ is attained at one point. The lower bound $f_{N, \text { grad }}^{*}$ is monotonically increasing as $N$ increases, since the feasible domain of (4.1.5) is also increasing, i.e.,

$$
\begin{equation*}
\cdots f_{N, \text { grad }}^{*} \leq f_{N+1, \text { grad }}^{*} \leq \cdots \leq f^{*} \tag{4.1.8}
\end{equation*}
$$

The convergence of these lower bounds is summarized in the following theorem.

Theorem 4.1.1. Let $f(x)$ be a polynomial in $n$ real variables which attains its infimum $f^{*}$ over $\mathbb{R}^{n}$. Then $\lim _{N \rightarrow \infty} f_{N, \text { grad }}^{*}=f^{*}$. Furthermore, if the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ is radical, then $f^{*}$ is attainable, i.e., there exists an integer $N$ such that $f_{N, \text { grad }}^{*}=f_{g r a d}^{*}=f^{*}$.

The proof of this theorem will be given in Section 4.3.

### 4.2 Polynomials over their gradient varieties

Consider a polynomial $f \in \mathbb{R}[X]$ and its gradient ideal $\mathcal{I}_{\text {grad }}(f)$ as in (4.1.4). A natural idea in solving (4.1.1) is to apply Theorem 2.3 .4 to the ideal $I=\mathcal{I}_{\text {grad }}(f)$, since the minimum of $f$ over $\mathbb{R}^{n}$ will be attained at a subset of $V^{\mathbb{R}}(I)$ if
it is attained at all. However, the hypothesis of Theorem 2.3.4 requires that $I$ be zerodimensional, which means that the complex variety $V_{\text {grad }}(f)=V(I)$ of all critical points must be finite. Our results in this section remove this restrictive hypothesis. We shall prove that every nonnegative $f$ is $\operatorname{SOS}$ in $\mathbb{R}[X] / I$ as long as the gradient ideal $I=\mathcal{I}_{\text {grad }}(f)$ is radical.

Theorem 4.2.1. Assume that the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ is radical. If the real polynomial $f(x)$ is nonnegative over $V_{g r a d}^{\mathbb{R}}(f)$, then there exist real polynomials $q_{i}(x)$ and $\phi_{j}(x)$ so that

$$
\begin{equation*}
f(x)=\sum_{i=1}^{s} q_{i}(x)^{2}+\sum_{j=1}^{n} \phi_{j}(x) \frac{\partial f}{\partial x_{j}} . \tag{4.2.9}
\end{equation*}
$$

The proof of this theorem will be based on the following two lemmas. The first is a generalization of the Lagrange Interpolation Theorem from sets of points to disjoint varieties.

Lemma 4.2.2. Let $V_{1}, \ldots, V_{r}$ be pairwise disjoint varieties in $\mathbb{C}^{n}$. Then there exist polynomials $p_{1}, \ldots, p_{r} \in \mathbb{R}[X]$ such that $p_{i}\left(V_{j}\right)=\delta_{i j}$, where $\delta_{i j}$ is the Kronecker delta function.

Proof. Our definition of variety requires that each $V_{j}$ is actually defined by polynomials with real coefficients. If $I_{j}=I\left(V_{j}\right)$ is the radical ideal of $V_{j}$ then we have $V_{j}=V\left(I_{j}\right)$. Fix an index $j$ and let $W_{j}$ denote the union of the varieties $V_{1}, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{r}$. Then

$$
I\left(W_{j}\right)=I_{1} \cap \cdots \cap I_{j-1} \cap I_{j+1} \cap \cdots \cap I_{r} .
$$

Our hypothesis implies that $V_{j} \cap W_{j}=\emptyset$. By Hilbert's Weak Nullstellensatz (Theorem 2.2.2), there exist polynomials $p_{j} \in I\left(W_{j}\right)$ and $q_{j} \in I_{j}$ such that $p_{j}+q_{j}=1$. This
identity shows that $p_{j}\left(V_{j}\right)=1$ and $p_{j}\left(V_{k}\right)=0$ for $k \neq j$. Hence the $r$ polynomials $p_{1}, \ldots, p_{r}$ have the desired properties.

Now consider the behavior of the polynomial $f(x)$ over its gradient variety $V_{\text {grad }}(f)$. We make use of the fact that $V_{\text {grad }}(f)$ is a finite union of irreducible subvarieties $([13, \S 2])$.

Lemma 4.2.3. Let $W$ be an irreducible subvariety of $V_{\text {grad }}(f)$ and suppose that $W$ contains at least one real point. Then $f(x)$ is constant on $W$.

Proof. If we replace our polynomial ring $\mathbb{R}[X]$ by $\mathbb{C}[X]$ then $W$ either remains irreducible or it becomes a union of two irreducible components $W=W_{1} \cup W_{2}$ which are exchanged under complex conjugation. Let us first consider the case when $W$ is irreducible in the Zariski topology induced by $\mathbb{C}[X] . W$ is connected in $\mathbb{C}^{n}$ (see [104]). Any two points in a connected algebraic variety in $\mathbb{C}^{n}$ can be connected by an algebraic curve. This curve may be singular, but it is a projection of some nonsingular curve. Let $x, y$ be two arbitrary points in $W$. Hence there exists a smooth path $\varphi(t)(0 \leq t \leq 1)$ lying inside $W$ such that $x=\varphi(0)$ and $y=\varphi(1)$. By the Mean Value Theorem of Calculus, it holds that for some $t^{*} \in(0,1)$

$$
f(y)-f(x)=\nabla f\left(\varphi\left(t^{*}\right)\right)^{T} \varphi^{\prime}\left(t^{*}\right)=0
$$

since $\nabla f$ vanishes on $W$. We conclude that $f(x)=f(y)$, and hence $f$ is constant on $W$.

Now consider the case when $W=W_{1} \cup W_{2}$ where $W_{1}$ and $W_{2}$ are exchanged by complex conjugation. We had assumed that $W$ contains a real point $p$. Since $p$ is fixed under complex conjugation, $p \in W_{1} \cap W_{2}$. By the same argument as above, $f(x)=f(p)$ for all $x \in W$.

Proof of Theorem 4.2.1. Consider the irreducible decomposition of $V_{\text {grad }}(f)$. We group together all components which have no real point and all components on which $f$ takes the same real value. Hence the gradient variety has a decomposition

$$
\begin{equation*}
V_{\text {grad }}(f)=W_{0} \cup W_{1} \cup W_{2} \cup \cdots \cup W_{r} \tag{4.2.10}
\end{equation*}
$$

such that $W_{0}$ has no real point and $f$ is a real constant on each other variety $W_{i}$, say,

$$
f\left(W_{1}\right)>f\left(W_{2}\right)>\cdots>f\left(W_{r}\right) \geq 0
$$

The varieties $W_{i}$ are pairwise disjoint, so by Lemma 4.2 .2 there exist polynomials $p_{i} \in \mathbb{R}[X]$ such that $p_{i}\left(W_{j}\right)=\delta_{i j}$. By Theorem 2.3 .5 , there exists a sum of squares $\operatorname{sos}(x) \in \mathbb{R}[X]$ such that $f(x)=\operatorname{sos}(x)$ for all $x \in W_{0}$. Using the non-negative real numbers $\alpha_{j}:=\sqrt{f\left(W_{j}\right)}$, we define

$$
\begin{equation*}
q(x)=\operatorname{sos}(x) \cdot p_{0}^{2}(x)+\sum_{i=1}^{r}\left(\alpha_{i} \cdot p_{i}(x)\right)^{2} \tag{4.2.11}
\end{equation*}
$$

By construction, $f(x)-q(x)$ vanishes on the gradient variety $V_{\text {grad }}(f)$. The gradient ideal $\mathcal{I}_{\text {grad }}(f)$ was assumed to be radical. Using Hilbert's Strong Nullstellensatz (Theorem 2.2.3), we conclude that $f(x)-q(x)$ lies in $\mathcal{I}_{\text {grad }}(f)$. Hence the desired representation (4.2.9) exists.

In Theorem 4.2.1, the assumption that $\mathcal{I}_{\text {grad }}(f)$ is radical cannot be removed. This is shown by the following counterexample.

Example 4.2.4. Let $n=3$ and consider the polynomial

$$
f(x, y, z)=x^{8}+y^{8}+z^{8}+M(x, y, z)
$$

where $M(x, y, z)=x^{4} y^{2}+x^{2} y^{4}+z^{6}-3 x^{2} y^{2} z^{2}$ is the Motzkin polynomial, which is is non-negative but not a sum of squares in $\mathbb{R}[X] / \mathcal{I}_{\text {grad }}(f)$ (see [71]).

In cases (like Example 4.2.4) when the gradient ideal is not radical, the following still holds.

Theorem 4.2.5. Let $f(x) \in \mathbb{R}[X]$ be a polynomial which is strictly positive on its real gradient variety $V_{\text {grad }}^{\mathbb{R}}(f)$, Then $f(x)$ is SOS modulo its gradient ideal $\mathcal{I}_{\text {grad }}(f)$.

Proof. We retain the notation from the proof of Theorem 4.2.1. Consider the decomposition of the gradient variety in (4.2.10). Each $W_{i}$ is the union of several irreducible components. Consider a primary decomposition of the ideal $\mathcal{I}_{\text {grad }}(f)$, and define $J_{i}$ to be the intersection of all primary ideals in that decomposition whose variety is contained in $W_{i}$. Then we have $\mathcal{I}_{\text {grad }}(f)=J_{0} \cap J_{1} \cap \cdots \cap J_{r}$, where $W_{i}=V\left(J_{i}\right)$ and, since the $W_{i}$ are pairwise disjoint, we have $J_{i}+J_{k}=\mathbb{R}[X]$ for $i \neq k$. The Chinese Remainder Theorem [29] implies

$$
\begin{equation*}
\mathbb{R}[X] / \mathcal{I}_{\text {grad }}(f) \simeq \mathbb{R}[X] / J_{0} \times \mathbb{R}[X] / J_{1} \times \cdots \times \mathbb{R}[X] / J_{r} \tag{4.2.12}
\end{equation*}
$$

Here $V^{\mathbb{R}}\left(J_{0}\right)=\emptyset$. Hence, by Theorem 2.3.5, there exists a sum of squares $\operatorname{sos}(x) \in$ $\mathbb{R}[X]$ such that $f(x)-\operatorname{sos}(x) \in J_{0}$. By assumption, $\alpha_{i}^{2}=f\left(W_{i}\right)$ is strictly positive for all $i \geq 1$. The polynomial $f(x) / \alpha_{i}^{2}-1$ vanishes on $W_{i}$. By Hilbert's Strong Nullstellensatz, there exists an integer $m>0$ such that $\left(f(x) / \alpha_{i}^{2}-1\right)^{m}$ is in the ideal $J_{i}$. We construct a square root of $f(x) / \alpha_{i}^{2}$ in the residue ring $\mathbb{R}[X] / J_{i}$ using the familiar Taylor series expansion for the square root function:

$$
\left(1+\left(f(x) / \alpha_{i}^{2}-1\right)\right)^{1 / 2}=\sum_{k=0}^{m-1}\binom{1 / 2}{k}\left(f(x) / \alpha_{i}^{2}-1\right)^{k} \bmod J_{i} .
$$

Multiplying this polynomial by $\alpha_{i}$, we get a polynomial $q_{i}(x)$ such that $f(x)-q_{i}^{2}(x)$ is in the ideal $J_{i}$. We have shown that $f(x)$ maps to the vector $\left(\operatorname{sos}(x), q_{1}(x)^{2}, q_{2}(x)^{2}, \ldots, q_{r}(x)^{2}\right)$ under the isomorphism (4.2.12). That vector is clearly a sum of squares in the ring
on the right hand side of (4.2.12). We conclude that $f(x)$ is a sum of squares in $\mathbb{R}[X] / \mathcal{I}_{\text {grad }}(f)$.

Example 4.2.6. Let $f$ be the polynomial in Example 4.2 .4 and let $\epsilon$ be any positive constant. Theorem 4.2 .5 says that $f+\epsilon$ is $\operatorname{SOS}$ modulo $\mathcal{I}_{\text {grad }}(f)$. Such a representation can be found by symbolic computation as follows. Primary decomposition over $\mathbb{Q}[x, y, z]$ yields

$$
\mathcal{I}_{\text {grad }}(f)=J_{0} \cap J_{1},
$$

where $V^{\mathbb{R}}\left(J_{0}\right)=\emptyset$ and and $\sqrt{J_{1}}=\langle x, y, z\rangle$. The ideal $J_{1}$ has multiplicity 153 , and it contains the square $f^{2}$ of our given polynomial. The ideal $J_{0}$ has multiplicity 190 . Its variety $V\left(J_{0}\right)$ consists of 158 distinct points in $\mathbb{C}^{3}$. By elimination, we can reduce to the univariate case. Using the algorithm of $[8,9]$ for real radicals in $\mathbb{Q}[z]$, we find a sum of squares $\operatorname{sos}(z) \in \mathbb{Q}[z]$ such that $f-\operatorname{sos}(z) \in J_{0}$. Running Buchberger's algorithm for $J_{0}+J_{1}=\langle 1\rangle$, we get polynomials $p_{0} \in J_{0}$ and $p_{1} \in J_{1}$ such that $p_{0}+p_{1}=1$. The following polynomial is a sum of squares,

$$
\begin{equation*}
p_{1}^{2} \cdot(\operatorname{sos}(z)+\epsilon)+p_{0}^{2} \cdot \epsilon \cdot\left(1+\frac{1}{2 \epsilon} f\right)^{2}, \tag{4.2.13}
\end{equation*}
$$

and it is congruent to $f(x, y, z)+\epsilon$ modulo $\mathcal{I}_{\text {grad }}(f)=J_{0} \cap J_{1}=J_{0} \cdot J_{1}$. Note that the coefficients of the right hand polynomial in the SOS representation (4.2.13) tend to infinity as $\epsilon$ approaches zero. This is consistent with the conclusion of Example 4.2.4.

### 4.3 Convergence analysis and the algorithm

We are now ready to give the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. Since $f(x)$ attains its infimum, the global minima of $f(x)$ must occur on the real gradient variety $V_{\text {grad }}^{\mathbb{R}}(f)$. It is obvious that any real number $\gamma$ which satisfies the SOS constraint in (4.1.5) is a lower bound of $f(x)$, and we have the sequence of inequalities in (4.1.8). Consider an arbitrary small real number $\varepsilon>0$. The polynomial $f(x)-f^{*}+\varepsilon$ is strictly positive on its real gradient variety $V_{\text {grad }}^{\mathbb{R}}(f)$. By Theorem 4.2.5, $f(x)-f^{*}+\varepsilon$ is SOS modulo $\mathcal{I}_{\text {grad }}(f)$. Hence there exists an integer $N(\epsilon)$ such that

$$
f_{N, \text { grad }}^{*} \geq f^{*}-\varepsilon \quad \text { for all } \quad N \geq N(\epsilon) .
$$

Since the sequence $\left\{f_{N, \text { grad }}^{*}\right\}$ is monotonically increasing, it follows that $\lim _{N \rightarrow \infty} f_{N, \text { grad }}^{*}=$ $f^{*}$.

Now suppose $\mathcal{I}_{\text {grad }}(f)=\mathcal{I}_{\text {grad }}\left(f-f^{*}\right)$ is a radical ideal. The nonnegative polynomial $f(x)-f^{*}$ is SOS modulo $\mathcal{I}_{\text {grad }}(f)$ by Theorem 4.2.1. Hence $f_{N, \text { grad }}^{*}=f^{*}$ for some $N \in \mathbb{Z}_{>0}$.

Remark 4.3.1. (i) The condition that $f(x)$ attains its infimum cannot be removed. Otherwise the infimum $f_{\text {grad }}^{*}$ of $f(x)$ on $V_{\text {grad }}^{\mathbb{R}}(f)$ need not be a lower bound for $f(x)$ on $\mathbb{R}^{n}$. A counterexample is $f(x)=x^{3}$. Obviously $f(x)$ has infimum $f^{*}=-\infty$ on $\mathbb{R}^{1}$. However, $f_{\text {grad }}^{*}=f_{\text {grad }, N}^{*}=0$ for all $N \geq 1$ because $f(x)=\left(\frac{x}{3}\right) f^{\prime}(x)$ is in the gradient ideal $\mathcal{I}_{\text {grad }}(f)=\left\langle f^{\prime}(x)\right\rangle$.
(ii) It is also not always the case that $f_{\text {grad }}^{*}=f^{*}$ when $f^{*}$ is finite. Consider the bivariate polynomial $f(x, y)=x^{2}+(1-x y)^{2}$. We can see that $f^{*}=0$ is not attained, but $f_{\text {grad }}^{*}=1>f^{*}$.
(iii) If $f(x)$ attains its infimum but $\mathcal{I}_{\text {grad }}(f)$ is not radical, we have only that $\lim _{N \rightarrow \infty} f_{N, \text { grad }}^{*}=f^{*}$. But there is typically no integer $N$ with $f_{N, g r a d}^{*}=f^{*}$, as shown in Example 4.2.4.

In the rest of this section, we discuss the duality of problem (4.1.5), and show how to extract the global minimizer(s) (if any). Given any multi-indexed vector $y=\left(y_{\alpha}\right)$, i.e., indexed by integer vectors $\alpha \in \mathbb{N}^{n}$, its moment matrix $M_{N}(y)$ to defined to be (see Section 2.3)

$$
M_{N}(y)=\left(y_{\alpha+\beta}\right)_{0 \leq|\alpha|,|\beta| \leq N}
$$

For polynomial $p(x)=\sum_{\beta} p_{\beta} x^{\beta}$, define the linear map $y \mapsto p * y$ such that the monomial-indexed vector $p * y$ has coordinates $(p * y)_{\alpha}=\sum_{\beta} p_{\beta} y_{\alpha+\beta}$. Denote by $f$ the vector of coefficients of $f(x)$. Let $f_{i}$ denote the vector of coefficients of the $i$-th partial derivative $\frac{\partial f}{\partial x_{i}}$. We rewrite (4.1.5) as follows:

$$
\begin{align*}
& f_{N, \text { grad }}^{*}=\max _{\substack{\gamma \in \mathbb{R}, \sigma \in \mathbb{R}[X]_{2 N} \\
\phi_{j}(x) \in \mathbb{R}[X]_{2 N-d+1}}} \gamma \quad \text { subject to } \quad \sigma(x) \succeq_{\text {sos }} 0  \tag{4.3.14}\\
& \text { and } f(x)-\gamma=\sigma(x)+\sum_{j=1}^{n} \phi_{j}(x) \frac{\partial f}{\partial x_{j}} . \tag{4.3.15}
\end{align*}
$$

The dual of above problem is the following (see also[52, 55])

$$
\begin{align*}
f_{N, \text { mom }}^{*}=\min _{y} & f^{T} y  \tag{4.3.16}\\
\text { s.t. } & M_{N-d / 2}\left(f_{i} * y\right)=0, i=1, \cdots, n  \tag{4.3.17}\\
& M_{N}(y) \succeq 0, y_{0}=1 \tag{4.3.18}
\end{align*}
$$

The following theorem relates the primal and dual objective function values $f_{N, \text { mom }}^{*}$ and $f_{N, g r a d}^{*}$, and it shows how to extract a point $x^{*}$ in $\mathbb{R}^{n}$ at which the minimum of $f(x)$ is attained.

Theorem 4.3.2. Assume $f(x)$ attains its infimum $f^{*}$ over $\mathbb{R}^{n}$ (hence $d$ is even).
Then we have:
(i) $f_{N, \text { mom }}^{*} \geq f_{N, \text { grad }}^{*}$ and hence $\lim _{N \rightarrow \infty} f_{N, \text { mom }}^{*}=f^{*}$.
(ii) Suppose $f_{N, \text { grad }}^{*}=f^{*}$ for some $N$. If $x^{*} \in \mathbb{R}^{n}$ minimizes $f(x)$, then $y^{*}=$ $\operatorname{mon}_{2 N}\left(x^{*}\right) \in \mathbb{R}^{\binom{n+2 N}{2 N}}$ solves the primal SDP.
(iii) If $y$ is a solution to the primal problem with $\operatorname{rank}\left(M_{N}(y)\right)=1$, then factoring $M_{N}(y)$ as column vector times row vector yields one global minimizer $x^{*}$ of the polynomial $f(x)$.
(iv) Suppose that $f_{N, \text { grad }}^{*}=f^{*}$ and $\sigma(x)=\sum_{j=1}^{\ell}\left(q_{j}(x)\right)^{2}$ solves the dual SDP. Then the set of all global minima of $f(x)$ equals the set of solutions $x \in \mathbb{R}^{n}$ to the following equations:

$$
\begin{aligned}
q_{j}(x) & =0, \quad j=1, \ldots, \ell \\
\frac{\partial f(x)}{\partial x_{i}} & =0, \quad i=1, \ldots, n
\end{aligned}
$$

Proof. Parts (i) and (ii) are basically a direct application of Theorem 4.2 in [52]. The hypotheses of that theorem are verified by an "epsilon argument" and applying our Theorem 4.2.5. Let us prove part (iii). Since the moment matrix $M_{N}(y)$ has rank one, there exists a vector $x^{*} \in \mathbb{R}^{n}$ such that $y=\operatorname{mon}_{N}\left(x^{*}\right)$. The strong duality result in (i) implies that

$$
f\left(x^{*}\right)=f^{T} y=f_{N, \text { mom }}^{*}=f_{N, \text { grad }}^{*}
$$

Since $f_{N, g r a d}^{*}$ is a lower bound for $f(x)$, we conclude that this lower bound is attained at the point $x^{*}$. Therefore, $f_{N, g r a d}^{*}=f^{*}$ and $x^{*}$ is a global minimizer. Part (iv) is straightforward.

From Theorem 4.3.2 (ii), we can see that there exists one optimal solution $y^{*}$ to the primal SDP such that $\operatorname{rank}\left(M_{N}\left(y^{*}\right)\right)=1$ if $f_{N, g r a d}^{*}=f^{*}$ for some integer
$N$. However, interior-point solvers for SDP will find a solution with moment matrix of maximum rank. So, if there are several global minimizers, the moment matrix $M_{N}\left(y^{*}\right)$ at relaxation $N$ for which the global minimum is reached, will have rank $>1$. However, if some flat extension condition holds at order $N$, i.e.,

$$
\begin{equation*}
\operatorname{rank} M_{N}\left(y^{*}\right)=\operatorname{rank} M_{N-d / 2}\left(y^{*}\right)=r \tag{4.3.19}
\end{equation*}
$$

where $y^{*}$ is one optimal solution to the dual problem, we still can extract minimizers. The rank condition (4.3.19) can be verified very accurately by Singular Value Decomposition (SVD). Then as a consequence of Theorem 1.6 in [24], there exist $r$ vectors $x^{*}(1), \cdots, x^{*}(r) \in \mathbb{R}^{n}$ such that

$$
M_{N}\left(y^{*}\right)=\sum_{j=1}^{r} \nu_{j} \operatorname{mon}_{N}\left(x^{*}(j)\right) \cdot \operatorname{mon}_{N}\left(x^{*}(j)\right)^{T}
$$

where $\sum_{j=1}^{r} \nu_{j}=1$ and $\nu_{j}>0$ for all $j=1, \cdots, r$. Henrion and Lasserre [41] proposed a detailed algorithm to find all such vectors $x^{*}(j)$. The condition (4.3.19) can be satisfied for some $N$ when $V_{\text {grad }}(f)$ is finite; see [55] for a proof. We refer to [41] and [71] for more details about extracting minimizers.

Summarizing the discussion above, we get the following algorithm for minimizing polynomials globally.

Algorithm 4.3.3. Computing the global minimizer(s) (if any) of a polynomial.

Input: A polynomial $f(x)$ of even degree $d$ in $n$ variables $x=\left(x_{1}, \ldots, x_{n}\right)$.
Output: Global minimizers $x^{*}(1), \cdots, x^{*}(r) \in \mathbb{R}^{n}$ of $f(x)$ for some $r \geq 1$.

Algorithm: Initialize $N=d / 2$.

Step 1 Solve the pair of primal SDP (4.3.14)-(4.3.15) and dual SDP (4.3.16)(4.3.18).

Step 2 Check rank condition (4.3.19). If it is satisfied, extract $r$ solutions $x^{*}(1), \cdots, x^{*}(r)$ by using the method in [41], where $r$ is the rank of $M_{N}\left(y^{*}\right)$, and then stop.

