SEMIDEFINITE REPRESENTATION OF THE K-ELLIPSE

JIAWANG NIE*, PABLO A. PARRILO†, AND BERND STURMFELS‡

Abstract. The k-ellipse is the plane algebraic curve consisting of all points whose sum of distances from k given points is a fixed number. The polynomial equation defining the k-ellipse has degree 2^k if k is odd and degree $2^k - \binom{k}{k/2}$ if k is even. We express this polynomial equation as the determinant of a symmetric matrix of linear polynomials. Our representation extends to weighted k-ellipses and k-ellipsoids in arbitrary dimensions, and it leads to new geometric applications of semidefinite programming.

Key words. k-ellipse, algebraic degree, semidefinite representation, Zariski closure, tensor sum.

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1. Introduction. The *circle* is the plane curve consisting of all points (x, y) whose distance from a given point (u_1, v_1) is a fixed number d. It is the zero set of the quadratic polynomial

$$p_1(x,y) \quad = \quad \det \begin{bmatrix} d + x - u_1 & y - v_1 \\ y - v_1 & d - x + u_1 \end{bmatrix}. \tag{1.1}$$

The *ellipse* is the plane curve consisting of all points (x, y) whose sum of distances from two given points (u_1, v_1) and (u_2, v_2) is a fixed number d. It is the zero set of

$$p_2(x,y) = \det \begin{bmatrix} d{+}2x{-}u_1{-}u_2 & y - v_1 & y - v_2 & 0 \\ y - v_1 & d{+}u_1{-}u_2 & 0 & y - v_2 \\ y - v_2 & 0 & d{-}u_1{+}u_2 & y - v_1 \\ 0 & y - v_2 & y - v_1 & d{-}2x{+}u_1{+}u_2 \end{bmatrix}. \tag{1.2}$$

In this paper we generalize these determinantal formulas for the circle and the ellipse. Fix a positive real number d and fix k distinct points $(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)$ in $\mathbb{R}^2$. The *k-ellipse* with *foci* (u_i, v_i) and *radius* d is the following curve in the plane:

$$\left\{ (x,y) \in \mathbb{R}^2 \,:\, \sum_{i=1}^{k} \sqrt{(x-u_i)^2 + (y-v_i)^2} \;=\; d \right\}. \tag{1.3}$$

The k-ellipse is the boundary of a convex set $\mathcal{E}_k$ in the plane, namely, the set of points whose sum of distances to the k given points is at most d.

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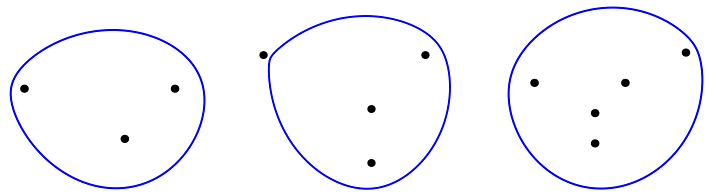

FIG. 1. *A 3-ellipse, a 4-ellipse, and a 5-ellipse, each with its foci.*

These convex sets are of interest in computational geometry [16] and in optimization, e.g. for the Fermat-Weber facility location problem [1, 3, 7, 14, 18]. In the classical literature (e.g. [15]), k-ellipses are known as *Tschirnhaus'sche Eikurven* [11]. Indeed, they look like "egg curves" and they were introduced by Tschirnhaus in 1686.

We are interested in the irreducible polynomial $p_k(x,y)$ that vanishes on the k-ellipse. This is the unique (up to sign) polynomial with coprime integer coefficients in the unknowns x and y and the parameters d, $u_1, v_1, \ldots, u_k, v_k$. By the *degree of the k-ellipse* we mean the degree of $p_k(x,y)$ in x and y. To compute it, we must eliminate the square roots in the representation (1.3). Our solution to this problem is as follows:

THEOREM 1.1. *The k-ellipse has degree 2^k if k is odd and degree $2^k - \binom{k}{k/2}$ if k is even. Its defining polynomial has a determinantal representation*

$$p_k(x,y) \quad = \quad \det\bigl(\, x \cdot A_k + y \cdot B_k + C_k \,\bigr) \qquad (1.4)$$

where A_k, B_k, C_k are symmetric $2^k \times 2^k$ matrices. The entries of A_k and B_k are integer numbers, and the entries of C_k are linear forms in the parameters $d, u_1, v_1, \ldots, u_k, v_k$.

For the circle ($k = 1$) and the ellipse ($k = 2$), the representation (1.4) is given by the formulas (1.1) and (1.2). The polynomial $p_3(x,y)$ for the 3-ellipse is the determinant of

$$\left[\begin{array}{cccc}
d{+}3x{-}u_1{-}u_2{-}u_3 & y{-}v_1 & y{-}v_2 & 0 \\
y-v_1 & d{+}x{+}u_1{-}u_2{-}u_3 & 0 & y{-}v_2 \\
y-v_2 & 0 & d{+}x{-}u_1{+}u_2{-}u_3 & y{-}v_1 \\
0 & y-v_2 & y-v_1 & d{-}x{+}u_1{+}u_2{-}u_3 \\
y-v_3 & 0 & 0 & 0 \\
0 & y-v_3 & 0 & 0 \\
0 & 0 & y-v_3 & 0 \\
0 & 0 & 0 & y-v_3
\end{array}\right.$$