Step 3 If (4.3.19) is not satisfied, $N=N+1$ and then go to Step 1.

As we pointed out after (4.3.19) ([55]), this algorithm will terminate if $V_{\text {grad }}(f)$ is finite. If $V_{\text {grad }}(f)$ is infinite, it is possible to have infinitely many global minimizers and the extraction method in [41] can not be applied generally (it may work sometimes). In such situations we need to solve the equations in (iv) of Theorem 4.3.2 to obtain the minimizers.

### 4.3.1 What if the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ is not radical ?

The lack of radicalness of the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ would be an obstacle for our algorithm. Fortunately, this does not happen often in practice because the ideal $\mathcal{I}_{\text {grad }}(f)$ is generically radical, as shown by Proposition 4.3.4. It can be proved by standard arguments of algebraic geometry. We omit the proof.

Proposition 4.3.4. For almost all polynomials $f$ in the finite-dimensional vector space $\mathbb{R}[X]_{d}$, the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ is radical and the gradient variety $V_{\text {grad }}(f)$ is a finite subset of $\mathbb{C}^{n}$.

Proposition 4.3.4 means that, for almost all polynomials $f$ which attain their minimum $f^{*}$, Algorithm 4.3.3 will compute the minimum in finitely many steps. An a priori bound for a degree $N$ with $f_{N, g r a d}^{*}=f^{*}$ is given in [55].

Let us now consider the unlucky case when $\mathcal{I}_{\text {grad }}(f)$ is not radical. This happened for instance, in Example 4.2.4. In theory, one can replace the gradient ideal $\mathcal{I}_{\text {grad }}(f)$ by its radical $\sqrt{\mathcal{I}_{\text {grad }}(f)}$ in our SOS optimization problem. This is justified by the following result.

Corollary 4.3.5. If a polynomial $f(x)$ attains its infimum $f^{*}$ over $\mathbb{R}^{n}$ then $f(x)-f^{*}$ is SOS modulo the radical $\sqrt{\mathcal{I}_{\text {grad }}(f)}$ of the gradient ideal.

Proof. Consider the decomposition (4.2.10) and form the SOS polynomial $q(x)$ in (4.2.11). Since $f(x)-q(x)$ vanishes on the gradient variety $V\left(\mathcal{I}_{\text {grad }}(f)\right)=V\left(\sqrt{\mathcal{I}_{\text {grad }}(f)}\right)$, Hilbert's Strong Nullstellensatz implies that $f(x)-q(x) \in \sqrt{\mathcal{I}_{\text {grad }}(f)}$.

There are some known algorithms for computing radicals (see e.g. [33, 51]), and they are implemented in various computer algebra systems. But running these algorithms is usually very time-consuming. In practice, replacing $\mathcal{I}_{\text {grad }}(f)$ by its radical $\sqrt{\mathcal{I}_{\text {grad }}(f)}$ is not a viable option for efficient optimization algorithms. However, if some polynomials in $\sqrt{\mathcal{I}_{\text {grad }}(f)} \backslash \mathcal{I}_{\text {grad }}(f)$ are known to the user (for instance, from the geometry of the problem at hand), including these polynomials in (4.1.5) will probably speedup convergence of Algorithm 4.3.3.

### 4.4 Numerical experiments

In this section, we show some numerical examples by implementing Algorithm 4.3.3. Firstly we show examples where Algorithm 4.3 .3 provides much better lower bounds than the standard SOS relaxations, which is consistent with Theorem 4.1.1. Secondly, we show that Algorithm 4.3.3 is more computationally efficient than the standard SOS relaxation.

### 4.4.1 Comparison of lower bounds

The following examples demonstrate the effectiveness of our Algorithm 4.3.3 for a sample of polynomials that have been discussed in the polynomial optimization literature.

Example 4.4.1 (Homogeneous Polynomials). Let $f(x)$ be a homogeneous polynomial. Regardless of whether $f(x)$ is non-negative, we always have $f_{N, \text { grad }}^{*}=0$ for any $N \geq d / 2$. This comes from the identity $f(x)=\frac{1}{d} \cdot \sum_{i} x_{i} \frac{\partial f}{\partial x_{i}}$, which implies that $f(x)$ lies in its gradient ideal $\mathcal{I}_{\text {grad }}(f)$. In order to test global non-negativity of a homogeneous polynomial $f(x)$, we can apply Algorithm 4.3.3 to a dehomogenization of $f(x)$, as shown in examples below.

Example 4.4.2. $f(x, y)=x^{2} y^{2}\left(x^{2}+y^{2}-1\right)$. This polynomial is taken from [52]. It has global minimum value $f^{*}=-1 / 27=-0.03703703703703 \ldots$. However, $f_{\text {sos }}^{*}=$ $-\infty$ is considerably smaller than $f^{*}$. If we minimize $f(x)$ over its gradient ideal with $N=4$, then we get $f_{4, \text { grad }}^{*}=-0.03703703706212$. The difference equals $f^{*}-f_{4, \text { grad }}^{*} \approx$ $2.50 \cdot 10^{-11}$. The solutions extracted by GloptiPoly $([41])$ are $( \pm 0.5774, \pm 0.5774)$.

Example 4.4.3. The polynomial $f(x, y)=x^{4} y^{2}+x^{2} y^{4}+1-3 x^{2} y^{2}$ is obtained from the Motzkin polynomial by substituting $z=1$ as in [81]. We have $f^{*}=0>$ $f_{\text {sos }}^{*}=-\infty$. However, if we minimize $f(x, y)$ over its gradient ideal with $N=$ 4, we get $f_{4, \text { grad }}^{*}=-6.1463 \cdot 10^{-10}$. The solutions extracted by GloptiPoly are $( \pm 1.0000, \pm 1.0000)$.

Example 4.4.4. The polynomial $f(x, y)=x^{4}+x^{2}+z^{6}-3 x^{2} z^{2}$ is obtained from the Motzkin polynomial by substituting $y=1$. Now, $f^{*}=0>f_{\text {sos }}^{*}=-729 / 4096$. However, if we minimize $f(x, z)$ over its gradient ideal with $N=4$, we get $f_{4, \text { grad }}^{*}=$
$-9.5415 \cdot 10^{-12}$. The solutions extracted by GloptiPoly are $(0.0000,0.0000)$ and $( \pm 1.0000, \pm 1.0000)$.

### 4.4.2 Comparison of computational efficiency

We test the efficiency of Algorithm 4.3.3 on the Parrilo-Sturmfels family of polynomials of the form

$$
f\left(x_{1}, \cdots, x_{n}\right) \quad=\quad x_{1}^{d}+\cdots+x_{n}^{d}+g\left(x_{1}, \cdots, x_{n}\right)
$$

where $g \in \mathbb{R}[X]$ is a random polynomial of degree $\leq d-1$ whose coefficients are uniformly distributed between $-K$ and $K$, for a fixed positive integer $K$. This family of polynomials was considered in [80] where it was shown experimentally that the SOS formulation (4.1.2) almost always yields the global minimum. Without loss of generality, we can set $K=1$, because any $f(x)$ in the above form can be scaled to have coefficients between -1 and 1 by taking

$$
f_{s}\left(x_{1}, \cdots, x_{n}\right)=\alpha^{-d} \cdot f\left(\alpha x_{1}, \cdots, \alpha x_{n}\right)
$$

for some properly chosen $\alpha$. As observed in [80], this scaling will greatly increase the stability and speed of the numerical computations involved in solving the primal-dual SDP.

We ran a large number of randomly generated examples for various values of $d$ and $n$. The comparison results are in listed in Table 4.1 and Table 4.4. The computations were performed on a Dell Laptop with a 2.0 GHz Pentium IV and 512 MB of memory. Table 4.1 is the comparison of the lower bounds by formulation (4.1.2) and (4.1.5). Taking $N=d / 2$ in Algorithm 4.3 .3 appears to be good enough
in practice for minimizing the Parrilo-Sturmfels polynomials. Our experiments show that increasing $N$ above $d / 2$ will not increase the lower bound significantly.

From Table 4.1, we can see that the lower bounds $f_{\text {sos }}^{*}$ and $f_{N, \text { grad }}^{*}$ are close, agreeing to their leading 8 to 10 decimal digits, which confirms the observation made in [80] that almost all the polynomials gotten by subtracting their infima are SOS. Tables 4.2-4.4 are comparisons of running time in CPU seconds for formulations (4.1.2) and (4.1.5). The symbol "-" in the tables means that the computation takes more than one hour and we then terminate it. And "*" means we use a different scaling as described below.

Our formulation (4.1.5) uses about three quarters of the running time used by formulation (4.1.2). This may be unexpected since the use of gradients introduces many new variables. While we are not sure of the reason, one possible explanation is that adding gradients improves the conditioning and makes the interior-point algorithm for solving the SDP converge faster.

The numerical performance is subtle in this family of test polynomials. In the cases $(n, d)=(4,10)$ or $(n, d)=(5,10)$, our formulation (4.1.5) has numerical trouble, while (4.1.2) does not, and yet (4.1.5) is still faster than (4.1.2). However, for these two cases, if we scale $f\left(x_{1}, \ldots, x_{n}\right)$ so that the coefficients of $g\left(x_{1}, \ldots, x_{n}\right)$ belong to $[-0.1,0.1]$, both (4.1.2) and (4.1.5) do not have numerical trouble, and formulation (4.1.5) is still faster than (4.1.2). In Table 4.4 we see that the time ratio between (4.1.5) and (4.1.2) under this scaling is smaller than the time ratio for other values of $(n, d)$. So numerical comparisons in Tables 4.1-4.4 for $(n, d)=(4,10)$ or $(n, d)=(5,10)$ are implemented under this new scaling, while for other values of $(n, d)$ we still use the old scaling where the coefficients of $g\left(x_{1}, \ldots, x_{n}\right)$ belong
"-" means the computation is terminated if it takes more than one hour;
"*" means the coefficients of $g\left(x_{1}, \cdots, x_{n}\right)$ are scaled to belong to $[-0.1,0.1]$.

| $d \backslash n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 5 | 7 | 9 | 10 | 11 | 13 | 14 | 15 |
| 6 | 10 | 19 | 38 | 41 | 232 | - | - | - |
| 8 | 17 | 78 | 186 | 233 | - | - | - | - |
| 10 | 40 | $39^{*}$ | $102^{*}$ | - | - | - | - | - |

Table 4.1: The relative difference $\frac{\left|f_{N, g r a d}^{*}-f_{\text {sos }}^{*}\right|}{\left|f_{\text {sos }}^{*}\right|} \times 10^{10}$, with $N=d / 2$.

| $d \backslash n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 0.16 | 0.24 | 0.42 | 0.86 | 1.86 | 7.56 | 25.85 | 73.69 |
| 6 | 0.32 | 1.17 | 8.40 | 49.04 | 309.66 | - | - | - |
| 8 | 1.10 | 12.23 | 173.98 | 1618.86 | - | - | - | - |
| 10 | 3.15 | $64.48^{*}$ | $2144.04^{*}$ | - | - | - | - | - |

Table 4.2: Running time in CPU seconds via traditional SOS approach (4.1.2)

| $d \backslash n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 0.12 | 0.18 | 0.32 | 0.68 | 1.46 | 5.65 | 18.85 | 54.97 |
| 6 | 0.23 | 0.91 | 6.39 | 35.16 | 241.71 | - | - | - |
| 8 | 0.84 | 9.54 | 129.53 | 1240.23 | - | - | - | - |
| 10 | 2.59 | $45.14^{*}$ | $1539.80^{*}$ | - | - | - | - | - |

Table 4.3: Running time in CPU seconds via our approach (4.1.5), with $N=d / 2$.

| $d \backslash n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 4 | 0.75 | 0.75 | 0.76 | 0.79 | 0.78 | 0.74 | 0.73 | 0.75 |
| 6 | 0.72 | 0.77 | 0.76 | 0.72 | 0.78 | - | - | - |
| 8 | 0.76 | 0.78 | 0.74 | 0.76 | - | - | - | - |
| 10 | 0.82 | $0.70^{*}$ | $0.71^{*}$ | - | - | - | - | - |

Table 4.4: The ratio of CPU seconds between (4.1.2) and (4.1.5), with $N=d / 2$.
to $[-1,1]$. A stability analysis for the scaling and the speed-up caused by adding gradients may be a future research topic.

## Chapter 5

## SOS Methods based on the

## Kuhn-Karush-Tucker (KKT)

## Ideal

As shown in Chapter 1, a sequence of lower bounds of $f^{*}=\min _{x \in S} f(x)$ can be obtained by solving the SOS program

$$
f_{N}^{*}\left(p_{N}^{*} \text { resp. }\right)=\max \gamma \text { s.t. } f(x)-\gamma \in \mathcal{M}(S)_{2 N}\left(\mathcal{P}(S)_{2 N} \text { resp. }\right)
$$

Lasserre [52] showed convergence $\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$ under condition (1.1.14). If (1.1.14) fails but $S$ is compact, we still have $\lim _{N \rightarrow \infty} p_{N}^{*}=f^{*}$. When $S$ is not compact, we may not have convergence. In such situations, the gradient SOS methods introduced in Chapter 4 can be generalized to get a new sequence of lower bound of better properties. This chapter is based on joint work with Demmel and Powers [73].

### 5.1 Introduction

Consider the constrained polynomial optimization problem

$$
\begin{align*}
f^{*}=\min & f(x)  \tag{5.1.1}\\
\text { s.t. } & g_{i}(x)=0, \quad i=1, \cdots, s  \tag{5.1.2}\\
& h_{j}(x) \geq 0, \quad j=1, \cdots, t \tag{5.1.3}
\end{align*}
$$

where $x=\left[x_{1} \cdots x_{n}\right] \in \mathbb{R}^{n}$ and $f(x), g_{i}(x), h_{j}(x) \in \mathbb{R}[X]$, the ring of real multivariate polynomials in $X=\left(x_{1}, \cdots, x_{n}\right)$. Let $S$ be the feasible set defined by the constraints (5.1.2) - (5.1.3). Many optimization problems in practice can be formulated as (5.1.1)-(5.1.3). Finding the global optimal solutions to (5.1.1) - (5.1.3) is an NPhard problem, even if $f(x)$ is quadratic and $g_{i}, h_{j}$ are linear. For instance, the Maximum-Cut problem and nonconvex quadratic programming (QP) are NP-hard ([34, 64]).

Recently, the techniques of sum of squares (SOS) relaxations and moment matrix methods have made it possible to find globally optimal solutions to (5.1.1)(5.1.3) by SOS relaxations (also called SDP relaxations in some references). For more details about these methods and their applications, see $[45,52,53,54,55,70$, $71,80,81]$. To have convergence for these methods, it is often necessary to assume that the feasible region $S$ is compact or even finite. In [89], it is shown that SOS relaxations can solve (5.1.1)-(5.1.3) globally in finitely many steps in the case where $\left\{x \in \mathbb{C}^{n}: g_{1}(x)=\cdots=g_{s}(x)=0\right\}$ is finite and the ideal $\left\langle g_{1}(x), \cdots, g_{s}(x)\right\rangle$ is radical. If we only assume that $\left\{x \in \mathbb{C}^{n}: g_{1}(x)=\cdots=g_{s}(x)=0\right\}$ is finite, it is shown in [55] that the moment matrix method can solve (5.1.1)-(5.1.3) globally in finitely many steps. Finally, if $S$ is compact and its quadratic module $\mathcal{M}(\mathcal{F})$
is archimedean (see Theorem 2.3.2), then arbitrarily close lower bounds for $f^{*}$ can be obtained by SOS relaxations or moment matrix methods [52]. In this case, a convergence rate is given in Chapter 2.

The above global optimization methods are based on representation theorems from real algebraic geometry for polynomials positive and nonnegative on semialgebraic sets. On the other hand, the traditional local methods in optimization often follow the first order optimality conditions. The underlying idea in [71] and the present paper is to combine these two types of methods in order to more efficiently solve (5.1.1)-(5.1.3) globally. In [71], SOS relaxations are applied on the gradient ideal $\mathcal{I}_{\text {grad }}$ (the ideal generated by all the partial derivatives of $f(x)$ ) in the unconstrained case, and on the KKT (Kuhn-Karush-Tucker) ideal $I_{K K T}$ (defined below) in the constrained case, where only equality constraints are allowed. When $\mathcal{I}_{\text {grad }}$ or $I_{K K T}$ is radical, which is generically true in practice, the method in [71] can solve the optimization (5.1.1)-(5.1.2) globally; otherwise, arbitrarily close lower bounds of $f^{*}$ can be obtained. No assumptions about $S$ are made, i.e., it need not be finite or even compact. Jibetean and Laurent [45] also proposed a method to minimize polynomials by using the gradient ideal.

The KKT system of problem (5.1.1)-(5.1.3) is

$$
\begin{gather*}
F:=\nabla f(x)+\sum_{i=1}^{s} \lambda_{i} \nabla g_{i}(x)-\sum_{j=1}^{t} \nu_{j} \nabla h_{j}(x)=0,  \tag{5.1.4}\\
h_{j}(x) \geq 0, \nu_{j} h_{j}(x)=0, \quad j=1, \cdots, t,  \tag{5.1.5}\\
g_{i}(x)=0, \quad i=1, \cdots, s \tag{5.1.6}
\end{gather*}
$$

where vectors $\lambda=\left[\lambda_{1} \cdots \lambda_{s}\right]^{T}$ and $\nu=\left[\nu_{1} \cdots \nu_{t}\right]^{T}$ are called Lagrange multipliers. See [75] for some regularity conditions that make the KKT system hold at local or
global minimizers. For an example where the KKT system fails to define the global minimum, see Example 5.3.2 in Section 5.3.

Notice that we do not require $\nu \geq 0$ above; this makes the SOS relaxations simpler and does not affect the convergence of the method, since omitting the constraint $\nu \geq 0$ means simply that there are more feasible points for (5.1.4)-(5.1.6), including maxima as well as minima.

Define the KKT ideal $I_{K K T}$ and its varieties as follows:

$$
\begin{aligned}
I_{K K T} & =\left\langle F_{1}, \cdots, F_{n}, g_{1}, \cdots, g_{s}, \nu_{1} h_{1}, \cdots, \nu_{t} h_{t}\right\rangle, \\
V_{K K T} & =\left\{(x, \lambda, \nu) \in \mathbb{C}^{n} \times \mathbb{C}^{s} \times \mathbb{C}^{t}: p(x, \lambda, \nu)=0, \quad \forall p \in I_{K K T}\right\}, \\
V_{K K T}^{\mathbb{R}} & =\left\{(x, \lambda, \nu) \in \mathbb{R}^{n} \times \mathbb{R}^{s} \times \mathbb{R}^{t}: p(x, \lambda, \nu)=0, \quad \forall p \in I_{K K T}\right\} .
\end{aligned}
$$

Here $F=\left[F_{1}, \cdots, F_{n}\right]^{T}$ is defined in (5.1.4). Let

$$
\mathcal{H}=\left\{(x, \lambda, \nu) \in \mathbb{R}^{n} \times \mathbb{R}^{s} \times \mathbb{R}^{t}: h_{j}(x) \geq 0, \quad j=1, \cdots, t\right\}
$$

The preorder cone $\mathcal{P}_{K K T}$ associated with the KKT system is defined as

$$
\mathcal{P}_{K K T}=\left\{\sum_{\theta \in\{0,1\}^{t}} \sigma_{\theta} h_{1}^{\theta_{1}} h_{2}^{\theta_{2}} \cdots h_{t}^{\theta_{t}} \mid \sigma_{\theta} \text { are } \mathrm{SOS}\right\}+I_{K K T} .
$$

The quadratic module (sometimes called linear cone) associated with the KKT system is defined to be

$$
\mathcal{M}_{K K T}=\left\{\sigma_{0}+\sum_{j=1}^{t} \sigma_{j} h_{j} \mid \sigma_{0}, \cdots, \sigma_{t} \text { are SOS }\right\}+I_{K K T}
$$

Notice that $I_{K K T} \subseteq \mathcal{M}_{K K T} \subseteq P_{K K T} \subseteq \mathbb{R}[x, \lambda, \nu]$.
In solving SOS programs, we often set an upper bound on the degrees of the involved polynomials. Define the truncated KKT ideal

$$
I_{N, K K T}=\left\{\sum_{k=1}^{n} \phi_{k} F_{k}+\sum_{i=1}^{s} \varphi_{i} g_{i}+\sum_{j=1}^{t} \psi_{j} \nu_{j} h_{j} \mid \operatorname{deg}\left(\phi_{k} F_{k}\right), \operatorname{deg}\left(\varphi_{i} g_{i}\right), \operatorname{deg}\left(\psi_{j} \nu_{j} h_{j}\right) \leq N\right\} .
$$

and truncated preorder and linear cones

$$
\begin{aligned}
P_{N, K K T} & =\left\{\sum_{\theta \in\{0,1\}^{t}} \sigma_{\theta} h_{1}^{\theta_{1}} h_{2}^{\theta_{2}} \cdots h_{t}^{\theta_{t}} \mid \operatorname{deg}\left(\sigma_{\theta} h_{1}^{\theta_{1}} \cdots h_{t}^{\theta_{t}}\right) \leq N\right\}+I_{N, K K T} . \\
\mathcal{M}_{N, K K T} & =\left\{\sigma_{0}+\sum_{j=1}^{t} \sigma_{j} h_{j} \left\lvert\, \begin{array}{c}
\sigma_{0}, \cdots, \sigma_{t} \operatorname{are} \operatorname{SOS} \\
\operatorname{deg}\left(\sigma_{0}\right), \operatorname{deg}\left(\sigma_{j} h_{j}\right) \leq N
\end{array}\right.\right\}+I_{N, K K T} .
\end{aligned}
$$

A sequence $\left\{p_{N}^{*}\right\}$ of lower bounds of (5.1.1)-(5.1.3) can be obtained by SOS relaxations:

$$
\begin{align*}
p_{N}^{*}=\sup & \gamma  \tag{5.1.7}\\
\text { s.t. } & f(x)-\gamma \in P_{N, K K T} . \tag{5.1.8}
\end{align*}
$$

Since $P_{N, K K T}$ has a summation over $2^{t}$ terms like $\sigma_{\theta} h_{1}^{\theta_{1}} h_{2}^{\theta_{2}} \cdots h_{t}^{\theta_{t}}$, it is usually very expensive to solve the SOS program (5.1.7)-(5.1.8) in practice. So it is natural to replace the truncated preorder cone $P_{N, K K T}$ by the truncated linear cone $\mathcal{M}_{N, K K T}$, which leads to the SOS relaxations:

$$
\begin{align*}
& f_{N}^{*}=\max _{\gamma \in \mathbb{R}} \gamma  \tag{5.1.9}\\
& \text { s.t. }  \tag{5.1.10}\\
& f(x)-\gamma \in \mathcal{M}_{N, K K T} .
\end{align*}
$$

Thus we get monotonically increasing lower bounds $\left\{f_{N}^{*}\right\}_{N=2}^{\infty}$ and $\left\{p_{N}^{*}\right\}_{N=2}^{\infty}$ such that $f_{N}^{*} \leq p_{N}^{*} \leq f^{*}$. The following is the convergence theorem, which will be proved in Section 5.3.

Theorem 5.1.1. Assume $f(x)$ has a minimum $f^{*}:=f\left(x^{*}\right)$ at one KKT point $x^{*}$ of (5.1.1)-(5.1.3). Then $\lim _{N \rightarrow \infty} p_{N}^{*}=f^{*}$. Furthermore, if $I_{K K T}$ is radical, then there exists some $N \in \mathbb{N}$ such that $p_{N}^{*}=f^{*}$, i.e., the SOS relaxations (5.1.7)-(5.1.8) converge in finitely many steps.

The lower bounds $\left\{f_{N}^{*}\right\}$ are not guaranteed converge to $f^{*}$. However, if $\mathcal{M}_{K K T}$ is archimedean (see Section 2.3), then we have convergence $\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$ by Theorem 2.3.2 (Putinar's Positivestellensatz). We will return to this claim again in Section 5.3.

### 5.2 Representations in $\mathcal{P}_{K K T}$ and $\mathcal{M}_{K K T}$

This section discusses the representations of objective polynomial $f(x)$ in cones $\mathcal{P}_{\text {KKT }}$ and $\mathcal{M}_{\text {KKT }}$.

Theorem 5.2.1. Assume $I_{K K T}$ is radical. If $f(x)$ is nonnegative on $V_{K K T}^{\mathbb{R}} \cap \mathcal{H}$, then $f(x)$ belongs to $\mathcal{P}_{K K T}$.

To prove Theorem 5.2.1, we need the following lemma, which is a generalization of Lemma 4.2.3.

Lemma 5.2.2. Let $W$ be an irreducible component of $V_{K K T}$. Then $f(x)$ is constant on $W$.

Proof. Since $W$ is irreducible and contains a real point, it remains irreducible if we replace $\mathbb{R}[X, \lambda]$ by $\mathbb{C}[X, \lambda]$. Thus $W$ is connected in the strong topology on $\mathbb{C}^{n+s}$ and hence is path-connected (see e.g. [107, 4.1.3]).

We notice that the Lagrangian function

$$
\mathcal{L}(x, \lambda, \nu)=f(x)+\sum_{i=1}^{s} \lambda_{i} g_{i}(x)+\sum_{j=1}^{t} \nu_{j} h_{j}(x)
$$

is equal to $f(x)$ on $V_{K K T}$, which contains $W$. Choose two arbitrary points $\left(x^{(1)}, \lambda^{(1)}, \nu^{(1)}\right)$, $\left(x^{(2)}, \lambda^{(2)}, \nu^{(2)}\right)$ in $W$. We claim that $f\left(x^{(1)}\right)=f\left(x^{(2)}\right)$.

Firstly assume both $\left(x^{(1)}, \lambda^{(1)}, \nu^{(1)}\right)$ and $\left(x^{(2)}, \lambda^{(2)}, \nu^{(2)}\right)$ are nonsingular points. The set of nonsingular points consists a manifold. Since $W$ is path-connected, there exists a piecewise-smooth path $\varphi(\tau)=(x(\tau), \lambda(\tau), \nu(\tau))(0 \leq \tau \leq 1)$ lying inside $W$ such that $\varphi(0)=\left(x^{(1)}, \lambda^{(1)}, \nu^{(1)}\right)$ and $\varphi(1)=\left(x^{(2)}, \lambda^{(2)}, \nu^{(2)}\right)$. Let $\mu_{j}(\tau)$ be the principle square root of $\nu_{j}(\tau), 1 \leq j \leq t$ (for a complex number $z=|z| \exp (\sqrt{-1} \theta)$ with $0 \leq \theta<2 \pi$, its principle square root is defined to be $\left.\sqrt{|z|} \exp \left\{\frac{1}{2} \sqrt{-1} \theta\right\}\right)$. From the KKT system (5.1.4)-(5.1.6), we can see that the function

$$
f(x)+\sum_{i=1}^{s} \lambda_{i} g_{i}(x)+\sum_{j=1}^{t} \mu_{j}^{2} h_{j}(x)
$$

has zero gradient on the path $\varphi(\tau)(0 \leq \tau \leq 1)$. By the Mean Value Theorem, we have $f\left(x^{(1)}\right)=f\left(x^{(2)}\right)$.

Secondly consider the case that at least one of $\left(x^{(1)}, \lambda^{(1)}, \mu^{(1)}\right)$ and $\left(x^{(2)}, \lambda^{(2)}, \mu^{(2)}\right)$ is singular. Since the set of nonsingular points of $W$ is dense and open in $W$ ([107, Chap. 4]), we can choose arbitrarily close nonsingular points to approximate $\left(x^{(1)}, \lambda^{(1)}, \mu^{(1)}\right)$ and $\left(x^{(2)}, \lambda^{(2)}, \mu^{(2)}\right)$. By continuity of $f(x)$, we immediately have $f\left(x^{(1)}\right)=f\left(x^{(2)}\right)$ and hence that $f$ is constant on $W$.

Proof of Theorem 5.2.1. Decompose $V_{K K T}$ into its irreducible components, then by Lemma 5.2.2, $f(x)$ is constant on each of them. Let $W_{0}$ be the union of all the components whose intersection with $\mathcal{H}$ is empty, and group together the components on which $f(x)$ attains the same value, say $W_{1}, \ldots, W_{r}$. Suppose $f(x)=\alpha_{i} \geq 0$ on $W_{i}$.

We have $V_{K K T}=W_{0} \cup W_{1} \cup \cdots \cup W_{r}$, and $W_{i}$ are pairwise disjoint. Note that by our definition of irreducible, each $W_{i}$ is conjugate symmetric. By Lemma 4.2.2, there exist polynomials $p_{0}, p_{1}, \cdots, p_{r} \in \mathbb{R}[x, \lambda, \nu]$ such that $p_{i}\left(W_{j}\right)=\delta_{i j}$, where $\delta_{i j}$
is the Kronecker delta function.
By assumption, $W_{0} \cap \mathcal{H}=\emptyset$ and so, by Theorem 2.3.5, there are SOS polynomials $v_{\theta}\left(\theta \in\{0,1\}^{t}\right)$ such that

$$
-1 \equiv \sum_{\theta \in\{0,1\}^{t}} v_{\theta} h_{1}^{\theta_{1}} \cdots h_{t}^{\theta_{t}} \stackrel{\text { def }}{=} v_{0} \bmod \quad I\left(W_{0}\right)
$$

We have $f=\left(f+\frac{1}{2}\right)^{2}-\left(f^{2}+\left(\frac{1}{2}\right)^{2}\right)=f_{1}+v_{0} \cdot f_{2}$ for the SOS polynomials $f_{1}=$ $\left(f+\frac{1}{2}\right)^{2}, f_{2}=f^{2}+\left(\frac{1}{2}\right)^{2}$. Then

$$
f \equiv f_{1}+v_{0} f_{2} \equiv \sum_{\theta \in\{0,1\}^{t}} u_{\theta} h_{1}^{\theta_{1}} \cdots h_{t}^{\theta_{t}} \stackrel{\text { def }}{=} q_{0} \quad \bmod \quad I\left(W_{0}\right)
$$

for some $\operatorname{SOS}$ polynomials $u_{\theta}\left(\theta \in\{0,1\}^{t}\right)$. Recall that $f(x)=\alpha_{i}$, a constant, on each $W_{i}(1 \leq i \leq r)$. Set $q_{i}(x)=\sqrt{\alpha_{i}}$, then $f(x)=q_{i}(x)^{2}$ on $I\left(W_{i}\right)$.

Now let $q=q_{0} \cdot\left(p_{0}\right)^{2}+\left(\sum_{i=1}^{r} q_{i} p_{i}\right)^{2}$. Then $f-q$ vanishes on $V_{K K T}$ and hence $f-q \in I_{K K T}$ since $I_{K K T}$ is radical. It follows that $f \in \mathcal{P}_{K K T}$.

Remark 5.2.3. The assumption that $I_{K K T}$ is radical is needed in Theorem 5.2.1, as shown by Example 3.4 in [71]. However, when $I_{K K T}$ is not radical, the conclusion also holds if $f(x)$ is strictly positive on $V_{K K T}^{\mathbb{R}}$.

Theorem 5.2.4. If $f(x)$ is strictly positive on $V_{K K T}^{\mathbb{R}} \cap \mathcal{H}$ then $f(x)$ belongs to $\mathcal{P}_{K K T}$.

Proof. As in the proof of Theorem 5.2.1, we decompose $V_{K K T}$ into subvarieties $W_{0}, W_{1}, \cdots, W_{r}$ such that $W_{0} \cap \mathcal{H}=\emptyset$, and for $i=1, \ldots r, W_{i} \cap \mathcal{H} \neq \emptyset$ and $f$ is constant on $W_{i}$. Since each $W_{i}, i>0$ contains at least one real point and $f(x)>0$ on $V_{K K T}^{\mathbb{R}}$, each $\alpha_{i}>0$. The $W_{i}$ were chosen so that each $\alpha_{i}$ is distinct, hence the $W_{i}$ 's are pairwise disjoint.

Consider the primary decomposition $I_{K K T}=\cap_{i=0}^{r} J_{i}$ corresponding to our decomposition of $V_{K K T}$, i.e., $V\left(J_{i}\right)=W_{i}$ for $i=0,1, \cdots, r$. Since $W_{i} \cap W_{j}=\emptyset$, we
have $J_{i}+J_{j}=\mathbb{R}[x, \lambda, \nu]$ by Theorem 2.2.2. The Chinese Remainder Theorem, see e.g. [29, 2.13], implies that there is an isomorphism

$$
\rho: \mathbb{R}[x, \lambda, \nu] / I_{K K T} \rightarrow \mathbb{R}[x, \lambda, \nu] / J_{0} \times \mathbb{R}[x, \lambda, \nu] / J_{1} \times \cdots \times \mathbb{R}[x, \lambda, \nu] / J_{r} .
$$

For any $p \in \mathbb{R}[x, \lambda, \nu]$, let $[p]$ and $\rho([p])_{i}$ denote the equivalence classes of $p$ in $\mathbb{R}[x, \lambda, \nu] / I_{K K T}$ and $\mathbb{R}[x, \lambda, \nu] / J_{i}$ respectively.

Recall that that $V\left(J_{0}\right) \cap \mathcal{H}=\emptyset$, hence by Theorem 2.3.5 there exist SOS polynomials $u_{\theta}\left(\theta \in\{0,1\}^{t}\right)$ such that

$$
-1 \equiv \sum_{\theta \in\{0,1\}^{t}} u_{\theta} \rho\left(\left[h_{1}^{\theta_{1}}\right]\right)_{0} \cdots \rho\left(\left[h_{t}^{\theta_{t}}\right]\right)_{0} \stackrel{\text { def }}{=} u_{0} \quad \bmod \quad J_{0} .
$$

As in the proof of Theorem 5.2.1, we write $f=f_{1}-f_{2}$ for SOS polynomials $f_{1}, f_{2}$ and then we have

$$
f \equiv f_{1}+u_{0} f_{2} \equiv \sum_{\theta \in\{0,1\}^{t}} v_{\theta}\left(\rho\left(\left[h_{1}^{\theta_{1}}\right]\right)\right)_{0} \cdots\left(\rho\left(\left[h_{t}^{\theta_{t}}\right]\right)\right)_{0} \stackrel{\text { def }}{=} q_{0} \bmod J_{0}
$$

for some SOS polynomials $v_{\theta}\left(\theta \in\{0,1\}^{t}\right)$. Thus the preimage $\rho^{-1}\left(\left(q_{0}, 0, \cdots, 0\right)\right) \in$ $\mathcal{P}_{\text {KKT }}$.

Now on each $W_{i}, 1 \leq i \leq r, f(x)=\alpha_{i}>0$, and hence $\left(f(x) / \alpha_{i}\right)-1$ vanishes on $W_{i}$. Then by Theorem 2.2.3 there is $\ell \in \mathbb{N}$ such that $\left(f(x) / \alpha_{i}-1\right)^{\ell} \in J_{i}$. From the binomial theorem, it follows that

$$
\left(1+\left(f(x) / \alpha_{i}-1\right)\right)^{1 / 2} \equiv \sum_{k=1}^{\ell-1}\binom{1 / 2}{k}\left(f(x) / \alpha_{i}-1\right)^{k} \stackrel{\text { def }}{=} q_{i} / \sqrt{\alpha_{i}} \bmod J_{i} .
$$

Thus $(\rho([f]))_{i}=q_{i}^{2}$ is $\operatorname{SOS}$ in $\mathbb{R}[x, \lambda, \nu] / J_{i}$, and hence $\rho^{-1}\left(q_{i}^{2} e_{i+1}\right)$ is $\operatorname{SOS}$ in $\mathbb{R}[x, \lambda, \nu] / I_{K K T}$, where $e_{i+1}$ is the $(i+1)$-st standard unit vector in $\mathbb{R}^{r+1}$.

Finally, we see that $\rho([f])=\left(q_{0}, q_{1}^{2}, \cdots, q_{r}^{2}\right)$. The preimage of the latter is

$$
\left.\rho^{-1}\left(\left(q_{0}, q_{1}^{2}, \cdots, q_{r}^{2}\right)\right)=\rho^{-1}\left(q_{0} e_{1}\right)\right)+\sum_{i=1}^{r} \rho^{-1}\left(q_{i}^{2} e_{i+1}\right)
$$

which implies that $f \in \mathcal{P}_{K K T}$.

Remark 5.2.5. The conclusions in Theorem 5.2.1 and Theorem 5.2.4 can not be strengthened to show that $f(x) \in \mathcal{M}_{K K T}$. The following is a counterexample.

Example 5.2.6. Consider the optimization

$$
\begin{array}{ll}
\min & f(x)=\left(x_{3}-x_{1}^{2} x_{2}\right)^{2}-1+\epsilon \\
\text { s.t. } & h_{1}(x)=1-x_{1}^{2} \geq 0 \\
& h_{2}(x)=x_{2} \geq 0 \\
& h_{3}(x)=x_{3}-x_{2}-1 \geq 0
\end{array}
$$

where $0<\epsilon<1$. From the constraints, we can easily observe that the global minimum $f^{*}=\epsilon>0$ which is attained at $x^{*}=(0,0,1)$. Its KKT ideal

$$
\begin{aligned}
I_{K K T}= & \left\langle 2 x_{1} x_{2}\left(x_{3}-x_{1}^{2} x_{2}\right)-\nu_{1} x_{1}, 2 x_{1}^{2}\left(x_{3}-x_{1}^{2} x_{2}\right)+\nu_{2}-\nu_{3}\right. \\
& \left.2\left(x_{3}-x_{1}^{2} x_{2}\right)-\nu_{3}, \nu_{1}\left(1-x_{1}^{2}\right), \nu_{2} x_{2}, \nu_{3}\left(x_{3}-x_{2}-1\right)\right\rangle
\end{aligned}
$$

is radical (verified in Macaulay 2 [30]). However, we can not find SOS polynomials $\sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}$ and general polynomials $\phi_{1}, \phi_{2}, \phi_{3}$ such that
$f(x)=\sigma_{0}+\sigma_{1} h_{1}+\sigma_{2} h_{2}+\sigma_{3} h_{3}+\phi_{1}\left(\frac{\partial f}{\partial x_{1}}-\nu_{1} x_{2}\right)+\phi_{2}\left(\frac{\partial f}{\partial x_{2}}-\nu_{2}+\nu_{3}\right)+\phi_{3}\left(\frac{\partial f}{\partial x_{3}}-\nu_{3}\right)$.
Suppose to the contrary that they exist. Plugging $\nu=(0,0)$ in the above identity yields

$$
0=1-\epsilon+\sigma_{0}+\sigma_{1}\left(1-x_{1}^{2}\right)+\sigma_{2} x_{2}+\sigma_{3}\left(x_{3}-x_{2}-1\right)+\phi\left(x_{3}-x_{1}^{2} x_{2}\right)
$$

where $\phi=-4 x_{1} \phi_{1}-x_{1}^{2} \phi_{2}+2 \phi_{3}-\left(x_{3}-x_{1}^{2} x_{2}\right)$. Now substitute $x_{3}=x_{1}^{2} x_{2}$ in the above, yielding

$$
\sigma_{3}\left(\left(1-x_{1}^{2}\right) x_{2}+1\right)=1-\epsilon+\sigma_{0}+\sigma_{1}\left(1-x_{1}^{2}\right)+\sigma_{2} x_{2}
$$

Here $\sigma_{0}, \sigma_{1}, \sigma_{2}, \sigma_{3}$ are now considered as SOS polynomials in $\left(x_{1}, x_{2}\right)$. Since $1-\epsilon>0$, $\sigma_{3}$ can not be the zero polynomial. If $\sigma_{3}=\sigma_{3}\left(x_{1}\right)$ is independent of $x_{2}$, we can derive a contradiction using an argument identical to the argument in the proof of of [86, Thm. 2]. Thus $2 m=\operatorname{deg}_{x_{2}} \sigma_{3}\left(x_{1}, x_{2}\right) \geq 2$ and $2 d=\operatorname{deg}_{x_{1}} \sigma_{3}\left(x_{1}, x_{2}\right) \geq 0$. On the left hand side, the leading term is of the form $A \cdot x_{1}^{2 d+2} x_{2}^{2 m+1}$ with coefficient $A<0$. Since the degree in $x_{2}$ on the left hand side is odd, the leading term on the right hand side must come from $\sigma_{2}\left(x_{1}, x_{2}\right) x_{2}$, and is of the form like $B \cdot x_{1}^{2 d} x_{2}^{2 m+1}$ with $B>0$. This is a contradiction. Therefore we can conclude that $f(x) \notin \mathcal{M}_{K K T}$.

### 5.3 Convergence of the lower bounds

In this section, we give the proof of Theorem 5.1.1. To get the convergence of $\left\{f_{N}^{*}\right\}$, we need some extra assumptions.

Proof of Theorem 5.1.1. The sequence $\left\{p_{N}^{*}\right\}$ is monotonically increasing, and $p_{N}^{*} \leq f^{*}$ for all $N \in \mathbb{N}$, since $f^{*}$ is attained by $f(x)$ in the KKT system (5.1.4)(5.1.6) by assumption and the constraint (5.1.10) implies that $\gamma \leq f^{*}$. Now for arbitrary $\epsilon>0$, let $\gamma_{\epsilon}=f^{*}-\epsilon$ and replace $f(x)$ by $f(x)-\gamma_{\epsilon}$ in (5.1.1)-(5.1.3). The KKT system remains unchanged, and $f(x)-\gamma_{\epsilon}$ is strictly positive on $V_{K K T}^{\mathbb{R}}$. By Theorem 5.2.4, $f(x)-\gamma_{\epsilon} \in \mathcal{P}_{K K T}$. Since $f(x)-\gamma_{\epsilon}$ is fixed, there must exist some integer $N_{1}$ such that $f(x)-\gamma_{\epsilon} \in P_{N_{1}, K K T}$. Hence $f^{*}-\epsilon \leq p_{N_{1}}^{*} \leq f^{*}$. Therefore we have that $\lim _{N \rightarrow \infty} p_{N}^{*}=f^{*}$.

Now assume that $I_{K K T}$ is radical. Replace $f(x)$ by $f(x)-f^{*}$ in (5.1.1)(5.1.3). The KKT system still remains the same, and $f(x)-f^{*}$ is now nonnegative on $V_{K K T}^{\mathbb{R}}$. By Theorem 5.2.1, $f(x)-f^{*} \in \mathcal{P}_{K K T}$. So there exists some integer $N_{2}$
such that $f(x)-f^{*} \in P_{N_{2}, K K T}$, and hence $P_{N_{2}}^{*} \geq f^{*}$. Then $p_{N}^{*} \leq f^{*}$ for all $N$ implies that $p_{N_{2}}^{*}=f^{*}$.

Remark 5.3.1. (i) In Lasserre's method [52], a sequence of lower bounds that converge to $f^{*}$ asymptotically can be obtained when the feasible region $S$ is compact; but those lower bounds usually do not converge in finitely many steps. However, from Theorem 5.1.1, we see that when $I_{K K T}$ is radical then the lower bounds $\left\{p_{N}^{*}\right\}$ converge in finitely many steps, even if $S$ is not compact. This implies that the lower bounds $\left\{p_{N}^{*}\right\}$ may have better convergence even in the case where $S$ is compact. (ii) The assumption in Theorem 5.1.1 is non-trivial and can not be removed, which is illustrated by the following example.

Example 5.3.2. Consider the optimization: $\min x$ s.t. $x^{3} \geq 0$. Obviously $f^{*}=0$ and the global minimizer $x^{*}=0$. However, the KKT system

$$
1-\nu \cdot 3 x^{2}=0, \quad \nu \cdot x^{3}=0, \quad x^{3} \geq 0, \quad \nu \geq 0
$$

is not satisfied, since $V_{K K T}=\emptyset$. Actually we can see that the lower bounds $\left\{f_{N}^{*}\right\}$ given by (5.1.9)-(5.1.10) tend to infinity. By Theorem 2.2.2, $V_{K K T}=\emptyset$ implies that $1 \in \mathcal{P}_{K K T}$, i.e.,

$$
\left(1+3 \nu x^{2}\right)\left(1-3 \nu x^{2}\right)+9 \nu^{2} x \cdot \nu x^{3}=1 .
$$

In the SOS relaxation (5.1.9)-(5.1.10), for arbitrarily large $\gamma, x-\gamma \in \mathcal{P}_{K K T}$, since

$$
x-\gamma=(x-\gamma)\left(1+3 \nu x^{2}\right)\left(1-3 \nu x^{2}\right)+9 \nu^{2} x(x-\gamma) \cdot \nu x^{3} \in \mathcal{P}_{K K T} .
$$

Thus $p_{8}^{*}=\infty$. In this example, the conclusion in Theorem 5.1.1 does not hold.
The convergence of lowers bounds $\left\{f_{N}^{*}\right\}$ cannot be guaranteed, as we see in Example 5.2.6. In that example, replace the objective by the perfect square
$\left(x_{3}-x_{1}^{2} x_{2}\right)^{2}$. Then $f^{*}=1$, but we do not have $\lim _{N \rightarrow \infty} f_{N}^{*}=1$. From the arguments there, we can see that $f(x)-(1-\epsilon) \notin \mathcal{M}_{K K T}$ for all $0<\epsilon<1$, which implies that $f_{N}^{*} \leq 0$. But $f_{N}^{*} \geq 0$ is obvious since $\left(x_{3}-x_{1}^{2} x_{2}\right)^{2}$ is a perfect square. Therefore $\lim _{N \rightarrow \infty} f_{N}^{*}=0<1=f^{*}$, i.e., the lower bounds $\left\{f_{N}^{*}\right\}$ obtained from (5.1.9)-(5.1.10) may not converge.

On the other hand, the situation is often not that bad in practice. In the examples in the rest of this paper, it always happens that $\lim _{N \rightarrow \infty} p_{N}^{*}=\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$. If we further assume that $\mathcal{M}_{K K T}$ is archimedean then it must hold that $\lim _{N \rightarrow \infty} p_{N}^{*}=$ $\lim _{N \rightarrow \infty} f_{N}^{*}=f^{*}$ from Theorem 2.3.2 (Putinar). This is the generalization of assumption 4.1 in [52]. See also the remark after Theorem 2.3.2.

The SOS relaxation (5.1.9)-(5.1.10) can be solved using software SOSTOOLS [88], or GloptiPoly [40]. The SOS relaxations (5.1.9)-(5.1.10) not only give the lower bounds $f_{N}^{*}$, but also the information about global minimizers $x^{*}$ and their Lagrange multipliers $\left(\lambda^{*}, \nu^{*}\right)$. SOSTOOLS can extract the minimizer when the moment matrix has rank one. Gloptipoly can also find the lower bounds, and extract the global minimizers when the moment matrix satisfies some rank condition ([41]). Gloptipoly does not need the moment matrix to be rank one.

Example 5.3.3 (Exercise 2.18, [43]). Consider the global optimization:

$$
\begin{aligned}
\min & \left(-4 x_{1}^{2}+x_{2}^{2}\right)\left(3 x_{1}+4 x_{2}-12\right) \\
\text { s.t. } & 3 x_{1}-4 x_{2} \leq 12, \quad 2 x_{1}-x_{2} \leq 0, \quad-2 x_{1}-x_{2} \geq 0 .
\end{aligned}
$$

The global minimum is $f^{*} \approx-18.6182$ and the minimizer is $x^{*}=(-24 / 55,128 / 55) \approx$ $(-0.4364,2.3273)$. The lower bound obtained from (5.1.9)-(5.1.10) is $f_{4}^{*} \approx-18.6182$. The extracted minimizer is $\hat{x} \approx(-0.4364,2.3273)$, which coincides with $x^{*}$.

Example 5.3.4. Consider the Quadratically Constrained Quadratic Program (QCQP):

$$
\begin{array}{ll}
\min & -\frac{4}{3} x_{1}^{2}+\frac{2}{3} x_{2}^{2}-2 x_{1} x_{2} \\
\text { s.t. } & x_{2}^{2}-x_{1}^{2} \geq 0, \quad-x_{1} x_{2} \geq 0 .
\end{array}
$$

The global minimum is $f^{*}=0$ and minimizer is $x^{*}=(0,0)$. The feasible region $S$ defined by the constraints is non-compact. The lower bound returned by (5.1.9)-(5.1.10) is $f_{4}^{*} \approx-2.6 \times 10^{-15}$ (Note: this computation was done in double precision floating point, with round off error bounded by $2^{-53} \approx 10^{-16}$ ). The extracted minimizer is $\hat{x} \approx\left(6.1 \times 10^{-16},-9.0 \times 10^{-17}\right)$ and the Lagrange multiplier is $\hat{\nu} \approx(0.3884,0.3909)$.

### 5.4 Structures over some special constraints

In SOS relaxation (5.1.9)-(5.1.10), the polynomials are in $(x, \lambda, \nu) \in \mathbb{R}^{n+s+t}$. It is very expensive to implement when there are many constraints. In practice, if the polynomials $g_{i}(x)$ or $h_{j}(x)$ are of special forms, the KKT system (5.1.4)(5.1.6) can be simplified and so can (5.1.9)-(5.1.10). In this section, we consider the case where the constraints include the nonnegative orthant $\mathbb{R}_{+}^{n}$ or some box $[a, b]_{n}=\left\{x \in \mathbb{R}^{n}: a \leq x \leq b\right\}$.

### 5.4.1 Nonnegative orthant $\mathbb{R}_{+}^{n}$

In this subsection, suppose the inequality constraints (5.1.3) are the nonnegative orthant $\mathbb{R}_{+}^{n}$. Then (5.1.2)-(5.1.3) have the form

$$
g_{1}(x)=\cdots=g_{s}(x)=0, \quad x \in \mathbb{R}_{+}^{n}
$$

The KKT system (5.1.4)-(5.1.6) becomes

$$
\begin{aligned}
\nabla f(x) & +\sum_{i=1}^{s} \lambda_{i} \nabla g_{i}(x)-\nu=0, \\
g_{1}(x) & =\cdots=g_{s}(x)=0, \\
x_{k} \nu_{k} & =0, \quad k=1, \cdots, n, \\
x & \in \mathbb{R}_{+}^{n}, \quad \nu \in \mathbb{R}^{n} .
\end{aligned}
$$

We can see that Lagrange multiplier $\nu$ can be solved for explicitly. By eliminating $\nu$, the above system simplifies to

$$
\begin{align*}
& x_{k}\left(\frac{\partial f}{\partial x_{k}}+\sum_{i=1}^{s} \lambda_{i} \frac{\partial g_{i}}{\partial x_{k}}\right)=0, \quad k=1, \cdots, n  \tag{5.4.11}\\
& g_{1}(x)=\cdots=g_{s}(x)=0 \tag{5.4.12}
\end{align*}
$$

We define cones $\mathcal{M}_{K K T}^{\mathbb{R}_{+}^{n}}$ and $\mathcal{M}_{N, K K T}^{\mathbb{R}_{+}^{n}}$, associated to the above simplified system, similar to the definition of $\mathcal{M}_{K K T}$ and $\mathcal{M}_{N, K K T}$. Note that $\mathcal{M}_{K K T}^{\mathbb{R}_{+}^{n}}, \mathcal{M}_{K K T}^{\mathbb{R}_{+}^{n}} \subset$ $\mathbb{R}[x, \lambda]$ and the Lagrange multiplier $\nu$ does not appear. Similar to (5.1.9)-(5.1.10), a sequence $\left\{\hat{f}_{N}^{*}\right\}$ of lower bounds of (5.1.1)-(5.1.3) can be obtained by the following SOS relaxations:

$$
\begin{align*}
& \hat{f}_{N}^{*}=\max _{\gamma \in \mathbb{R}} \gamma  \tag{5.4.13}\\
& \text { s.t. }  \tag{5.4.14}\\
& f(x)-\gamma \in \mathcal{M}_{N, K K T}^{\mathbb{R}_{+}^{n}} .
\end{align*}
$$

Now the indeterminates in the above SOS program are $(x, \lambda)$ instead of $(x, \lambda, \nu)$.
Since $\nu$ is eliminated by direct substitutions, systems (5.1.4)-(5.1.6) and (5.4.11)-(5.4.12) are equivalent. Thus we see that $f(x)-\gamma \in \mathcal{M}_{N_{1}, K K T}$ if and only if $f(x)-\gamma \in \mathcal{M}_{N_{2}, K K T}^{\mathbb{R}_{+}^{n}}$, for some integers $N_{1}$ and $N_{2}$. Therefore the lower bounds $\left\{\hat{f}_{N}^{*}\right\}$ have the same property of convergence as $\left\{f_{N}^{*}\right\}$ obtained from (5.1.9)-(5.1.10).

If, in addition, the constraints (5.1.2) are the standard simplex:

$$
A x=b, \quad x \geq 0
$$

where $A \in \mathbb{R}^{s \times n}, b \in \mathbb{R}^{s}$, the KKT system (5.1.4)-(5.1.6) can be furtherly reduced to

$$
\begin{gathered}
x_{k}\left(\frac{\partial f}{\partial x_{k}}+a_{k}^{T} \lambda\right)=0, \quad k=1, \cdots, n \\
A x=b, \quad x \geq 0
\end{gathered}
$$

where $a_{k} \in \mathbb{R}^{s}$ is the $k$-th column of matrix $A$.
Furthermore, if $A x=b$ consists of a single equation $a^{T} x=b \neq 0$, then $\lambda=-\frac{x^{T} \nabla f(x)}{b}$ and the KKT system has the simpler form

$$
\begin{aligned}
& x_{k}\left(\frac{\partial f}{\partial x_{k}}-\alpha_{k} \frac{x^{T} \nabla f(x)}{b}\right)=0, \quad k=1, \cdots, n \\
& \quad a^{T} x=b, x \geq 0
\end{aligned}
$$

where $a=\left[\alpha_{1}, \cdots, \alpha_{n}\right]^{T}$.
Based on the reduced KKT systems, simpler SOS relaxations can be obtained.

Example 5.4.1 (Test Problem 2.9, [32]). Consider the Maximum Clique Problem for $n=5$ :

$$
\begin{array}{ll}
\min & -\left(\sum_{i=1}^{4} x_{i} x_{i+1}+x_{1} x_{5}+x_{1} x_{4}+x_{2} x_{5}+x_{3} x_{5}\right) \\
\text { s.t. } & x_{1}+x_{2}+x_{3}+x_{4}+x_{5}=1 \\
& x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \geq 0 .
\end{array}
$$

The global minimum $f^{*}=-1 / 3$ and minimizers $x^{*}$ are $(1 / 3,1 / 3,0,0,1 / 3),(1 / 3,0,0,1 / 3,1 / 3)$, ( $0,1 / 3,1 / 3,0,1 / 3$ ), and ( $0,0,1 / 3,1 / 3,1 / 3$ ). The lower bound obtained from (5.4.13)(5.4.14) is $\hat{f}_{4}^{*} \approx-0.33333333378814$. The difference is $f^{*}-\hat{f}_{4}^{*} \approx 4.5 \times 10^{-10}$.

Example 5.4.2 (Exercise 1.20, [43]). Consider optimization problem:

$$
\begin{aligned}
\min & \sum_{i=1}^{n-1} x_{i}^{2} x_{i+1}+x_{n}^{2} x_{1} \\
\text { s.t. } & \sum_{x_{i}=1}^{n} x_{i}=1, \quad x \geq 0 .
\end{aligned}
$$

The global minimum $f^{*}=0$ and the minimizers are the vertices of the simplex defined by the constraints. The lower bound obtained from (5.4.13)-(5.4.14) is $\hat{f}_{4}^{*}=$ $-4.0 \cdot 10^{-8}$.

Example 5.4.3. $f(x)=x^{T} H x$ and the constraints are $0 \leq x \leq e$, where $x \in \mathbb{R}^{5}$ and $e=[1,1,1,1,1]^{T}$, and

$$
H=\left[\begin{array}{ccccc}
1 & -1 & 1 & 1 & -1 \\
-1 & 1 & -1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & -1 \\
-1 & 1 & 1 & -1 & 1
\end{array}\right]
$$

is a co-positive matrix $([79,81])$, i.e., $f(x) \geq 0 \forall x \geq 0$. If each $x_{i}$ is replaced by $x_{i}^{2}$, then the resulting quartic polynomial is nonnegative, but not SOS. Consider the Quadratic Program (QP):

$$
\begin{array}{cl}
\min & x^{T} H x \\
\text { s.t. } & x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \geq 0 .
\end{array}
$$

The lower bound obtained from (5.4.13)-(5.4.14) is $\hat{f}_{2}^{*}=-3.35 \times 10^{-9}$. Actually, we have the following decomposition

$$
x^{T} H x=0+\sum_{i=1}^{5} 2 \cdot\left(x_{i} \cdot h_{i}^{T} x\right)
$$

in (5.4.13)-(5.4.14). Here $h_{i}$ is the $i$-th column of matrix $H$.

### 5.4.2 $\quad$ Box $[a, b]_{n}$

Consider the case that (5.1.3) is given by $a \leq x \leq b$ where $a=\left[a_{1}, \cdots, a_{n}\right]^{T}$ and $b=\left[b_{1}, \cdots, b_{n}\right]^{T}$, and $a<b$. Now the KKT system (5.1.4)-(5.1.6) has the form

$$
\begin{aligned}
& \nabla f(x)+\sum_{i=1}^{s} \lambda_{i} \nabla g_{i}(x)-\nu+\mu=0 \\
& g_{1}(x)=\cdots=g_{s}(x)=0 \\
&\left(x_{k}-a_{k}\right) \nu_{k}=0, \quad\left(b_{k}-x_{k}\right) \mu_{k}=0, \quad k=1, \cdots, n, \\
& x-a \geq 0, \quad b-x \geq 0
\end{aligned}
$$

where $\nu_{i}\left(\mu_{i}, \lambda_{i}\right)$ is the $i$-th component of Lagrange multipliers $\nu(\mu, \lambda)$ respectively. One good property of this KKT system is that $(\nu, \mu)$ can be solved for explicitly. Eliminating $\nu$ and $\mu$, we have that

$$
\begin{aligned}
& \left(\frac{\partial f}{\partial x_{k}}+\sum_{i=1}^{s} \lambda_{i} \frac{\partial g_{i}}{\partial x_{k}}\right)\left(x_{k}-a_{k}\right)\left(b_{k}-x_{k}\right)=0, \quad k=1, \cdots, n \\
& g_{1}(x)=\cdots=g_{s}(x)=0, \quad x-a \geq 0, \quad b-x \geq 0
\end{aligned}
$$

Like the definitions of $\mathcal{M}_{K K T}^{\mathbb{R}_{+}^{n}}$ and $\mathcal{M}_{N, K K T}^{\mathbb{R}_{+}^{n}}$ (see the preceding subsection), define the cones $\mathcal{M}_{K K T}^{[a, b]_{n}}$ and $\mathcal{M}_{N, K K T}^{[a, b]_{n}}$ associated with the above simplified KKT system, where $\mathcal{M}_{K K T}^{[a, b]_{n}}, \mathcal{M}_{N, K K T}^{[a, b]_{n}} \subset \mathbb{R}[x, \lambda]$. Similar to (5.4.13)-(5.4.14), a sequence of lower bounds $\left\{\tilde{f}_{N}^{*}\right\}$ of (5.1.1)-(5.1.3) can be obtained by the following SOS relaxations:

$$
\begin{align*}
\tilde{f}_{N}^{*}=\max _{\gamma \in \mathbb{R}} & \gamma  \tag{5.4.15}\\
& \text { s.t. } \quad f(x)-\gamma \in \mathcal{M}_{N, K K T}^{[a, b]_{n}} . \tag{5.4.16}
\end{align*}
$$

Now a polynomial $u(x, \lambda)$ of degree $d$ in $\mathcal{M}_{d, K K T}^{[a, b]_{n}}$ has at most $\binom{n+s+d}{d}$ coefficients, which is much smaller than $\left(\begin{array}{c}n+s+2 n+d\end{array}\right)$, the number of coefficients of one polynomial of degree $d$ in $\mathcal{M}_{N, K K T}$. So (5.4.15)-(5.4.16) can be solved much more efficiently. Similarly as $\left\{\hat{f}_{N}^{*}\right\}$, the lower bounds $\left\{\tilde{f}_{N}^{*}\right\}$ have the same properties of convergence as $\left\{f_{N}^{*}\right\}$.

Consider the special case that $f(x)=\frac{1}{2} x^{T} H x+g^{T} x$ is a quadratic function and there are no equality constraints. Here $g \in \mathbb{R}^{n}$ and $H=H^{T} \in \mathbb{R}^{n \times n}$ is symmetric. The the above KKT system can be further reduced to

$$
\begin{aligned}
& \left(h_{k}^{T} x+g_{k}\right)\left(x_{k}-a_{k}\right)\left(b_{k}-x_{k}\right)=0, \quad k=1, \cdots, n, \\
& \quad x-a \geq 0, \quad b-x \geq 0
\end{aligned}
$$

Here $h_{k}$ is the $k$-th row of matrix $H$ and $g_{k}$ is the $k$-entry of $g$. Finding the global minimum of a general nonconvex quadratic function over a box is an NP-hard problem. The relaxations (5.4.15)-(5.4.16) provide a new approach for such nonconvex quadratic programming problem.

Example 5.4.4 (Test Problem 4.7, [32]). Consider optimization problem

$$
\begin{array}{ll}
\min & -12 x_{1}-7 x_{2}+x_{2}^{2} \\
\text { s.t. } & -2 x_{1}^{4}+2-x_{2}=0 \\
& 0 \leq x_{1} \leq 2, \quad 0 \leq x_{2} \leq 3 .
\end{array}
$$

The best known objective value $\approx-16.73889$. The lower bound obtained from (5.4.15)-(5.4.16) is $\tilde{f}_{6}^{*} \approx-16.73889$. So $f^{*} \approx \tilde{f}_{6}^{*}$. The extracted minimizer $\tilde{x} \approx$ $(0.7175,1.4698)$ and Lagrange multiplier $\tilde{\lambda} \approx-4.0605$.

Example 5.4.5 (Test Problem 2.1, [32]). Consider optimization problem

$$
\begin{array}{ll}
\min & 42 x_{1}+44 x_{2}+45 x_{3}+47 x_{4}+47.5 x_{5}-50 \sum_{i=1}^{5} x_{i}^{2} \\
\text { s.t. } & 20 x_{1}+12 x_{2}+11 x_{3}+7 x_{4}+4 x_{5} \leq 40 \\
& 0 \leq x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \leq 1 .
\end{array}
$$

The global minimum $f^{*} \approx-17$ and the minimizer $x^{*}=(1,1,0,1,0)$. The lower bound obtained from (5.4.15)-(5.4.16) is $\tilde{f}_{6}^{*} \approx-17.00$. The extracted minimizer $\tilde{x} \approx(1.00,1.00,0.00,1.00,0.00)$ and Lagrange multiplier $\tilde{\nu} \approx 0.1799$.

Example 5.4.6 (Exercise 2.22, [43]). Consider the Maximum Independent Set Problem

$$
\begin{aligned}
\min & -\sum_{i=1}^{n} x_{i}+\sum_{(i, j) \in E} x_{i} x_{j} \\
\text { s.t. } & 0 \leq x_{i} \leq 1, \quad i=1, \cdots, n .
\end{aligned}
$$

The negative of the global minimum $-f^{*}$ equals the cardinality of the maximum independent vertex set of $G=(V, E)$. Let $G$ be a pentagon with two diagonals which do not intersect in the interior. Now $n=5$ and $f^{*}=-2$. The lower bound obtained from (5.4.15)-(5.4.16) is $\tilde{f}_{4}^{*} \approx-2.00$.

Example 5.4.7 (Exercise 1.32, [43]). Consider optimization problem

$$
\begin{array}{ll}
\min & \prod_{i=1}^{n} x_{i}-\sum_{i=1}^{n} x_{i} \\
\text { s.t. } & 0 \leq a \leq x_{1}, \cdots, x_{n} \leq b
\end{array}
$$

The global minimum is $f^{*}=a^{n}-n a$ when $a \geq 1$. For $n=4, a=2, b=3$, the
lower bound obtained from $(5.4 .15)-(5.4 .16)$ is $\tilde{f}_{6}^{*} \approx 8.00$. The extracted minimizer is $\tilde{x} \approx(2.00,2.00,2.00,2.00)$.

## Chapter 6

## Minimizing Rational Functions

This chapter discusses the global minimization of rational functions. Consider the problem of minimizing a rational function

$$
\begin{array}{rl}
r^{*}=\min _{x \in \mathbb{R}^{n}} & r(x):=\frac{f(x)}{g(x)} \\
& \text { s.t. }  \tag{6.0.2}\\
h_{1}(x) \geq 0, \cdots, h_{m}(x) \geq 0
\end{array}
$$

where $f(x), g(x), h_{i}(x) \in \mathbb{R}[X]$. The motivation is to find the global minimum $r^{*}$ of the rational function $r(x)$, and if possible, one or more global minimizer(s) $x^{*}$ such that $r\left(x^{*}\right)=r^{*}$, subject to constraints. This contains a broad class of nonlinear global optimization problems. Without loss of generality, assume that $g(x)$ is nonnegative and not identically zero on the feasible set; as long as $g(x)$ is not identically zero, we can replace $\frac{f(x)}{g(x)}$ by $\frac{f(x) g(x)}{g^{2}(x)}$. The sum of squares (SOS) methods can be generalized to solve this problem. Some special features arise that differ from the polynomial case. The difficulty appears when the minimum occurs on the common zeros of $f(x)$ and $g(z)$.

### 6.1 SOS relaxation for unconstrained minimization

In this section, we discuss the global minimization of (6.0.1) without any constraints.

Obviously, $\gamma$ is a lower bound for $r^{*}$ if and only if the polynomial $f(x)-\gamma g(x)$ is nonnegative. Now by approximating the nonnegativity of $f(x)-\gamma g(x)$ by a sum of squares, we get the following SOS relaxation

$$
\begin{aligned}
& r_{\text {sos }}^{*}:= \sup _{\gamma} \\
& \gamma \\
& \text { s.t. } \\
& \quad f(x)-\gamma g(x) \in \sum \mathbb{R}[X]^{2} .
\end{aligned}
$$

For any $\gamma$ feasible in the above formulation, we immediately have $r(x) \geq \gamma$ for every $x \in \mathbb{R}^{n}$. Thus every feasible $\gamma$ (and hence including $r_{\text {sos }}^{*}$ ) is a lower bound for $r(x)$, i.e., $r_{s o s}^{*} \leq r^{*}$.

Let $2 d=\max (\operatorname{deg}(f), \operatorname{deg}(g))$ (it must be even for $r(x)$ to have a finite minimum) and $m(x)$ be the column vector of monomials up to degree $d$

$$
m(x)^{T}=\left[1, x_{1}, \cdots, x_{n}, x_{1}^{2}, x_{1} x_{2}, \cdots, x_{n}^{2}, x_{1}^{3}, \cdots, x_{n}^{d}\right]
$$

Notice that the length of vector $m(x)$ is $\binom{n+d}{d}$. As discussed in Section 2.1, the polynomial $f(x)-\gamma g(x)$ is SOS if and only there exists a symmetric matrix $W \succeq 0$ of length $\binom{n+d}{d}$ such that the following identity holds:

$$
\begin{equation*}
f(x)-\gamma g(x) \equiv m(x)^{T} W m(x) \tag{6.1.3}
\end{equation*}
$$

Now we write $f(x)($ resp. $g(x))$ as $\sum_{\alpha \in F} f_{\alpha} x^{\alpha}$ (resp. $\sum_{\alpha \in F} g_{\alpha} x^{\alpha}$ ), where $F$ is a finite subset of $\mathbb{N}^{n}$. i.e., $F$ is the support of polynomials $f(x)$ and $g(x)$.

Throughout this chapter, we index the rows and columns of matrix $W$ by monomials up to degree $d$, i.e., the indices for the entries in $W$ have the form $(\alpha, \beta)$ where $\alpha, \beta \in \mathbb{N}^{n}$. For any $\alpha \in \mathbb{N}^{n}$, denote by $B_{\alpha}$ the coefficient matrix of $x^{\alpha}$ in $m(x) m(x)^{T}$ (see Section 2.1) When $n=1$, the $B_{\alpha}$ are Hankel matrices. Now we can see that (6.1.3) holds if and only if

$$
f_{\alpha}-\gamma g_{\alpha}=<B_{\alpha}, W>, \quad \forall \alpha \in F
$$

Therefore the SOS relaxation of problem (6.0.1) is essentially the following (SDP):

$$
\begin{align*}
r_{\text {sos }}^{*}:=\sup _{\gamma, W} & \gamma  \tag{6.1.4}\\
& \text { s.t. } \quad f_{\alpha}-\gamma g_{\alpha}  \tag{6.1.5}\\
& =<B_{\alpha}, W>, \quad \forall \alpha \in F  \tag{6.1.6}\\
& W \succeq 0 .
\end{align*}
$$

Notice that the decision variables are $\gamma$ and $W$ instead of $x$.

Now let us derive the dual problem to SDP (6.1.4)-(6.1.5). Its Lagrange function is

$$
\begin{aligned}
\mathcal{L}(\gamma, W, y, S) & =\gamma+\sum_{\alpha \in F}\left(f_{\alpha}-\gamma g_{\alpha}-<B_{\alpha}, W>\right) y_{\alpha}+W \bullet S \\
& =\sum_{\alpha \in F} f_{\alpha} y_{\alpha}+\left(1-\sum_{\alpha \in F} g_{\alpha} y_{\alpha}\right) \gamma+\left(S-\sum_{\alpha \in F} y_{\alpha} B_{\alpha}\right) \bullet W
\end{aligned}
$$

where $y=\left(y_{\alpha}\right)$ and $W$ are dual decision variables (Lagrange multipliers). The vector $y$ is monomial-indexed, and $S$ is a symmetric matrix of the same size as $W$. And
$S \succeq 0$ corresponds to the constraint $W \succeq 0$. Obviously the following holds

$$
\sup _{\gamma, W} \mathcal{L}(\gamma, W, y, S)= \begin{cases}\sum_{\alpha \in F} f_{\alpha} y_{\alpha} & \text { if } \sum_{\alpha \in F} g_{\alpha} y_{\alpha}=1 \\ & \sum_{\alpha \in F} y_{\alpha} B_{\alpha}=S \\ +\infty & \text { otherwise }\end{cases}
$$

Therefore, the dual problem of (6.1.4)-(6.1.6) is

$$
\begin{align*}
r_{\text {mom }}^{*}:=\inf _{y} & \sum_{\alpha \in F} f_{\alpha} y_{\alpha}  \tag{6.1.7}\\
\text { s.t. } & \sum_{\alpha} g_{\alpha} y_{\alpha}=1  \tag{6.1.8}\\
& M_{d}(y) \succeq 0 . \tag{6.1.9}
\end{align*}
$$

where the matrix $M_{d}(y):=\sum_{\alpha} y_{\alpha} B_{\alpha}$ is the $d$-th moment matrix of $y$. (6.1.7)-(6.1.9) can also be considered as an generalization of moment approaches in [52] except the equality (6.1.8).

From the derivation of dual problem (6.1.7)-(6.1.9) we immediately have that $r_{\text {sos }}^{*} \leq r_{\text {mom }}^{*}$, which is referred to as weak duality in optimization duality theory. Actually we have stronger properties for the SOS relaxation (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9) as summarized in the following theorem.

Theorem 6.1.1. Assume that the SOS relaxation (6.1.4)-(6.1.6) has a feasible solution $(\gamma, W)$. Then the following properties hold for the primal problem (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9):
(i) Strong duality holds, i.e., $r_{\text {sos }}^{*}=r_{m o m}^{*}$, and $f(x)-r_{\text {sos }}^{*} g(x)$ is SOS.
(ii) The lower bound $r_{\text {sos }}^{*}$ obtained from the SOS relaxation (6.1.4)-(6.1.6) is exact, i.e., $r_{\text {sos }}^{*}=r^{*}$, if and only if $f(x)-r^{*} g(x)$ is SOS.
(iii) When $r_{\text {sos }}^{*}=r^{*}$ and $u^{(j)}(j=1, \cdots, t)$ are global minimizers, then every vector $y$ in the set

$$
y \in\left\{\sum_{j=1}^{t} \theta_{j} m_{2 d}\left(u^{(j)}\right): \theta_{j} \geq 0, \sum_{j=1}^{t} \theta_{j}=1\right\}
$$

is an optimal solution of (6.1.7)-(6.1.9).
Proof. (i) The result can be obtained from the standard duality theory of convex programs [96, §30], if we can show that there exists a vector $\hat{y}$ such that $\sum_{\alpha} g_{\alpha} \hat{y}_{\alpha}=1$ and $M_{d}(\hat{y}) \succ 0$. Let $\mu$ be a Lebesgue measure on $\mathbb{R}^{n}$ with strictly positive density everywhere on $\mathbb{R}^{n}$ and finite moments, i.e., $\left|\int x^{\alpha} d \mu\right|<\infty$ for all $\alpha \in \mathbb{N}^{n}$ (e.g., one density function can be chosen as $\left.\exp \left(-\sum_{i=1}^{n} x_{i}^{2}\right)\right)$. Define the vector $y=\left(y_{\alpha}\right)$ as follows:

$$
y_{\alpha}=\int x^{\alpha} d \mu<\infty
$$

Then we can claim that

$$
0<\tau:=\sum_{\alpha} g_{\alpha} y_{\alpha}=\int g(x) d \mu<\infty .
$$

The second inequality is obvious since all the moments of $\mu$ are finite. For the first inequality, for a contradiction, suppose $\tau \leq 0$, that is,

$$
\int g(x) d \mu \leq 0 .
$$

Since $g(x)$ is assumed to be nonnegative everywhere and $\mu$ has positive density everywhere, we must have that $g(x)$ should be identically zero, which is a contradiction. Then we prove that $M_{d}(y)$ is positive definite. For any monomial-indexed nonzero vector $q$ with the same length as $M_{d}(y)$ (it corresponds to a nonzero polynomial $q(x)$ ), it holds that

$$
q^{T} M_{d}(y) q=\sum_{0 \leq|\alpha|,|\beta| \leq d} y_{\alpha+\beta} q_{\alpha} q_{\beta}=\int\left(\sum_{0 \leq|\alpha|,|\beta| \leq d} x^{\alpha+\beta} q_{\alpha} q_{\beta}\right) d \mu=\int q(x)^{2} d \mu>0 .
$$

Now let $\hat{y}=y / \tau$, which obviously satisfies $\sum g_{\alpha} \hat{y}_{\alpha}=1$ and $M_{d}(\hat{y}) \succ 0$. In other words, the problem (6.1.7)-(6.1.9) has an interior point. Therefore, from the duality theory of convex optimization, we know that the strong duality holds, i.e., $r_{s o s}^{*}=r^{*}$ and the optimal solution set of (6.1.4)-(6.1.6) is nonempty.

As already shown in (i), the optimal solution set of (6.1.4)-(6.1.6) is nonempty, which implies the conclusion in (ii) immediately.
(iii) When $r_{\text {sos }}^{*}=r^{*}$, the optimal value in (6.1.7)-(6.1.9) is also $r^{*}$, by strong duality as established in (i). Now choose an arbitrary monomial-indexed vector $y$ of the form

$$
y=\sum_{j=1}^{t} \theta_{j} m_{2 d}\left(u^{(j)}\right)
$$

for any $\theta$ such that $\theta_{j} \geq 0, \sum_{j=1}^{t} \theta_{j}=1$. Then we have

$$
\sum_{\alpha \in F} f_{\alpha} y_{\alpha}=\sum_{j=1}^{t} \theta_{j} f\left(u^{(j)}\right)=\sum_{j=1}^{t} \theta_{j} r^{*}=r^{*} .
$$

And obviously $M_{d}(y)=\sum_{j=1}^{t} \theta_{j} m_{d}\left(u^{(j)}\right) m_{d}\left(u^{(j)}\right)^{T} \succeq 0$. So $y$ is a feasible solution with optimal objective value. Thus $y$ is a optimal solution to (6.1.7)-(6.1.9).

The information about the minimizers of (6.0.1) can be found from the optimal solutions to the dual problem (6.1.7)-(6.1.9). Suppose $y^{*}=\left(y_{\alpha}^{*}\right)$ with $y_{(0, \cdots, 0)}^{*} \neq 0$ is one minimizer of (6.1.7)-(6.1.9) such that the moment matrix $M_{d}\left(y^{*}\right)$ has rank one. Then there is a vector $w$, with the same length as $M_{d}\left(y^{*}\right)$, such that

$$
M_{d}\left(y^{*}\right) / y_{(0, \cdots, 0)}^{*}=w w^{T}
$$

where the left hand side is the called normalized moment matrix, with $(1,1)$ entry being 1. Set $x^{*}:=w(2: n+1)$. So for any monomial-index $\alpha$, it holds that
$w(\alpha)=\left(x^{*}\right)^{\alpha}$. Now plug the point $x^{*}$ into the rational function $r(x)$, evaluate it, then we can see that

$$
r\left(x^{*}\right)=\frac{f\left(x^{*}\right)}{g\left(x^{*}\right)}=\frac{\sum_{\alpha} f_{\alpha}\left(x^{*}\right)^{\alpha}}{\sum_{\alpha} g_{\alpha}\left(x^{*}\right)^{\alpha}}=\frac{\sum_{\alpha} f_{\alpha} y_{\alpha}^{*}}{\sum_{\alpha} g_{\alpha} y_{\alpha}^{*}}=r_{m o m}^{*}=r_{\text {sos }}^{*} .
$$

In other words, we get a point $x^{*}$ at which the evaluation of objective $r(x)$ equals the lower bound $r_{\text {sos. }}^{*}$. Therefore, $x^{*}$ is a global minimizer and $r_{\text {sos }}^{*}$ equals the global minimum $r^{*}$. When $M_{d}\left(y^{*}\right)\left(\right.$ with $\left.y_{(0, \cdots, 0)}^{*} \neq 0\right)$ has rank more than one and satisfies the flat extension condition, there is more than one global minimizer, and they can be found numerically by solving a particular eigenvalue problem. We refer to [24, 41] for more details about the flat extension condition and extracting minimizers. When it happens that $y_{(0, \cdots, 0)}^{*}=0$, we can not normalize the moment matrix $M_{d}\left(y^{*}\right)$ to represent some measure, which might be due to the case that the infimum of $r(x)$ is attained at infinity. For instance, consider the example that $r(x):=1 /\left(1+x_{1}^{2}\right)$. The optimal solution is $y^{*}=(0,0,1)$, which can not be normalized.

In the rest of this section, we show some numerical examples. The problem (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9) are solved by YALMIP [57] which is based on $\operatorname{SeDuMi}$ [111].

Example 6.1.2. Consider the global minimization of the rational function

$$
\frac{\left(x_{1}^{2}+1\right)^{2}+\left(x_{2}^{2}+1\right)^{2}}{\left(x_{1}+x_{2}+1\right)^{2}} .
$$

Solving (6.1.4)-(6.1.6) yields the lower bound $r_{\text {sos }}^{*} \approx 0.7639$. The solution $y^{*}$ to (6.1.7)-(6.1.9) is

$$
\begin{aligned}
y^{*} \approx & (0.2000,0.1236,0.1236,0.0764,0.0764,0.0764,0.0472,0.0472, \\
& 0.0472,0.0472,0.0292,0.0292,0.0292,0.0292,0.0292) .
\end{aligned}
$$

The rank of moment matrix $M_{2}\left(y^{*}\right)$ is one, and we can extract one point $x^{*} \approx$ ( $0.6180,0.6180$ ). The evaluation of $r(x)$ at $x^{*}$ shows that $r\left(x^{*}\right) \approx 0.7639$. So $x^{*}$ is a global minimizer and 0.7639 is the global minimum (approximately, or ignoring rounding errors).

Example 6.1.3. Consider the global minimization of the rational function

$$
\frac{x_{1}^{4}-2 x_{1}^{2} x_{2} x_{3}+\left(x_{2} x_{3}+1\right)^{2}}{x_{1}^{2}}
$$

The lower bound given by (6.1.4)-(6.1.6) is $r_{\text {sos }}^{*} \approx 2.0000$. The solution $y^{*}$ to (6.1.7)(6.1.9) is

$$
\begin{aligned}
y^{*} \approx & (1.0859,-0.0000,-0.0000,-0.0000,1.0000,0.0000,-0.0000,0.8150,-0.0859 \\
& 0.8150,-0.0000,-0.0000,-0.0000,-0.0000,0.0000,-0.0000,-0.0000,-0.0000 \\
& -0.0000,-0.0000,1.0859,0.0000,-0.0000,0.8150,0.0859,0.8150,0.0000,0.0000 \\
& -0.0000,-0.0000,2.3208,-0.0000,0.1719,0.0000,2.3208) .
\end{aligned}
$$

The moment matrix $M_{2}\left(y^{*}\right)$ does not satisfy the flat extension condition, and no minimizers can be extracted. Actually one can see that 2 is the global minimum by observing the identity

$$
f(x)-2 g(x)=\left(x_{1}^{2}-x_{2} x_{3}-1\right)^{2} .
$$

The lower bound 2 is achieved at $(1,0,0)$ and hence is the global minimum. There are infinitely many global minimizers.

The relationship between the bounds is $r_{\text {mom }}^{*}=r_{\text {sos }}^{*} \leq r^{*}$ But it may happen that $r_{\text {sos }}^{*}<r^{*}$, just like SOS relaxations for minimizing polynomials. Let us see the following example.

Example 6.1.4. Consider the global minimization of the rational function

$$
\frac{x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+x_{3}^{6}}{x_{1}^{2} x_{2}^{2} x_{3}^{2}}
$$

The lower bound given by (6.1.4)-(6.1.6) is $r_{\text {sos }}^{*}=0$, and the solution $y^{*}$ to (6.1.7)(6.1.9) is

$$
y_{(2,2,2)}^{*}=1, \quad y_{\alpha}^{*}=0, \forall \alpha \neq(2,2,2) .
$$

The global minimum $r^{*}=3$ because

$$
x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+x_{3}^{6}-3 x_{1}^{2} x_{2}^{2} x_{3}^{2} \geq 0 \quad \forall x \in \mathbb{R}^{3}
$$

and $r(1,1,1)=3$. So in this example, the SOS lower bound $r_{\text {sos }}^{*}<r^{*}$. Actually for any $0<\gamma \leq 3$, the polynomial

$$
x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+x_{3}^{6}-\gamma x_{1}^{2} x_{2}^{2} x_{3}^{2}
$$

is nonnegative but not SOS. The proof is the the same as to prove that Motzkin polynomial

$$
x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+x_{3}^{6}-3 x_{1}^{2} x_{2}^{2} x_{3}^{2}
$$

is not SOS [95].

### 6.2 What if $r_{s o s}^{*}<r^{*}$ ?

From Theorem 6.1.1, we know that $r_{s o s}^{*}=r^{*}$ if and only if the polynomial $f(x)-r^{*} g(x)$ is a sum of squares. But sometimes $f(x)-r^{*} g(x)$ might not be SOS, as we observed in Example 6.1.4. In this subsection, we discuss how to minimize a rational function $r(x)$ when $r_{s o s}^{*}<r^{*}$. Here we generalize the big ball technique
introduced in [52], but we should be very careful about the zeros of the denominator $g(x)$ in $r(x)$.

Suppose we know in advance that at least one global minimizer of $r(x)$ belongs to the ball $B(c, \rho):=\left\{x \in \mathbb{R}^{n}: \rho^{2}-\|x-c\|_{2}^{2} \geq 0\right\}$ with center $c$ and radius $\rho>0$. Let $\pi(x):=\rho^{2}-\|x-c\|_{2}^{2}$. Then we immediately have that $r^{*}=\min _{x \in \mathbb{R}^{n}} r(x)=$ $\min _{x \in B(c, \rho)} r(x)$. In practice, we often choose the center $c=0$ and radius $\rho$ big enough. So the original unconstrained minimization problem (6.0.1) becomes the constrained problem

$$
\min _{x \in B(c, \rho)} r(x)
$$

One natural SOS relaxation of this constrained problem is

$$
\begin{align*}
r_{N}^{*}:=\sup _{\gamma} & \gamma  \tag{6.2.1}\\
& \text { s.t. }  \tag{6.2.2}\\
& f(x)-\gamma g(x) \equiv \sigma_{0}(x)+\sigma_{1}(x) \pi(x)  \tag{6.2.3}\\
& \operatorname{deg}\left(\sigma_{1}\right) \leq 2(N-1), \sigma_{0}(x), \sigma_{1}(x) \in \sum \mathbb{R}[X]^{2}
\end{align*}
$$

Similar to the dual of (6.1.4)-(6.1.6), the dual problem of (6.2.1)-(6.2.3) can be found to be

$$
\begin{align*}
\hat{r}_{N}^{*}:=\inf _{y} & \sum_{\alpha \in F} f_{\alpha} y_{\alpha}  \tag{6.2.4}\\
\text { s.t. } & \sum_{\alpha} g_{\alpha} y_{\alpha}=1  \tag{6.2.5}\\
& M_{N}(y) \succeq 0  \tag{6.2.6}\\
& M_{N-1}(\pi * y) \succeq 0 \tag{6.2.7}
\end{align*}
$$

where $\pi$ is the vector of the coefficients of polynomial $\pi(x)$. For a general polynomial $p(x)=\sum_{\alpha} p_{\alpha} x^{\alpha}$, the generalized moment matrix $M_{k}(p * y)$ is defined as (see

Section 2.3)

$$
M_{k}(p * y)(\beta, \tau):=\sum_{\alpha} p_{\alpha} y_{\beta+\tau+\alpha}, 0 \leq|\beta|,|\tau| \leq k .
$$

We have the following theorem for the SOS relaxation (6.2.1)-(6.2.3) and its dual (6.2.4)-(6.2.7).

Theorem 6.2.1. Assume that $r^{*}>-\infty$ and at least one global minimizer of $r(x)$ lies in the ball $B(c, \rho)$. If the $f(x)$ and $g(x)$ in $r(x)$ have no common real zeros on $B(c, \rho)$, then
(i) The lower bounds converge: $\lim _{N \rightarrow \infty} r_{N}^{*}=r^{*}$.
(ii) For $N$ large enough, there is no duality gap between (6.2.1)-(6.2.3) and its dual (6.2.4)-(6.2.7), i.e., $r_{N}^{*}=\hat{r}_{N}^{*}$.
(iii) For $N$ large enough, $r_{N}^{*}=r^{*}$ if and only if $f(x)-r^{*} g(x)=\sigma_{0}(x)+\sigma_{1}(x) \pi(x)$ for some SOS polynomials $\sigma_{0}, \sigma_{1}$ with $\operatorname{deg}\left(\sigma_{1}\right) \leq 2(N-1)$.
(iv) If $r_{N}^{*}=r^{*}$ for some integer $N$ and $u^{(j)}(j=1, \cdots, t)$ are global minimizers on $B(c, \rho)$, then every vector $y$ in the set

$$
y \in\left\{\sum_{j=1}^{t} \theta_{j} m_{2 N}\left(u^{(j)}\right): \theta_{j} \geq 0, \sum_{j=1}^{t} \theta_{j}=1\right\}
$$

is an optimal solution to (6.2.4)-(6.2.7).
Proof. (i) For any fixed $\gamma<r^{*}$, we can see that $f(x)-\gamma g(x)>0$ on $B(c, \rho)$ if $g(x) \neq 0$ (we have assumed that $g(x)$ is nonnegative). When $g(x)=0$, we must have $f(x) \geq 0$. Otherwise assume $f(u)<0$ at some point $u$ with $g(u)=0$. Then $r(x)$ is unbounded from the below, which contradicts the assumption that $r^{*}>-\infty$. Thus
$g(x)=0$ implies $f(x) \geq 0$ on $B(c, \rho)$. So we have that

$$
f(x)-\gamma g(x) \geq 0, \quad \forall x \in B(c, \rho) .
$$

Since $\gamma<r^{*}, f(x)-\gamma g(x)=0$ implies that $f(x)=g(x)=0$, which is not possible. Therefore, the polynomal $f(x)-\gamma g(x)$ is positive on ball $B(c, \rho)$. Now by Putinar's Positivstellensatz (Theorem 2.3.2), there exist SOS polynomials $\sigma_{0}, \sigma_{1}$ with degree high enough such that

$$
f(x)-\gamma g(x) \equiv \sigma_{0}(x)+\sigma_{1}(x) \pi(x)
$$

So in (6.2.1)-(6.2.3), $\gamma$ can be chosen arbitrarily close to $r^{*}$. Therefore we proved the convergence of lower bounds $r_{N}^{*}$.
(ii) Similar to the proof of Theorem 6.1.1, it suffices to show that the problem (6.2.4)-(6.2.7) has a strictly feasible solution. Let $\mu$ be a probability measure with uniform distribution on $B(c, \rho)$. Define the monomial-indexed vector $y=\left(y_{\alpha}\right)$ in the following way:

$$
y:=\int x^{\alpha} d u
$$

Now we show that $M_{N}(y)$ and $M_{N-1}(\pi * y)$ are positive definite. $M_{N}(y) \succ 0$ can be shown in the same way as in the proof of (i) in Theorem 6.1.1. Now we show that $M_{N-1}(\pi * y) \succ 0$. For any nonzero monomial-indexed vector $q$ of the same length as $M_{N-1}(\pi * y)$ (it corresponds to a nonzero polynomial $q(x)$ up to degree $N-1$ ), it holds that

$$
q^{T} M_{N-1}(\pi * y) q=\int q(x)^{2} \pi(x) d \mu=\frac{1}{\operatorname{Vol}(B(c, \rho))} \int_{B(c, \rho)} q(x)^{2} \pi(x) d x>0
$$

which implies that $M_{N-1}(\pi * y)$ is positive definite. In the above, $\operatorname{Vol}(B(c, \rho))$ denotes the volume of the ball $B(c, \rho)$. Since $g(x)$ is not identically zero and always
nonnegative, $g(x)$ can not be always zero on $B(c, \rho)$ and hence

$$
\sum_{\alpha} g_{\alpha} y_{\alpha}=\int g(x) d \mu=\frac{1}{\operatorname{Vol}(B(c, \rho))} \int_{B(c, \rho)} g(x) d x>0
$$

Now set the vector $\hat{y}$ as $y / \sum_{\alpha} g_{\alpha} y_{\alpha}$. Then can see that $\hat{y}$ is an interior point for the dual problem (6.2.4)-(6.2.7).
(iii) For any fixed $\hat{\gamma}<r^{*}$, from the previous arguments we know that the polynomial $f(x)-\gamma g(x)$ is positive on $K$. Then by Putinar's Theorem, there exist SOS polynomials $s_{0}(x), s_{1}(x)$ with $\operatorname{deg}\left(\sigma_{1}\right)$ high enough such that

$$
f(x)-\hat{\gamma} g(x) \equiv s_{0}(x)+s_{1}(x) \pi(x)
$$

This means that the primal convex problem (6.2.1)-(6.2.3) has a feasible solution. From (ii) we know its dual problem (6.2.4)-(6.2.7) has a strict interior point. Now applying the duality theory of standard convex programming, we know the solution set of (6.2.1)-(6.2.3) is nonempty. And notice that $r^{*}$ is obviously an upper bound for all $r_{N}^{*}$.

When $r_{N}^{*}=r^{*}$, we know $r_{N}^{*}$ is optimal. For $N$ sufficiently large, by (ii), the primal problem (6.2.1)-(6.2.3) is guaranteed to have a solution. So there exist SOS polynomials $\sigma_{0}(x), \sigma_{1}(x)$ with $\operatorname{deg}\left(\sigma_{1}\right) \leq 2(N-1)$ such that

$$
f(x)-r^{*} g(x) \equiv \sigma_{0}(x)+\sigma_{1}(x) \pi(x)
$$

The "if" direction is obvious.
The proof of (iv) is the same as (iii) of Theorem 6.1.1.

Remark 6.2.2. In Theorem 6.2 .1 , we need the assumption that the numerator $f(x)$ and denominator $g(x)$ have no common real zeros on ball $B(c, \rho)$ to show the
convergence $\lim _{N \rightarrow \infty} r_{N}^{*}=r^{*}$. When they have common real zeros, for any $\gamma<r^{*}$, the polynomial $f(x)-\gamma g(x)$ is not strictly positive on $B(c, \rho)$ and hence Putinar's Theorem can not be applied. In such situations, the convergence is not guaranteed (see Remark 6.3.5). However, in case of two variables, i.e., $n=2$, if $f(x)$ and $g(x)$ have at most finitely many real common zeros on $B(c, \rho)$, we still have $\lim _{N \rightarrow \infty} r_{N}^{*}=r^{*}$; furthermore, if the global minimizers of $r(x)$ are finite, finite convergence holds, i.e., there exists $N \in \mathbb{N}$ such that $r_{N}^{*}=r^{*}$. Please see Theorem 6.3.7 in Section 4. Notice that the ball $B(c, \rho)$ satisfies both conditions (i) and (ii) there.

Remark 6.2.3. When $f(x)$ and $g(x)$ have common zeros on $B(c, \rho)$, the solution to dual problem (6.2.4)-(6.2.7) is not unique. To see this fact, suppose $w \in B(c, \rho)$ is such that $f(w)=g(w)=0$, and $y^{*}$ is an optimal solution to (6.2.4)-(6.2.7). Now let $\hat{y}=m_{2 N}(w)$, which is not zero since $\hat{y}_{(0, \cdots, 0)}=1$. Then $\sum_{\alpha} f_{\alpha} \hat{y}_{\alpha}=\sum_{\alpha} g_{\alpha} \hat{y}_{\alpha}=0$ and $M_{N}(\hat{y}) \succeq 0, M_{N-1}(\pi * \hat{y}) \succeq 0$. So we can see that $y^{*}+\hat{y}$ is another feasible solution with the same optimal value. In such situations, some extracted points from the moment matrix $M_{N}\left(y^{*}+\hat{y}\right)$ might not be global minimizers and they may be the common zeros of $f(x)$ and $g(x)$. See Example 6.2.5.

Example 6.2.4. Consider the global minimization of the rational function (obtained by plugging $x_{3}=1$ in Example 6.1.4)

$$
\frac{x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+1}{x_{1}^{2} x_{2}^{2}}
$$

Choose $c=0$ and $\rho=2$. For $N=3$, the lower bound given by (6.2.1)-(6.2.3) is $r_{3}^{*}=3$, and the solution to (6.2.4)-(6.2.6) is

$$
y^{*}=(1,0,0,1,0,1,0,0,0,0,1,0,1,0,1,0,0,0,0,0,0,1,0,1,0,1,0,1) .
$$

The moment matrix $M_{3}\left(y^{*}\right)$ has rank 4, and satisfies the flat extension condition. The following four points are extracted: $( \pm 1, \pm 1)$. They are all global minimizers.

Example 6.2.5. Consider the global minimization of the rational function (obtained by plugging $x_{2}=1$ in Example 6.1.4)

$$
\frac{x_{1}^{4}+x_{1}^{2}+x_{3}^{6}}{x_{1}^{2} x_{3}^{2}}
$$

Choose $c=0$ and $\rho=2$. For $N=4$, the lower bound given by (6.2.1)-(6.2.3) is $r_{4}^{*}=3.0000$, and the solution to (6.2.4)-(6.2.6) is

$$
\begin{array}{r}
y^{*} \approx(2.8377,0,0,1,0,0,1.0008,0,0,0,0,1,0,1,0,1,0,0,0,0,0, \\
1,0,1,0,1,0,1,0,0,0,0,0,0,0,0,1,0,1,0,1,0,1,0,1)
\end{array}
$$

The moment matrix has rank 6 and satisfies flat extension condition. Six points are extracted:

$$
( \pm 1.0000, \pm 1.0000), \quad(0.0000, \pm 0.0211)
$$

The evaluation of $r(x)$ at these points shows that the first four points are global minimizers. The last two points are not global minimizers, but they are approximately common zeros of the numerator and denominator. See Remark 6.2.3.

### 6.3 Constrained minimization

In this section, consider the constrained optimization problem

$$
\begin{array}{rl}
r^{*}:=\min _{x \in \mathbb{R}^{n}} & r(x):=\frac{f(x)}{g(x)} \\
& \text { s.t. }  \tag{6.3.2}\\
h_{1}(x) \geq 0, \cdots, h_{m}(x) \geq 0
\end{array}
$$

where $f(x), g(x), h_{i}(x)$ are all real multivariate polynomials in $x=\left(x_{1}, \cdots, x_{n}\right)$. Without confusion, let $r^{*}$ still be the minimum objective value as in the unconstrained case. If some $h_{i}$ are rational functions, we can reformulate the constraints $h_{i}(x) \geq 0$ equivalently as some polynomial inequalities (one should be careful with the zeros of $\left.h_{i}(x)\right)$. Denote by $S$ the feasible set. Here we assume that $g(x)$ is not identically zero on $S$, and $g(x)$ is nonnegative on $S$ (otherwise, e.g., replace $\frac{f(x)}{g(x)}$ by $\frac{f(x) g(x)}{g^{2}(x)}$ ).

When $g(x) \equiv 1$ (or a nonzero constant), problem (6.3.1)-(6.3.2) becomes a standard constrained polynomial optimization problem. Lasserre [52] (also see Chapter 1) proposed a general procedure to solve this kind of optimization problem by a sequence of sum of squares relaxations. When $g(x)$ is a nonconstant polynomial nonnegative on $S$, Lasserre's procedure can be generalized in a natural way. For each fixed positive integer $N$, consider the SOS relaxation

$$
\begin{align*}
r_{N}^{*}:=\sup & \gamma  \tag{6.3.3}\\
\text { s.t. } & f(x)-\gamma g(x) \equiv \sigma_{0}(x)+\sum_{i=1}^{m} \sigma_{i}(x) h_{i}(x)  \tag{6.3.4}\\
& \operatorname{deg}\left(g_{i}\right) \leq 2 N-d_{i}, \quad \sigma_{i}(x) \in \sum \mathbb{R}[X]^{2} \tag{6.3.5}
\end{align*}
$$

where $d_{i}=\left\lceil\operatorname{deg}\left(h_{i}\right) / 2\right\rceil$. For any feasible $\gamma$ above, it is obvious that $f(x)-\gamma g(x) \geq 0$ on $S$ and so hence $r(x) \geq \gamma$. Thus every such $\gamma$ (and hence including $r_{N}^{*}$ ) is a lower bound of $r(x)$ on $S$.

We denote by $M(S)$ the set of polynomials which can be represented as

$$
\sigma_{0}(x)+\sigma_{1}(x) h_{1}(x)+\cdots+\sigma_{m}(x) h_{m}(x)
$$

with all $\sigma_{i}(x)$ being SOS. $M(S)$ is the quadratic module generated by polynomial tuple $\left(h_{1}, \cdots, h_{m}\right)$. Throughout this section, we make the following assumption for $M(S):$

Assumption 6.3.1 (Constraint Qualification Condition). There exist $R>0$ and SOS polynomials $s_{0}(x), s_{1}(x), \cdots, s_{m}(x) \in \sum \mathbb{R}[X]^{2}$ such that

$$
R-\|x\|_{2}^{2}=s_{0}(x)+s_{1}(x) h_{1}(x)+\cdots+s_{m}(x) h_{m}(x)
$$

Remark 6.3.2. When the assumption above is satisfied, the quadratic module $M(S)$ is said to be archimedean (see Section 2.3). Obviously, when this assumption holds, the semialgebraic set $S$ is contained in the ball $B(0, \sqrt{R})$ and hence compact; but the converse might not be true. See Example 6.3.1 in [27] for a counterexample. Under this assumption, Putinar [91] showed that every polynomial $p(x)$ positive on $S$ belongs to $M(S)$ (see Theorem 2.3.2).

Remark 6.3.3. When Assumption 6.3 .1 does not hold, we can add to $S$ one redundant constraint like $R-\|x\|_{2}^{2} \geq 0$ for $R$ sufficiently large (e.g., a norm bound is known in advance for one global minimizer). Then the new quadratic module is always archimedean.

Similar to the derivation of (6.1.7)-(6.1.9), the dual problem of (6.3.3)(6.3.4) can be found to be

$$
\begin{array}{ll}
\inf _{y} & \sum_{\alpha \in F} f_{\alpha} y_{\alpha} \\
\text { s.t. } & \sum_{\alpha} g_{\alpha} y_{\alpha}=1 \\
& M_{N}(y) \succeq 0 \\
& M_{N-d_{i}}\left(h_{i} * y\right) \succeq 0, i=1, \cdots, m . \tag{6.3.9}
\end{array}
$$

The properties of SOS relaxation (6.3.3)-(6.3.5) and (6.3.6)-(6.3.9) are summarized as follows:

Theorem 6.3.4. Assume that the minimum $r^{*}$ of $r(x)$ on $S$ is finite, and $f(x)=$ $g(x)=0$ has no solutions on $S$. Then the following holds:
(i) Convergence of the lower bounds: $\lim _{N \rightarrow \infty} r_{N}^{*}=r^{*}$.

If, furthermore, $S$ has nonempty interior, then (ii) and (iii) below are true.
(ii) For $N$ large enough, there is no duality gap between (6.3.3)-(6.3.5) and its dual (6.3.6)-(6.3.9).
(iii) For $N$ large enough, $r_{N}^{*}=r^{*}$ if and only if $f(x)-r^{*} g(x) \equiv \sigma_{0}(x)+\sum_{i=1}^{m} \sigma_{i} h_{i}(x)$ for SOS polynomials $\sigma_{i}(x)$ with $\operatorname{deg}\left(\sigma_{i} h_{i}\right) \leq 2 N$.
(iv) If $r_{N}^{*}=r^{*}$ for some integer $N$ and $u^{(j)}(j=1, \cdots, t)$ are global minimizers on $S$, then every vector $y$ in the set

$$
y \in\left\{\sum_{j=1}^{t} \theta_{j} m_{2 N}\left(u^{(j)}\right): \theta_{j} \geq 0, \sum_{j=1}^{t} \theta_{j}=1\right\}
$$

is an optimal solution to (6.3.6)-(6.3.9).
Proof. (i) For any $\gamma<r^{*}$, we have that the polynomial

$$
\vartheta_{\gamma}(x):=f(x)-\gamma g(x)
$$

is nonnegative on $S$. When $\vartheta_{\gamma}(u)=0$ for some point $u \in S$, we must have $f(u)=g(u)=0$, since otherwise $g(u)>0(g(x)$ is assumed to be nonnegative on $S$ ) and $r(u)=\gamma<r^{*}$, which is impossible. Therefore $\vartheta_{\gamma}(x)$ is positive on $S$. By Theorem 2.3.2, there exist SOS polynomials $\sigma_{i}(x)$ of degree high enough such that

$$
\vartheta_{\gamma}(x) \equiv \sigma_{0}(x)+\sum_{i=1}^{m} \sigma_{i}(x) h_{i}(x) .
$$

Therefore the claim in (i) is true.
(ii),(iii) \& (iv): The proof here is almost the same as the one of Theorem 6.2.1. In a similar way, show that (6.3.3)-(6.3.5) has a feasible solution, and (6.3.6)-(6.3.9) has an interior point. Then apply the duality theory of convex programming. In (iv), check every $y$ with given form is feasible and achieves the optimal objective value.

Remark 6.3.5. In Theorem 6.3.4, we made the assumption that $f(x)$ and $g(x)$ have no common zeros on $S$. But sometimes $f(x)$ and $g(x)$ may have common zeros, and it is also possible that the minimum $r^{*}$ is attained at the common zero(s) (in this case, $f(x)$ and $g(x)$ are of the same magnitude order around the common zero(s)). In such situations, we can not apply Theorem 2.3.2 and might not have convergence. For a counterexample, consider the global minimization (with $n=1$ )

$$
\begin{array}{ll}
\min & r(x):=\frac{1+x}{\left(1-x^{2}\right)^{2}} \\
\text { s.t. } & \left(1-x^{2}\right)^{3} \geq 0
\end{array}
$$

The global minimum is $r^{*}=\frac{27}{32}$ and the minimizer is $x^{*}=-\frac{1}{3}$. However, for any $\gamma<\frac{27}{32}$, there do not exist SOS polynomials $\sigma_{0}(x), \sigma_{1}(x)$ such that

$$
1+x-\gamma\left(1-x^{2}\right)^{2} \equiv \sigma_{0}(x)+\sigma_{1}(x)\left(1-x^{2}\right)^{3}
$$

Otherwise, for a contradiction, suppose they exist. Then the left hand side vanishes at $x=-1$ and so does the right hand side. So $x=-1$ is a zero of $\sigma_{0}(x)$ with multiplicity greater than one, since $\sigma_{0}$ is SOS. Hence $x=-1$ is a multiple zero of the left hand side, which is impossible since the derivative of $1+x-\gamma\left(1-x^{2}\right)^{2}$ at $x=-1$ is 1 . This counterexample is motivated by the one given by Stengle [109],
which shows that the polynomial $1-x^{2}$ does not belong to the quadratic module $M\left(\left(1-x^{2}\right)^{3}\right)$ since $1-x^{2}$ is not strictly positive on $\left\{x:\left(1-x^{2}\right)^{3} \geq 0\right\}$. On the other hand, if we can know in advance that the global minimum is not attained where the denominator $g(x)$ vanishes, one way to overcome this difficulty is to add more constraints which keep the global minimizers but kick out the zeros of $g(x)$.

Remark 6.3.6. When $f(x)$ and $g(x)$ have common zeros on $S$, the solution to dual problem (6.3.6)-(6.3.9) is not unique. In such situations, some extracted points from the moment matrix $M_{N}\left(y^{*}\right)$ may not be global minimizers and they might be the common zeros of $f(x)$ and $g(x)$. See Remark 6.2.3.

When $n=2$, i.e., in case of two variables, the distinguished representations of nonnegative polynomials by Scheiderer [97] are very useful. Under some conditions on the geometry of the feasible set $S$, the convergence or even finite convergence holds if $f(x)$ and $g(x)$ has finitely many common zeros on $S$. This leads to our next theorem.

Theorem 6.3.7. Suppose $n=2$. Let $Z(f, g)=\{u \in S: f(u)=g(u)=0\}$ and $\Theta$ be the set of global minimizer(s) of $r(x)$ on $S$. We have convergence $\lim _{N \rightarrow \infty} r_{N}^{*}=r^{*}$ if $\Omega=Z(f, g)$ is finite and satisfies at least one of the following two conditions:
(i) Each curve $\mathcal{C}_{i}=\left\{x \in \mathbb{C}^{2}: h_{i}(x)=0\right\}(i=1, \cdots, m)$ is reduced and no two of them share an irreducible component. No point in $\Omega$ is a singular point of the curve $\mathcal{C}_{1} \cup \cdots \cup \mathcal{C}_{m}$.
(ii) Each point of $\Omega$ is an isolated real common zero of $f(x)-r^{*} g(x)$ in $\mathbb{R}^{2}$, but not an isolated point of the feasible set $S$.

Furthermore, if $\Omega=Z(f, g) \cup \Theta$ is finite and satisfies at least one of (i) and (ii), then we have finite convergence, i.e., there exists an integer $N$ such that $r_{N}^{*}=r^{*}$.

Proof. Firstly, assume that $\Omega=Z(f, g)$ is finite and satisfies at least one of (i) and (ii). For any $\gamma<r^{*}$, we have that the polynomial

$$
\vartheta_{\gamma}(x):=f(x)-\gamma g(x)
$$

is nonnegative on $S$. When $\vartheta_{\gamma}(u)=0$ for some point $u \in S$, we must have $f(u)=$ $g(u)=0$, since otherwise $g(u)>0$ and $r(u)=\gamma<r^{*}$, which is impossible. By assumption in the theorem, the nonnegative polynomial $\vartheta_{\gamma}(x)$ has at most finitely many zeros on $S$. Now applying Corollary 3.7 (if (i) holds) or Corollary 3.10 (if (ii) holds) in [97], we know that there exist SOS polynomials $\sigma_{i}(x)$ of degree high enough such that

$$
\vartheta_{\gamma}(x) \equiv \sigma_{0}(x)+\sum_{i=1}^{m} \sigma_{i}(x) h_{i}(x) .
$$

Secondly, assume that $\Omega=Z(f, g) \cup \Theta$ is finite and satisfies at least one of (i) and (ii). Consider the polynomial $\vartheta_{r^{*}}(x):=f(x)-r^{*} g(x)$, which is nonnegative on $S$. When $\vartheta_{r^{*}}(u)=0$ for some $u \in S$, we must have either $f(u)=g(u)=0$ or $r(u)=r^{*}$. Thus polynomial $\vartheta_{r^{*}}(x)$ has at most finitely many zeros on $S$. Corollary 3.7(if (i) holds) or Corollary 3.10 (if (ii) holds) in [97] implies that there are SOS polynomials $\sigma_{i}(x)$ with $\operatorname{deg}\left(\sigma_{i} h_{i}\right) \leq 2 N$ ( N is large enough) such that

$$
\vartheta_{r^{*}}(x) \equiv \sigma_{0}(x)+\sum_{i=1}^{m} \sigma_{i}(x) h_{i}(x)
$$

which completes the proof.

Example 6.3.8. Consider the problem

$$
\begin{array}{ll}
\min _{x} & \frac{x_{1}^{4} x_{2}^{2}+x_{1}^{2} x_{2}^{4}+1}{x_{1}^{2} x_{2}^{2}} \\
\text { s.t. } & x_{1}, x_{2} \geq 0,1-x_{1}^{2}-x_{2}^{2} \geq 0 .
\end{array}
$$

The SOS relaxation (6.3.3)-(6.3.5) of order $N=3$ yields the lower bound $r_{3}^{*} \approx 5.000$, and we can extract one point $x^{*} \approx(0.7071,0.7071)$ from the dual solution to (6.3.6)(6.3.9). $r\left(x^{*}\right) \approx 5.0000$ shows that the point $x^{*}$ is a global minimizer.

Example 6.3.9. Consider the problem

$$
\begin{array}{ll}
\min _{x} & \frac{x_{1}^{4}+x_{1}^{2}+x_{3}^{6}}{x_{1}^{2} x_{3}^{2}} \\
\text { s.t. } & x_{1}, x_{3} \geq 0,1-x_{1}^{2}-x_{3}^{2} \geq 0 .
\end{array}
$$

The SOS relaxation (6.3.3)-(6.3.5) of order $N=3$ yields lower bound $r_{3}^{*} \approx 3.2324$, and we can extract one point $x^{*} \approx(0.6276,0.7785)$ from the dual solution to (6.3.6)(6.3.9). $r\left(x^{*}\right) \approx 3.2324$ shows that the point $x^{*}$ is a global minimizer.

Example 6.3.10. Consider the problem

$$
\begin{aligned}
\min _{x} & \frac{x_{1}^{3}+x_{2}^{3}+3 x_{1} x_{2}+1}{x_{1}^{2}\left(x_{2}+1\right)+x_{2}^{2}\left(1+x_{1}\right)+x_{1}+x_{2}} \\
\text { s.t. } & 2 x_{1}-x_{1}^{2} \geq 0,2 x_{2}-x_{2}^{2} \geq 0 \\
& 4-x_{1} x_{2} \geq 0, x_{1}^{2}+x_{2}^{2}-\frac{1}{2} \geq 0 .
\end{aligned}
$$

The SOS relaxation (6.3.3)-(6.3.5) of order $N=2$ yields lower bound $r_{2}^{*}=1$ and we can extract three points $(0,1), \quad(1,0), \quad(1,1)$ from the dual solution to (6.3.6)(6.3.9). The evaluations of $r(x)$ at these three points show that they are all global minimizers.

Example 6.3.11. Consider the problem

$$
\begin{array}{cl}
\min _{x} & \frac{x_{1}^{4}+x_{2}^{4}+x_{3}^{4}+x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+2 x_{1} x_{2} x_{3}\left(x_{1}+x_{2}+x_{3}\right)}{x_{1}^{3}+x_{2}^{3}+x_{3}^{3}+2 x_{1} x_{2} x_{3}} \\
\text { s.t. } & x_{1}^{4}+x_{2}^{4}+x_{3}^{4}=1+x_{1}^{2} x_{2}^{2}+x_{2}^{2} x_{3}^{2}+x_{3}^{2} x_{1}^{2} \\
& x_{3} \geq x_{2} \geq x_{1} \geq 0
\end{array}
$$

The SOS relaxation (6.3.3)-(6.3.5) of order $N=3$ yields $r_{3}^{*} \approx 2.0000$ and we can extract two points

$$
x^{*} \approx(0.0000,0.0000,1.0000), \quad x^{* *} \approx(-0.0032,0.9977,0.9974)
$$

from the dual solution to (6.3.6)-(6.3.9). $x^{*}$ is feasible and $r\left(x^{*}\right) \approx 2.0000$ implies that $x^{*}$ is a global minimizer. And $x^{* *}$ is not feasible, but if we round $x^{* *}$ to the nearest feasible point we get $(0,1,1)$, which is another global minimizer since $r(0,1,1)=2$.

Example 6.3.12. Consider the problem

$$
\begin{array}{cl}
\min _{x} & \frac{x_{1}^{2}+x_{2}^{2}+x_{3}^{2}+x_{4}^{2}+2\left(x_{2}+x_{3}+x_{1} x_{3}+x_{1} x_{4}+x_{2} x_{4}\right)+1}{x_{1}+x_{4}+x_{1} x_{2}+x_{2} x_{3}+x_{3} x_{4}} \\
\text { s.t. } & x_{1}^{2}+x_{2}^{2}-2 x_{3} x_{4}=0 \\
& 4-x_{1}^{2}-x_{2}^{2}-x_{3}^{2}-x_{4}^{2} \geq 0 \\
& x_{1}, x_{2}, x_{3}, x_{4} \geq 0 .
\end{array}
$$

The SOS relaxation (6.3.3)-(6.3.5) of order $N=3$ yields $r_{3}^{*} \approx 2.0000$ and we can extract one point

$$
x^{*} \approx(0.0002,0.0000,0.0000,0.9998) .
$$

from the dual solution to (6.3.6)-(6.3.9). $r\left(x^{*}\right) \approx 2.0000$ implies that $x^{*}$ is a global minimizer (approximately). Actually the exact global minimizer is $(0,0,0,1)$.

## Chapter 7

## Applications of Polynomial

## Optimization

This chapter shows some specific applications of polynomial optimization. Shape design of transfer functions, minimum ellipsoid bounds for polynomial systems, finding polynomials with a nontrivial GCD, maximum likelihood optimization, and sensor network localization will be discussed.

### 7.1 Shape optimization of transfer functions

Consider the linear time invariant (LTI) single-input-single-output (SISO) system

$$
\begin{align*}
& \dot{x}(t)=A x(t)+b u(t)  \tag{7.1.1}\\
& y(t)=c^{T} x(t)+d u(t) \tag{7.1.2}
\end{align*}
$$

where $A \in \mathbb{R}^{n \times n}, b, c \in \mathbb{R}^{n}, d \in \mathbb{R} . u(t)$ is the input, $x(t)$ is the state variable, and $y(t)$ is the output. The relationship between the Laplace transformations of $u(t)$ and $y(t)$ is that $\mathcal{L}(y)(s)=H(s) \mathcal{L}(u)(s)$ where

$$
H(s)=d+c^{T}(s I-A)^{-1} b
$$

is called the transfer function of system (7.1.1)-(7.1.2). $H(s)$ can also be written as the rational function

$$
\frac{\sum_{k=0}^{n} \alpha_{k} s^{k}}{\sum_{k=0}^{n} \beta_{k} s^{k}} \equiv \frac{q_{1}(s)}{q_{2}(s)} .
$$

Note that $\operatorname{deg}\left(q_{1}\right) \leq \operatorname{deg}\left(q_{2}\right) \leq n$. Actually any rational function $H(s)$ of this form is the transfer function of some particular LTI system (it is not unique). Any such LTI system is called a realization of $H(s)$. There are many such (algebraically equivalent) LTI systems [17, chap. 9].

In some engineering applications, designers want the transfer function to have certain desirable properties. For example, we may want the Bode plot (the graph of $|H(s)|$ versus the pure imaginary axis $s=j \cdot \omega)$ to have a certain shape corresponding to some kind of filtering. In this section, we discuss the shape optimization problem of choosing the coefficients of rational function $H(s)$ so that its Bode plot has some desired shape. For discrete LTI systems, i.e. the governing differential equation (7.1.1)-(7.1.2) is replaced by difference equations (see [17]), there are several papers $[1,35,116]$ that show how to formulate the filter design problem as the solution of the feasibility problem for certain convex sets. The main idea of this section is to apply the spectral factorization of trigonometric polynomials, a characterization of nonnegative univariate polynomials, and semi-infinite programming. This approach can be used to design the transfer function to be a bandpass filter, piecewise constant or polynomial, or even have an arbitrary shape.

Our contribution is to extend these results to the continuous time LTI SISO systems (7.1.1)-(7.1.2). In this case the transfer function is not a trigonometric polynomial and hence we cannot directly apply spectral factorization. Fortunately our transfer function is a univariate rational function, which lets us apply certain characterizations of nonnegative univariate polynomials over the whole axis $(-\infty, \infty)$, semi-axis $(0, \infty)$, or some finite interval $[a, b]$. Using these characterizations, we can solve the shape optimization problem for the following shapes: (i) standard bandpass filter design; (ii) arbitrary piecewise constant shape; (iii) arbitrary piecewise polynomial shape; (iv)general nonnegative function. The first three shape design problems can be solved by testing the feasibility of certain Linear Matrix Inequalities [14]. The fourth shape design can be obtained by semi-infinite programming (SIP) [82, 116]. In this section, we show how to get the first two kinds of designs. The designs for the latter two are similar, see [69]. There is a lot of related work in various kinds of filter design problems and characterizing nonnegative polynomials on lines, circles, or intervals. We refer to $[1,35,116,31,67,36]$.

Now we apply Theorem 2.3.7 to characterize the transfer function, which is similar to the spectral factorization for trigonometric polynomials. Observe that

$$
\begin{aligned}
|H(j \omega)|^{2}=\frac{\left|q_{1}(j \omega)\right|^{2}}{\left|q_{2}(j \omega)\right|^{2}} & =\frac{\left|q_{1, \text { even }}(j \omega)+q_{1, \text { odd }}(j \omega)\right|^{2}}{\left|q_{2, \text { even }}(j \omega)+q_{2, \text { odd }}(j \omega)\right|^{2}} \\
& =\frac{\left[q_{11}\left(\omega^{2}\right)\right]^{2}+\omega^{2}\left[q_{12}\left(\omega^{2}\right)\right]^{2}}{\left[q_{21}\left(\omega^{2}\right)\right]^{2}+\omega^{2}\left[q_{22}\left(\omega^{2}\right)\right]^{2}} \\
& \equiv \frac{p_{1}(w)}{p_{2}(w)} \text { where } w=\omega^{2}
\end{aligned}
$$

Here $q_{i, e v e n}$ and $q_{i, o d d}$ denote the even and odd parts of the polynomial $q_{i}$, and $q_{i j}, i, j=1,2$ are defined accordingly. Note that $p_{1}(w)$ and $p_{2}(w)$ are nonnegative polynomials on $w \in[0, \infty)$. Conversely, by Theorem 2.3.7, given any such nonneg-
ative $p_{1}(w)$ and $p_{2}(w)$, it is possible to reconstruct the $q_{i j}(w)$, and so $q_{i}(j \omega)$ and $H(j \omega)$. In other words, $p_{1}(w)$ and $p_{2}(w)$ with $\operatorname{deg}\left(p_{1}\right) \leq \operatorname{deg}\left(p_{2}\right)$ satisfy $|H(j \omega)|^{2}=$ $p_{1}(w) / p_{2}(w)$ where $w=\omega^{2}$ for some transfer function $H(j \omega)$ if and only if they are nonnegative on $[0, \infty)$.

First, let us design a bandpass filter. The goal is to design a transfer function $|H(j \omega)|^{2}=\frac{p_{1}(w)}{p_{2}(w)}$ which is close to one on some squared frequency $\left(w=\omega^{2}\right)$ interval $\left[w^{\ell}, w^{r}\right]$ and tiny in a neighborhood just outside this interval. The design rules can be formulated as

$$
\begin{aligned}
p_{1}(w), p_{2}(w) & \geq 0, \forall w \geq 0 \\
1-\alpha \leq \frac{p_{1}(w)}{p_{2}(w)} & \leq 1+\beta, \forall w \in\left[w^{\ell}, w^{r}\right] \\
\frac{p_{1}(w)}{p_{2}(w)} & \leq \delta, \forall w \in\left[w_{1}^{\ell}, w_{2}^{\ell}\right] \cup\left[w_{1}^{r}, w_{2}^{r}\right]
\end{aligned}
$$

where the interval $\left[w_{1}^{\ell}, w_{2}^{\ell}\right]$ is to the left of $\left[w^{\ell}, w^{r}\right]$, and $\left[w_{1}^{r}, w_{2}^{r}\right]$ is to the right. Here $\alpha, \beta, \delta$ are tiny tolerance parameters (say around .05). Let $p_{1}$ and $p_{2}$ be the vectors of coefficients of $p_{1}(w)$ and $p_{2}(w)$ respectively. Then the constraints above can we restated as

$$
\begin{aligned}
p_{1}, p_{2} & \in K_{0, \infty} \\
p_{1}-(1-\alpha) p_{2} & \in K_{w^{\ell}, w^{r}} \\
(1+\beta) p_{2}-p_{1} & \in K_{w^{\ell}, w^{r}} \\
\delta p_{2}-p_{1} & \in K_{w_{1}^{\ell}, w_{2}^{\ell}} \cap K_{w_{1}^{r}, w_{2}^{r}}
\end{aligned}
$$

where the cones $K_{[a, b]}$ are defined as

$$
K_{[a, b]}=\{p(t) \in \mathbb{R}[t]: p(w) \geq 0 \forall w \in[a, b]\} .
$$

The above cone constraints can be expressed as $A p \in K$ where

$$
A=\left[\begin{array}{cc}
I_{n+1} & 0 \\
0 & I_{n+1} \\
I_{n+1} & (\alpha-1) I_{n+1} \\
-I_{n+1} & (1+\beta) I_{n+1} \\
-I_{n+1} & \delta I_{n+1} \\
-I_{n+1} & \delta I_{n+1}
\end{array}\right], p=\left[\begin{array}{l}
p_{1} \\
p_{2}
\end{array}\right]
$$

and $K=K_{0, \infty} \times K_{0, \infty} \times K_{w^{\ell}, w^{r}} \times K_{w^{\ell}, w^{r}} \times K_{w_{1}^{\ell}, w_{2}^{\ell}} \times K_{w_{1}^{r}, w_{2}^{r}}$. Given $(\alpha, \beta, \delta)$, we solve a cone feasibility problem and then recover the coefficient of $p$ (see [69]). As introduced in [35] for the discrete case, we can also consider the following objectives:

- minimize $\alpha+\beta$ for fixed $\delta$ and $n$
- minimize $\delta$ for fixed $\alpha, \beta$, and $n$
- minimize the degree $n$ of $p_{1}$ and $p_{2}$ for fixed $\alpha, \beta$, and $\delta$.

These optimization problems with objectives are no longer convex, but quasi-convex. This means that we can use bisection to find the solution by solving a sequence of LMI feasibility problems. A design example is shown in Figure 7.1 (see also [69]). The parameters in Figure 7.1 are $\left[w^{l}, w^{r}\right]=[2,3],\left[w_{1}^{l}, w_{2}^{l}\right]=[0,1.8],\left[w_{1}^{r}, w_{2}^{r}\right]=$ $[3.2,5], \alpha=\beta=0.05, \delta=0.05, n=10$.

Second, let us show how to design a piecewise constant shape. In other words, we want the transfer function to be close to given constant values $c_{1}, \ldots, c_{m}$ in a set of $m$ disjoint intervals $\omega^{2}=w \in\left[a_{k}, b_{k}\right]$, where $a_{1}<b_{1}<a_{2}<b_{2}<$ $\cdots<a_{m}<b_{m}$. More precisely we want the transfer function to lie in the interval


Figure 7.1: A bandpass filter.
$\left[(1-\alpha) c_{k},(1+\beta) c_{k}\right]$ for $w \in\left[a_{k}, b_{k}\right]$. By picking enough intervals (picking $m$ large enough) we can approximate any continuous function as closely as we like.

These constraints may be written in the form

$$
\begin{aligned}
p_{1}(w), p_{2}(w) & \geq 0, \forall w \geq 0 \\
(1-\alpha) c_{k} \leq \frac{p_{1}(w)}{p_{2}(w)} & \leq(1+\beta) c_{k}, \forall w \in\left[a_{k}, b_{k}\right], k=1, \cdots, m .
\end{aligned}
$$

Similarly, these constraints can also be written as cone constraints

$$
\begin{aligned}
p_{1}(w), p_{2}(w) & \in K_{0, \infty} \\
p_{1}-(1-\alpha) c_{k} p_{2},(1+\beta) c_{k} p_{2}-p_{1} & \in K_{a_{k}, b_{k}}, k=1, \cdots, m .
\end{aligned}
$$

Now the design problem becomes to find vector $p$ such that $A p \in K$ where

$$
A=\left[\begin{array}{cc}
I_{n+1} & 0 \\
0 & I_{n+1} \\
I_{n+1} & (\alpha-1) c_{1} I_{n+1} \\
(1+\beta) c_{1} I_{n+1} & -I_{n+1} \\
\vdots & \vdots \\
I_{n+1} & (\alpha-1) c_{m} I_{n+1} \\
(1+\beta) c_{m} I_{n+1} & -I_{n+1}
\end{array}\right], p=\left[\begin{array}{c}
p_{1} \\
p_{2}
\end{array}\right]
$$

and $K=K_{0, \infty}^{2} \times K_{a_{1}, b_{1}}^{2} \times \cdots \times K_{a_{m}, b_{m}}^{2}$. By solving a particular feasibility problem, we can find the coefficients $p$ (see [69] for details). Similar to bandpass filter design, various design objectives can be achieved by applying bisection. A design example for a step function with 3 steps is shown in Figure 7.2. The parameters in Figure 7.2


Figure 7.2: A 3 -step constant filter.
are $\left[a_{1}, b_{1}\right]=[0,1.8],\left[a_{2}, b_{2}\right]=[2,3],\left[a_{3}, b_{3}\right]=[3.2,5], c_{1}=1, c_{2}=3, c_{3}=$ $2, \alpha=\beta=0.05, n=10$.

Lastly, let us show how to use Theorems 2.3.7 and 2.3.8 to recover the transfer function from the polynomials $p_{1}(w)$ and $p_{2}(w)$ that are obtained by design
(see [69] for more details). For given polynomials $p_{1}(w)$ and $p_{2}(w)\left(w=\omega^{2}\right)$ such that $\frac{p_{1}}{p_{2}}$ has some desired shape, we need to find real polynomials $q_{1}$ and $q_{2}$ so that

$$
\frac{p_{1}(w)}{p_{2}(w)}=\left|\frac{q_{1}(j \omega)}{q_{2}(j \omega)}\right|^{2} .
$$

To this end, given a polynomial $p(w)$ that is nonnegative on $[0, \infty)$, we can find two polynomials $q_{e}(w)$ and $q_{o}(w)$ such that $p(w)=q_{e}^{2}(w)+w \cdot q_{o}^{2}(w)$ (see [69]). Then $q_{e}$ contains the even coefficients and $q_{o}$ the odd coefficients (modulo signs) of the desired polynomials $q_{1}, q_{2}$.

### 7.2 Minimum ellipsoid bounds for polynomial systems

This section shows how to find a minimum ellipsoid bound on the solution set of parameterized polynomial systems. A full version of this section can be found in [70]. Consider the polynomial system of equalities and inequalities of the form:

$$
\left\{\begin{array}{c}
\phi_{1}\left(x_{1}, \cdots, x_{n} ; \mu_{1}, \cdots, \mu_{r}\right)=0  \tag{7.2.3}\\
\vdots \\
\phi_{s}\left(x_{1}, \cdots, x_{n} ; \mu_{1}, \cdots, \mu_{r}\right)=0 \\
\rho_{1}\left(x_{1}, \cdots, x_{n} ; \mu_{1}, \cdots, \mu_{r}\right) \leq 0 \\
\vdots \\
\rho_{t}\left(x_{1}, \cdots, x_{n} ; \mu_{1}, \cdots, \mu_{r}\right) \leq 0
\end{array}\right.
$$

where $x=\left(x_{1}, \cdots, x_{n}\right)^{T} \in \mathbb{R}^{n}$ and $\mu=\left(\mu_{1}, \cdots, \mu_{r}\right)^{T} \in \mathbb{R}^{r}$. For each $1 \leq i \leq s$ and $1 \leq j \leq t, \phi_{i}$ and $\rho_{j}$ are multivariate polynomials in $(x, \mu) \in \mathbb{R}^{n+r} . \mu$ can be thought of as parameters perturbing the solution $x$. We are only interested in bounding $x$ for all $\mu$ determined by (7.2.3). $x$ can also be thought of as the projection of the solution
$(x, \mu) \in \mathbb{R}^{n+r}$ of (7.2.3) into the subspace $\mathbb{R}^{n}$. We consider only real solutions, since many practical problems concern only real solutions.

Our goal is to bound the projected solution set defined as

$$
\mathcal{S}=\left\{x \in \mathbb{R}^{n}: \exists \mu \in \mathbb{R}^{r} \text { s.t. }(x, \mu) \text { satisfies system (7.2.3) }\right\} .
$$

For a given $\mu$, there may be no real $x$ satisfying (7.2.3), or one unique such $x$, or several such $x$, or infinitely many such $x$. So $S$ can be quite complicated.

The traditional approach in perturbation analysis of a system of equations is to find the maximum distance of the perturbed solutions to the unperturbed solution, i.e. to find a bounding ball of smallest radius with the unperturbed solution at the center. This approach works well when the solution set is almost a ball and the unperturbed solution lies near the center. Unfortunately, this is often not the case in practice, when the solution set is very elongated. Instead, we seek a bounding ellipsoid of smallest volume (in a sense defined below), which can more effectively bound many elongated sets.

The particular idea for finding minimum ellipsoids was introduced in [18, 19], where the authors try to find the minimum ellipsoids for linear systems whose coefficients are rational functions of perturbing parameters. In this section, we show how to find the minimum ellipsoid bounds for the projected solution set $\mathcal{S}$.

An open ellipsoid in $\mathbb{R}^{n}$ can be described as

$$
\begin{equation*}
\mathcal{E}(P, z)=\left\{x \in \mathbb{R}^{n}:(x-z)^{T} P^{-1}(x-z)<1\right\} \tag{7.2.4}
\end{equation*}
$$

where $P \in S_{++}^{n}$ is the shape matrix, and $z \in \mathbb{R}^{n}$ is the center. By taking the Schur
complement, the ellipsoid can be written in LMI form

$$
\mathcal{E}(P, z)=\left\{x \in \mathbb{R}^{n}:\left[\begin{array}{cc}
P & x-z  \tag{7.2.5}\\
(x-z)^{T} & 1
\end{array}\right] \succ 0\right\}
$$

For example, the ellipsoid in the 2D plane given by

$$
\frac{\left(x_{1}-z_{1}\right)^{2}}{a^{2}}+\frac{\left(x_{2}-z_{2}\right)^{2}}{b^{2}}<1
$$

has the shape matrix $P=\left[\begin{array}{cc}a^{2} & 0 \\ 0 & b^{2}\end{array}\right]$.
How do we measure the "size" of an ellipsoid? The "best" measure would appear to be its volume, which is proportional to $\sqrt{\operatorname{det} P}$. However, we will instead choose $\operatorname{trace}(P)$ to measure the size, for two reasons: 1) $\operatorname{trace}(P)$ is an affine function, whereas $\sqrt{\operatorname{det} P}$ is not; 2) $\operatorname{trace}(P)$ is zero if and only if all the axes are zero, but $\sqrt{\operatorname{det} P}$ is zero if any axis is zero.

The minimum ellipsoid bound can be found by solving the optimization problem:

$$
\left.\begin{array}{rl}
\inf _{P \in S_{++}^{n}, z \in \mathbb{R}^{n}} & \operatorname{trace}(P) \\
& (x-z)^{T} P^{-1}(x-z)<1  \tag{7.2.7}\\
\text { s.t. } & \text { for all }(x, \mu) \text { satisfying } \\
& \phi_{i}(x, \mu)=0, \rho_{j}(x, \mu) \leq 0
\end{array}\right\} .
$$

In the rest of this section, we will show how to relax the constraint (7.2.7) by the technique of Sum of Squares (SOS), which can be reduced to solving some SDP.

The constraint (7.2.7) holds if and only if

$$
1-(x-z)^{T} P^{-1}(x-z)>0 \text { for all }\left\{x \in \mathbb{R}^{n} \left\lvert\, \begin{array}{l}
\phi_{i}(x, \mu)=0, i=1, \cdots, s \\
\rho_{j}(x, \mu) \leq 0, j=1, \cdots, t
\end{array}\right.\right\}
$$

A certificate for the above can be obtained immediately by applying Putinar's Positivstellensatz (Theorem 2.3.2).

Theorem 7.2.1. Suppose Putinar's constraint qualification (see Section 2.3) holds for the polynomial system $\left\{ \pm \phi_{1}, \cdots, \pm \phi_{s},-\rho_{1}, \cdots,-\rho_{t}\right\}$. Then, if the constraint (7.2.7) holds, there exist polynomials $\lambda_{i}=\lambda_{i}(x, \mu), \sigma_{j}=\sigma_{j}(x, \mu)$ such that

$$
\begin{array}{r}
1-(x-z)^{T} P^{-1}(x-z)+\sum_{i=1}^{s} \lambda_{i} \phi_{i}+\sum_{j=1}^{t} \sigma_{j} \rho_{j} \succeq_{s o s} 0 \\
\sigma_{1}, \cdots, \sigma_{t} \succeq_{\text {sos }} 0
\end{array}
$$

where the inequality $q(x, \mu) \succeq_{\text {sos }} 0$ means that the polynomial $q(x, \mu)$ is SOS.

Proof. Let $p=1-(x-z)^{T} P^{-1}(x-z)$ and $\left\{ \pm \phi_{1}, \cdots, \pm \phi_{s},-\rho_{1}, \cdots,-\rho_{t}\right\}$ be the polynomials defining the semi-algebraic set in Theorem 2.3.2. Notice that $p(x)$ is strictly positive on the semialgebraic set

$$
\left\{(x, \mu): \phi_{1} \geq 0, \cdots, \phi_{s} \geq 0,-\phi_{1} \geq 0, \cdots,-\phi_{s} \geq 0,-\rho_{1} \geq 0, \cdots,-\rho_{t} \geq 0\right\} .
$$

Then by Theorem 2.3.2, there exist SOS polynomials $\varphi, \tau_{i}, \nu_{i}(i=1, \cdots, s)$, and $\sigma_{j}(j=1, \cdots, t)$ such that

$$
1-(x-z)^{T} P^{-1}(x-z)=\varphi+\sum_{i=1}^{s}\left(\tau_{i}-\nu_{i}\right) \phi_{i}-\sum_{j=1}^{t} \sigma_{j} \rho_{j} .
$$

Let $\lambda_{i}=\nu_{i}-\tau_{i}$. Then we get the result in the theorem.

Remark 7.2.2. If $\left\{ \pm \phi_{1}, \cdots, \pm \phi_{s},-\rho_{1}, \cdots,-\rho_{t}\right\}$ does not satisfy the constraint qualification condition for Putinar's Theorem, we can add a redundant ball condition like $x^{2}+\mu^{2} \leq R$ for $R$ sufficiently large. Then Putinar's Theorem can be applied.

Now we are ready to show how to find the minimum ellipsoid bounds. Denote by $R_{N}[x, \mu]$ the set of polynomials in $(x, \mu)$ with degrees at most $N$. By Theorem 7.2.1, the problem (7.2.6)-(7.2.7) can be relaxed as

$$
\begin{aligned}
\hat{E}_{N}: & \min _{\substack{P \in S_{+1}^{n}, z \in \mathbb{R}^{n} \\
\lambda_{i}, \sigma_{j} \in R_{N}}} \operatorname{trace},(P) \quad \text { subject to } \\
& 1-(x-z)^{T} P^{-1}(x-z)+\sum_{i=1}^{s} \lambda_{i} \phi_{i}+\sum_{j=1}^{t} \sigma_{j} \rho_{j} \succeq_{\text {sos }} 0, \quad \sigma_{1}, \cdots, \sigma_{t} \succeq_{\text {sos }} 0
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\substack{\min _{\begin{subarray}{c}{P \in S_{+}^{n}+z \in \mathbb{R}^{n} \\
\lambda_{i}, \sigma_{j} \in R_{N}[x, \mu]} }}} & \operatorname{trace}(P) \\
\text { s.t. } & 1-\left[\begin{array}{l}
x \\
1
\end{array}\right]^{T}\left[\begin{array}{ll}
I & -z
\end{array}\right]^{T} P^{-1}\left[\begin{array}{ll}
I & -z
\end{array}\right]\left[\begin{array}{l}
x \\
1
\end{array}\right]+ \\
& \sum_{i=1}^{s} \lambda_{i} \phi_{i}+\sum_{j=1}^{t} \sigma_{j} \rho_{j} \succeq_{\text {sos }} 0, \quad \sigma_{1}, \cdots, \sigma_{t} \succeq_{\text {sos }} 0 .
\end{aligned}
$$

Now by introducing a new matrix variable $Q$, this becomes

$$
\begin{aligned}
& \min _{\substack{Q, P \in S_{+}^{n}, z \in \mathbb{R}^{n} \\
\lambda_{i}, \sigma_{j} \in R_{N}[x, \mu]}} \operatorname{trace}(P) \quad \text { subject to } \\
& 1-\left[\begin{array}{l}
x \\
1
\end{array}\right]^{T} Q\left[\begin{array}{l}
x \\
1
\end{array}\right]+\sum_{i=1}^{s} \lambda_{i}(x, \mu) \phi_{i}+\sum_{j=1}^{t} \sigma_{j}(x, \mu) \rho_{j} \succeq_{\text {sos }} 0 \\
& {\left[\begin{array}{ll}
I & -z
\end{array}\right]^{T} P^{-1}\left[\begin{array}{ll}
I & -z
\end{array}\right] \preceq Q, \quad \sigma_{1}, \cdots, \sigma_{t} \succeq_{\text {sos }} 0 .}
\end{aligned}
$$

Taking a Schur complement, this is equivalent to

$$
\begin{align*}
E_{N}: p_{N}^{*}= & \min _{\substack{Q, P \in S^{n}, z \in \mathbb{R}^{n} \\
\lambda_{i}, \sigma_{j} \in R_{N}[x, \mu]}} \operatorname{trace}(P) \quad \text { subject to }  \tag{7.2.8}\\
1- & {\left[\begin{array}{c}
x \\
1
\end{array}\right]^{T} Q\left[\begin{array}{l}
x \\
1
\end{array}\right]+\sum_{i=1}^{s} \lambda_{i}(x, \mu) \phi_{i}+\sum_{j=1}^{t} \sigma_{j}(x, \mu) \rho_{j} \succeq_{\operatorname{sos}} 0 }  \tag{7.2.9}\\
& {\left[\begin{array}{c}
P \\
\left(\begin{array}{ll}
I & -z)^{T}
\end{array}\right. \\
\end{array} \begin{array}{ll}
I & -z) \\
& Q
\end{array}\right] \succeq 0, \quad \sigma_{1}, \cdots, \sigma_{t} \succeq_{\text {sos }} 0 } \tag{7.2.10}
\end{align*}
$$

The objective is an affine function of $P$, and the constraints are either LMIs or SOS inequalities, which are also essentially LMIs ([81]). Therefore it can be solved by a standard SDP routine.

As we can see, when the degree $N$ is higher, the ellipsoid bound by solving $E_{N}$ is tighter. The convergence of $\mathcal{E}_{N}$ is described as follows.

Theorem 7.2.3. Suppose the polynomial system 7.2.3 satisfies Putinar's constraint qualification condition (1.1.14). Then the trace $p_{N}^{*}$ of the ellipsoid $\mathcal{E}_{N}$ found in $E_{N}$ converges to trace $p^{*}$ of the minimum ellipsoid containing the solution set $\mathcal{S}$ when the degree $N$ tends to infinity.

Proof. Let $\mathcal{E}^{*}=\left\{x \in \mathbb{R}^{n}:\left(x-z^{*}\right)^{T}\left(P^{*}\right)^{-1}\left(x-z^{*}\right) \leq 1\right\}$ be the minimum ellipsoid containing the solution set $\mathcal{S}$, with $\operatorname{trace}\left(P^{*}\right)=p^{*}$. Then for arbitrary $\epsilon>0$, the polynomial $1-\left(x-z^{*}\right)^{T}\left(P^{*}+\epsilon I_{n}\right)^{-1}\left(x-z^{*}\right)$ is strictly positive on the set of $(x, \mu)$ defined by (7.2.3). By Theorem 2.3.2, there exist some general polynomials $\lambda_{i}(x, \mu)(i=1, \cdots, s)$ and SOS polynomials $\sigma_{j}(x, \mu)(j=1, \cdots, t)$ such that

$$
1-\left(x-z^{*}\right)^{T}\left(P^{*}+\epsilon I_{n}\right)^{-1}\left(x-z^{*}\right)+\sum_{i=1}^{s} \lambda_{i} \phi_{i}-\sum_{j=1}^{t} \sigma_{j} \rho_{j} \succeq_{\text {sos }} 0
$$

As we showed previously, problems $\hat{E}_{N}$ and $E_{N}$ are equivalent formulations. So they have the same optimal objective values. When $N$ is large enough, then in $\hat{E}_{N}$ we find one feasible solution with objective value $p^{*}+n \epsilon$. Thus it must be true that $p_{N}^{*} \leq p^{*}+n \epsilon$. Here $n$ is the dimension of $x$, which is a constant. Since $\mathcal{E}^{*}$ is minimum, it holds that $p_{N}^{*} \geq p^{*}$. Therefore we have $\lim _{N \rightarrow \infty} p_{N}^{*}=p^{*}$.

Last, let us show some examples. All of them are solved via SOSTOOLS [88].

Example 7.2.4. Consider the following polynomial system of two equations and two inequalities.

$$
\begin{align*}
\left(1+\mu_{1}^{2}\right) x_{1}^{2}+\mu_{2} x_{1} x_{2}+\left(1-\mu_{2}^{2}\right) x_{2}^{2}+\left(\mu_{1}+\mu_{2}\right) x_{1}+\left(\mu_{1}-\mu_{2}\right) x_{2}-1 & =0  \tag{7.2.11}\\
\left(1-\mu_{1}^{2}\right) x_{1}^{2}+\mu_{1} x_{1} x_{2}+\left(1+\mu_{2}^{2}\right) x_{2}^{2}+\left(\mu_{1}-\mu_{2}\right) x_{1}+\left(\mu_{1}+\mu_{2}\right) x_{2}-1 & =0  \tag{7.2.12}\\
\mu_{1}^{2}-\epsilon^{2} \leq 0, \mu_{2}^{2}-\epsilon^{2} & \leq 0 \tag{7.2.13}
\end{align*}
$$

where $\epsilon=0.1$. We formulate the optimization (7.2.8)-(7.2.10) for this polynomial system, and then solve it by SOSTOOLS. In this problem, $n=2, r=2, D=4$. We choose $N=2$ since any nonconstant SOS polynomials have degree at least 2 . The resulting 2D-ellipsoid is at the top of Figure 7.3. The asterisks are the solutions $\left(x_{1}, x_{2}\right)$ when $\left(\mu_{1}, \mu_{2}\right)$ are chosen randomly according to the two inequalities. As you can see, the computed ellipsoid is much larger than the set of real solutions. This is because the solution set is not connected.

However, if we want more information about one branch, we can add one more inequality of the form $\left(x_{1}-a\right)^{2}+\left(x_{2}-b\right)^{2} \leq r^{2}$, where $a, b, r$ are chosen according to the user's interests for the solution region, and then solve the optimization problem again. The role of this new inequality is that it can help to find the ellipsoid bound


Figure 7.3: The ellipsoid for polynomial system (7.2.11)-(7.2.13)
for just one solution component, and it also assures that the Putinar's constraint qualification is satisfied. See Figure 7.4 for the minimum ellipsoid bounds for each component. The left ellipsoid bound is obtained by adding inequality $\left(x_{1}+0.6\right)^{2}+$ $\left(x_{2}+0.6\right)^{2} \leq 0.6^{2}$. The right ellipsoid is found by adding inequality $\left(x_{1}-0.9\right)^{2}+$ $\left(x_{2}-0.8\right)^{2} \leq 0.8^{2}$.

Example 7.2.5. This example demonstrates how to find a minimum ellipsoid bounding a very elongated set, as indicated in the introduction. Consider the following example:

$$
\begin{align*}
x_{1}^{2} x_{2}^{2}-2 x_{1} x_{2}+x_{2}^{2}-3 / 4 & \leq 0  \tag{7.2.14}\\
x_{1}^{2}-6 x_{1}+x_{2}^{2}+2 x_{2}-6 & \leq 0 \tag{7.2.15}
\end{align*}
$$

Here $n=2, r=2, D=4$. We also choose $N=2$ as in Example 1. The computed ellipsoid is shown by gray curve in Figure 7.5. The center of the ellipsoid is


Figure 7.4: Ellipsoid bound for each component
(4.2970 0.2684) and its shape matrix is

$$
\left[\begin{array}{cc}
6.6334 & -0.3627 \\
-0.3627 & 0.2604
\end{array}\right]
$$

The short axis is 0.9795 and the long axis is 5.1591 . The asterisks are the solutions $\left(x_{1}, x_{2}\right)$ satisfying the system defined by the above polynomial inequalities. As you can be see, all the asterisks are contained inside the ellipsoid and a few are near the boundary.

### 7.3 Nearest greatest common divisor

This section discusses the application of minimizing rational polynomials in finding the smallest perturbation of two univariate polynomials that causes them to have a nontrivial GCD, i.e., a common root. We call this probem "finding the nearest GCD" for short.

Let $p(z)$ and $q(z)$ be two monic complex univariate polynomials of degree


Figure 7.5: Ellipsoid bound for polynomial system (7.2.11)-(7.2.15).
$m$ of the form

$$
\begin{align*}
& p(z)=z^{m}+p_{m-1} z^{m-1}+p_{m-2} z^{m-2}+\cdots+p_{1} z+p_{0}  \tag{7.3.1}\\
& q(z)=z^{m}+q_{m-1} z^{m-1}+q_{m-2} z^{m-2}+\cdots+q_{1} z+q_{0} . \tag{7.3.2}
\end{align*}
$$

Their coefficients $p_{i}, q_{j}$ are all complex numbers. When $p(z), q(z)$ have common divisors, their greatest common divisor (GCD) can be computed exactly by using Euclid's algorithm or other refined algorithms [16, 20]. These algorithms assume that all the coefficients of $p(z)$ and $q(z)$ are error-free, and return the exact GCD. However, in practice, it is more interesting to compute the GCD of two polynomials whose coefficients are uncertain. In such situations, we often get the trivial common divisor (the constant polynomial 1) if we apply exact methods like Euclid's algorithm.

For given $p(z)$ and $q(z)$, they may or may not have a common divisor, i.e., a common zero. But we may perturb their coefficients such that the perturbed polynomials have a common divisor, say, $z-c$. See [48, 49] and [110, §6.4] for a discussion
of this problem. The contribution of this section is to solve the associated global optimization problem for rational functions via SOS methods, instead of finding all the real critical points (zero gradient) as suggested in [48, 49].

Throughout this paper, we measure the polynomials $p(z), q(z)$ by $\|\cdot\|_{2}$ norm of their coefficients, i.e., $\|p\|_{2}=\sqrt{\sum_{k=0}^{m-1}\left|p_{k}\right|^{2}},\|q\|_{2}=\sqrt{\sum_{k=0}^{m-1}\left|q_{k}\right|^{2}}$. The perturbations made to $p(z), q(z)$ are measured similarly. The basic problem in this section is what is the minimum perturbation such that the perturbed polynomials have a common divisor? To be more specific, suppose the perturbed polynomials have the form

$$
\begin{align*}
& \hat{p}(z)=z^{m}+\hat{p}_{m-1} z^{m-1}+\hat{p}_{m-2} z^{m-2}+\cdots+\hat{p}_{1} z+\hat{p}_{0}  \tag{7.3.3}\\
& \hat{q}(z)=z^{m}+\hat{q}_{m-1} z^{m-1}+\hat{q}_{m-2} z^{m-2}+\cdots+\hat{q}_{1} z+\hat{q}_{0} . \tag{7.3.4}
\end{align*}
$$

with common zero $c$, i.e., $\hat{p}(c)=\hat{q}(c)=0$. The perturbation is measured by

$$
\mathcal{N}(c, \hat{p}, \hat{q})=\sum_{i=0}^{m-1}\left|p_{i}-\hat{p}_{i}\right|^{2}+\sum_{j=0}^{m-1}\left|q_{j}-\hat{q}_{j}\right|^{2}=\|p-\hat{p}\|_{2}^{2}+\|q-\hat{q}\|_{2}^{2}
$$

The problem of finding nearest GCD can be formulated as finding ( $c, \hat{p}, \hat{q}$ ) such that $\mathcal{N}(c, \hat{p}, \hat{q})$ is minimized subject to $\hat{p}(c)=\hat{q}(c)=0$.

We can see that $\mathcal{N}(c, \hat{p}, \hat{q})$ is a convex quadratic function in $(\hat{p}, \hat{q})$. But the constraints $\hat{p}(c)=\hat{q}(c)=0$ are nonconvex. However, if the common root $c$ is fixed, the constraints $\hat{p}(c)=\hat{q}(c)=0$ are linear with respect to $(\hat{p}, \hat{q})$, and there is a closed form solution. $\mathcal{N}(c, \hat{p}, \hat{q})$ is a convex quadratic function about $(\hat{p}, \hat{q})$. It can be shown that [49] that

$$
\min _{(\hat{p}, \hat{q}): \hat{p}(c)=\hat{q}(c)=0} \mathcal{N}(c, \hat{p}, \hat{q})=\frac{|p(c)|^{2}+|q(c)|^{2}}{\sum_{i=0}^{m-1}\left|c^{2}\right|^{i}}
$$

Therefore the problem of finding the nearest GCD become the global optimization of the rational function

$$
\begin{equation*}
\min _{c \in \mathbb{C}} \frac{|p(c)|^{2}+|q(c)|^{2}}{\sum_{i=0}^{m-1}\left|c^{2}\right|^{i}} \tag{7.3.5}
\end{equation*}
$$

over the complex plane. Karmarkar and Lakshman [49] proposed the following algorithm to find the nearest GCD:

Algorithm 7.3.1 (Nearest GCD Algorithm, [49]).

Input: Monic polynomials $p(z), q(z)$.

Step 1 Determine the rational function $r\left(x_{1}, x_{2}\right)$

$$
r\left(x_{1}, x_{2}\right):=\frac{|p(c)|^{2}+|q(c)|^{2}}{\sum_{k=0}^{m-1}\left(x_{1}^{2}+x_{2}^{2}\right)^{k}}, \quad c=x_{1}+\sqrt{-1} x_{2}
$$

Step 2 Solve the polynomial system $\frac{r\left(x_{1}, x_{2}\right)}{\partial x_{1}}=\frac{r\left(x_{1}, x_{2}\right)}{\partial x_{1}}=0$. Find all its real solutions inside the box: $-B \leq x_{1}, x_{2} \leq B$ where $B:=5 \max \left(\|p\|^{2},\|q\|^{2}\right)$. Choose the one $\left(\hat{x}_{1}, \hat{x}_{2}\right)$ such that $r\left(\hat{x}_{1}, \hat{x}_{2}\right)$ is minimum. Let $c:=\hat{x}_{1}+\sqrt{-1} \hat{x}_{2}$.

Step 3 Compute the coefficient perturbations

$$
\lambda_{j}:=\frac{\bar{c}^{j} p(c)}{\sum_{k=0}^{m-1}\left|c^{2}\right|^{k}}, \quad \mu_{j}:=\frac{\bar{c}^{j} q(c)}{\sum_{k=0}^{m-1}\left|c^{2}\right|^{k}}
$$

Output: The minimum perturbed polynomials with common divisors are returned as

$$
\hat{p}(z)=z^{m}+\sum_{k=0}^{m-1}\left(p_{k}-\lambda_{k}\right) z^{k}, \hat{q}(z)=z^{m}+\sum_{k=0}^{m-1}\left(q_{k}-\mu_{k}\right) z^{k}
$$

The most expensive part in the algorithm above is step 2. Karmarkar and Lakshman [49] proposed to use numerical methods like Arnon and McCallum [2] or

Manocha and Demmel [60] to find all the real solutions of a polynomial system inside a box.

However, in practice, it is very expensive to find all the real solutions of a polynomial system inside a box, although a polynomial complexity bound exists as stated in [49]. So in this section, we propose to solve (7.3.5) by SOS relaxations introduced in Chapter 6, instead of finding all the real solutions of a polynomial system. The SOS relaxation of problem (7.3.5) is the following:

$$
\begin{array}{ll}
\text { sup } & \gamma \\
\text { s.t. } & f\left(x_{1}, x_{2}\right)-\gamma\left(\sum_{i=0}^{m-1}\left(x_{1}^{2}+x_{2}^{2}\right)^{i}\right) \text { is SOS }
\end{array}
$$

where $f\left(x_{1}, x_{2}\right)=\left|p\left(x_{1}+\sqrt{-1} x_{2}\right)\right|^{2}+\left|q\left(x_{1}+\sqrt{-1} x_{2}\right)\right|^{2}$.

In the following examples, we solve this optimization problem via SOS relaxation (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9). In all the examples here, the global minimizers can be extracted and the big ball technique introduced in Section 6.2 is not required.

Example 7.3.2 (Example 2.1,[49]). Consider the following two polynomials

$$
\begin{aligned}
& p(z)=z^{2}-6 z+5 \\
& q(z)=z^{2}-6.30 z+5.72 .
\end{aligned}
$$

Solving SOS relaxation (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9), we find the global minimum and extract one minimizer:

$$
r^{*} \approx 0.0121, \quad c^{*}=x_{1}^{*}+\sqrt{-1} x_{2}^{*} \approx 5.0971 .
$$

which are the same as found in [49].

Example 7.3.3. Consider the following two polynomials

$$
\begin{aligned}
& p(z)=z^{3}-6 z^{2}+11 z-6 \\
& q(z)=z^{3}-6.24 z^{2}+10.75 z-6.50
\end{aligned}
$$

Solving SOS relaxation (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9), , we get a lower bound and extract one point

$$
r_{\text {sos }}^{*} \approx 0.0563, \quad\left(x_{1}^{*}, x_{2}^{*}\right) \approx(3.5725,0.0000) .
$$

Evaluation of $r(x)$ at $x^{*}$ shows that $r\left(x^{*}\right) \approx r_{\text {sos }}^{*}$, which implies that $c^{*} \approx 3.5725$ is a global minimizer for problem (7.3.5).

Example 7.3.4. Consider the following two polynomials

$$
\begin{aligned}
& p(z)=z^{3}+z^{2}-2 \\
& q(z)=z^{3}+1.5 z^{2}+1.5 z-1.25 .
\end{aligned}
$$

Solving SOS relaxation (6.1.4)-(6.1.6) and its dual (6.1.7)-(6.1.9), we find the lower bound $r_{\text {sos }}^{*} \approx 0.0643$ and extract two points

$$
x^{*} \approx(-1.0032,1.1011) \quad x^{* *} \approx(-1.0032,-1.1011)
$$

The evaluations of $r(x)$ at $x^{*}$ and $x^{* *}$ show that $r\left(x^{*}\right)=r\left(x^{* *}\right) \approx r_{s o s}^{*}$, which implies that $x^{*}$ and $x^{* *}$ are both global minimizers. So $c^{*}=-1.0032 \pm \sqrt{-1} \cdot 1.1011$ are the global minimizers of problem (7.3.5).

### 7.4 Maximum likelihood optimization

This section discusses another application of polynomial optimization. One important class of problems in statistics and computational biology is maximum
likelihood optimization. It can be formulated as

$$
\begin{align*}
\max _{x \in \mathbb{R}^{n}} & \prod_{i=1}^{r} f_{i}(x)^{m_{i}}  \tag{7.4.1}\\
\text { s.t. } & g_{1}(x), \cdots, g_{\ell}(x) \geq 0 \tag{7.4.2}
\end{align*}
$$

where $f_{i}(x), g_{j}(x)$ are all polynomials in $x \in \mathbb{R}^{n}$, and $m_{i}$ are positive integers. Here we assume that each $f_{i}(x)$ is nonnegative on the feasible set, which is often the case in statistics or computational biology (e.g., $f_{i}(x)$ represents some probability distribution).

Our goal is to find the global or approximately global solution to (7.4.1)(7.4.2). Theoretically, the SOS methods can be applied in this problem, since the objective and constraints are all described by polynomials. However, in practice, the exponents $m_{i}$ are big. It is very common that these integers are hundreds or even thousands. Then SOS methods are too expensive to be implemented, because the reduced SDP is too huge to be solved. So we need cheaper methods, and still want high quality solutions (e.g., approximately global).

Without changing the problem, we take the $\log$ of the objective in (7.4.1)(7.4.2) and get an equivalent problem:

$$
\begin{align*}
\max _{x \in \mathbb{R}^{n}} & \sum_{i=1}^{r} m_{i} \log f_{i}(x)  \tag{7.4.3}\\
\text { s.t. } & g_{1}(x), \cdots, g_{\ell}(x) \geq 0 . \tag{7.4.4}
\end{align*}
$$

However, the objective is no longer a polynomial, and hence SOS methods can not be applied directly. But moment matrix methods are still applicable.

Suppose $f_{i}(x)$ has the form $f_{i}(x)=\sum_{\alpha \in P_{i}} f_{i, \alpha} x^{\alpha}$, where $P_{i}$ is the support.

Then we can see that

$$
f_{i}(x)=\sum_{\alpha} f_{i, \alpha} y_{\alpha} \quad \text { when } y=\operatorname{mon}_{N}(x)
$$

where $N \geq \operatorname{deg}(f)$. We can also see that $g_{j}(x) \geq 0$ is the same as

$$
g_{j}(x) \cdot m_{N-d_{i}}(x) \cdot m_{N-d_{i}}(x)^{T} \succeq 0
$$

where $d_{i}=\left\lceil\operatorname{deg}\left(g_{i}\right) / 2\right\rceil$.

If we replace each $x^{\alpha}$ by $y_{\alpha}$, we get the following relaxation:

$$
\begin{aligned}
\max _{y=\left(y_{\alpha}\right)} & \sum_{i=1}^{r} m_{i} \log \left(\sum_{\alpha} f_{i, \alpha} y_{\alpha}\right) \\
\text { s.t. } & M_{N-d_{j}}\left(g_{j} * y\right) \succeq 0 \\
& M_{N}(y) \succeq 0 .
\end{aligned}
$$

This is still a convex optimization problem, and efficient techniques like interior-point methods are available. Let $y^{*}$ be the optimal solution to this problem. When the moment matrix $M_{N}\left(y^{*}\right)$ satisfies the flat extension condition, we can extract the $\operatorname{maximizer}(\mathrm{s}) x^{*}$. A very simple choice is $x_{i}^{*}=y_{e_{i}}^{*}$.

### 7.5 Sensor network localization

This section shows the application of sum of squares in sensor network localization. The basic description of this problem is as follows. For a sequence of unknown vectors (also called sensors) $x_{1}, x_{2}, \cdots, x_{n}$ in the Euclidean space $\mathbb{R}^{d}(d=$ $1,2, \cdots$ ), we need find their coordinates such that the distances (not necessarily all) between these sensors and the distances (not necessarily all) to other fixed sensors
$a_{1}, \cdots, a_{m}$ (they are also called anchors) are equal to some given numbers. To be more specific, let $\mathcal{A}=\left\{(i, j) \in[n] \times[n]:\left\|x_{i}-x_{j}\right\|_{2}=d_{i j}\right\}$, and $\mathcal{B}=\{(i, k) \in$ $\left.[n] \times[m]:\left\|x_{i}-a_{k}\right\|_{2}=e_{i k}\right\}$, where $d_{i j}, e_{i k}$ are given distances and $[n]=\{1,2, \cdots, n\}$. Then the problem of sensor network localization is to find vectors $\left\{x_{1}, x_{2}, \cdots, x_{n}\right\}$ such that $\left\|x_{i}-x_{j}\right\|_{2}=d_{i j}$ for all $(i, j) \in \mathcal{A}$ and $\left\|x_{i}-a_{k}\right\|_{2}=e_{i k}$ for all $(i, k) \in \mathcal{B}$.

This task can be formulated as an optimization problem. Let $x_{1}, \cdots, x_{n}$ be decisions variables, each $s_{i}$ being a vector in $\mathbb{R}^{d}$. Obviously, $x_{1}, \cdots, x_{m}$ provides the right sensor locations if and only if the optimal value of problem

$$
\min _{x_{1}, \cdots, x_{n} \in \mathbb{R}^{d}} \sum_{(i, j) \in \mathcal{A}}\left|\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right|+\sum_{(i, k) \in \mathcal{B}}\left|\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right|^{2} .
$$

is zero. This optimization problem is nonconvex, and it is often NP-hard to find global solutions. So approximation methods are of great interest. For example, SDP or second-order cone programming (SOCP) relaxations can be applied to solve the problem approximately. We refer to $[10,106,112]$ for work in this area. However, SDP relaxation is very expensive to implement for large problems (e.g., more than 100 sensors). SOCP relaxation is weaker than SDP relaxation, but can solve larger problems.

As we can see, the objective in the above involves absolute values and is not a polynomial. Hence, SOS methods can not be applied. However, if we replace the absolute value by squares, we can get a new optimization problem

$$
f(X):=\min _{X=\left[x_{1}, \cdots, x_{n}\right] \in \mathbb{R}^{d \times n}} \sum_{(i, j) \in \mathcal{A}}\left(\left\|x_{i}-x_{j}\right\|_{2}^{2}-d_{i j}^{2}\right)^{2}+\sum_{(i, k) \in \mathcal{B}}\left(\left\|x_{i}-a_{k}\right\|_{2}^{2}-e_{i k}^{2}\right)^{2} .
$$

The good property of this new objective is that it is a quartic polynomial. Therefore, the method of sum of squares is applicable.

On the other hand, we must be very careful in applying SOS method to solve this polynomial optimization problem. The total number of decision variables is $n \cdot d$. If we apply SOS relaxation directly, the size of matrix in reduced SDP is $\binom{n \cdot d+4}{4}$, which can be huge for even moderate $n$ and $d$. For instance, when $n=50$ and $d=2$, this number is as large as

$$
\binom{n \cdot d+4}{4} \geq 10^{4}
$$

So it seems that the applications of SOS relaxations are very limited in practice. So we can maximize $\gamma$ such that

$$
f(X)-\gamma \equiv \sum_{(i, j) \in \mathcal{A}} \sigma_{i j}\left(x_{i}, x_{j}\right)
$$

where $\sigma_{i j}\left(x_{i}, x_{j}\right)$ is some SOS polynomial in $\left(x_{i}, x_{j}\right)$. If we use this special representation, we can efficiently and accurately solve large scale sensor network localization problems that can not be solved by SDP relaxation. See the following example.

Example 7.5.1. We randomly generate test problems which are similar to those given in [11]. First, we generate $n=500$ points $x_{1}^{*}, \cdots, x_{n}^{*}$ from the unit square $[-0.5,0.5] \times[-0.50 .5]$. Choose anchors to be four points $( \pm 0.45, \pm 0.45)$. The edge set $\mathcal{A}$ is chosen as follows. Initially set $\mathcal{A}=\emptyset$. Then for each $i$ from 1 to 500 , compute the set $I_{i}=\left\{j \in[500]:,\left\|x_{i}^{*}-x_{j}^{*}\right\|_{2} \leq 0.3, j \geq i\right\}$; if $\left|I_{i}\right| \geq 10$, let $A_{i}$ the subset of $I_{i}$ consisting of the 10 smallest integers; otherwise, let $A_{i}=I_{i}$; then let $\mathcal{A}=\mathcal{A} \cup\left\{(i, j): j \in A_{i}\right\}$. The edge set $\mathcal{B}$ is chosen such that $\mathcal{B}=\{(i, k) \in[n] \times[m]:$ $\left.\left\|x_{i}^{*}-a_{k}\right\|_{2} \leq 0.3\right\}$, i.e., every anchor is connected to all the sensors that are within distance 0.3 . For every $(i, j) \in \mathcal{A}$ and $(i, k) \in \mathcal{B}$, let the distances be

$$
d_{i j}=\left\|x_{i}^{*}-x_{j}^{*}\right\|_{2}, \quad e_{i k}=\left\|x_{i}^{*}-a_{k}\right\|_{2} .
$$

There are no errors in the distances. The computed results are plotted in Figure 7.6. The true sensor locations (denoted by circles) and the computed locations (denoted by stars) are connected by solid lines.


Figure 7.6: 500 sensors, sparse SOS relaxation

From Figure 7.6, we find that all the stars are located inside circles, which implies that SOS relaxation provides high quality locations. The accuracy of the estimated points $\hat{x}_{1}, \cdots, \hat{x}_{n}$ will be measured by the Root Mean Square Distance (RMSD) which is defined as

$$
\operatorname{RMSD}=\left(\frac{1}{n} \sum_{i=1}^{n}\left\|\hat{x}_{i}-x_{i}^{*}\right\|_{2}^{2}\right)^{\frac{1}{2}} .
$$

The RMSD for this sparse SOS relaxation is $2.9 \cdot 10^{-6}$ (the computed locations will be exact if we ignore rounding errors involved in floating point operations). The interior-point method in SeDuMi consumes about 1079 CPU seconds (18 minutes). We generate this random examples 20 times. Every time the RMSD is in the order $O\left(10^{-6}\right)$ and the CPU time consumed by the sparse SOS relaxation is almost the same.

We refer to [74] for the sparse SOS relaxation for sensor network localization problem.

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## Glossary of Notations

$\mathbb{R}$ : the field of real numbers
$\mathbb{C}$ : the field of complex numbers
$\mathbb{N}$ : the set of nonnegative integers
$\mathbb{Z}$ : the ring of integers
$\mathbb{R}^{n}$ : Euclidean Space of dimension $n$
$\mathbb{R}_{+}^{n}$ : The nonnegative orthant of $\mathbb{R}^{n}$
$\mathbb{R}[x]: \quad$ the ring of polynomials in $\left(x_{1}, \cdots, x_{n}\right)$
$\mathbb{R}^{m \times n}:$ the vector space of matrices with dimension $m \times n$
$\mathcal{S}^{n}$ :the vector space of symmetric matrices with size $n$
$A \succeq 0$ : the symmetric matrix $A$ is positive semidefinite
$A \succ 0$ : the symmetric matrix $A$ is positive definite
$A \bullet B$ : the inner product of two matrices of same dimensions defined as $\operatorname{trace}\left(A^{T} B\right)$
$\mathcal{S}_{+}^{n}$ : the positive semidefinite cone of $\mathcal{S}^{n}$
$\mathcal{S}_{++}^{n}$ : the positive definite interior of $\mathcal{S}_{+}^{n}$
$\Sigma \mathbb{R}[x]^{2}$ : the cone of SOS polynomials $x=\left(x_{1}, \cdots, x_{n}\right):$ a $n$-dimensional vector
$p(x)$ : polynomial evaluated at the vector $x \in \mathbb{R}^{n}$
$\operatorname{deg}(\mathrm{p}):$ the degree of polynomial $p(x)$
$\operatorname{supp}(\mathrm{p})$ : the support of polynomial $p(x)$
$p(x) \succeq q(x)$ : the polynomial $p(x)-q(x)$
is SOS
$G C D$ : greatest common divisors
$S$ : a basic closed semialgebraic set
$\mathcal{P}(S)$ : the preorder cone associated with $S$
$\mathcal{M}(S)$ : the quadratic module associated with $S$
$\mathcal{P}_{K K T}$ : the preorder cone associated with KKT system
$\mathcal{M}_{\text {KKT }}$ : the quadratic module associated with KKT system
$\mathcal{M}(S)_{N}$ : the subset of $\mathcal{M}(S)$ with de-
gree at most $N$ in each summand
$\mathcal{M}(y)$ : the moment matrix induced by multi-indexed vector $y$
$\mathcal{M}_{N}(y)$ : the $\binom{N+n}{n}$-th leading submatrix of $M(y)$
$\mathcal{M}_{N}(g * y)$ : the moment matrix induced by multi-indexed vector $\hat{y}=\left(\sum_{\beta} g_{\beta} y_{\alpha+\beta}\right)$ where $g(x)=$ $\sum_{\beta} g_{\beta} x^{\alpha}$.

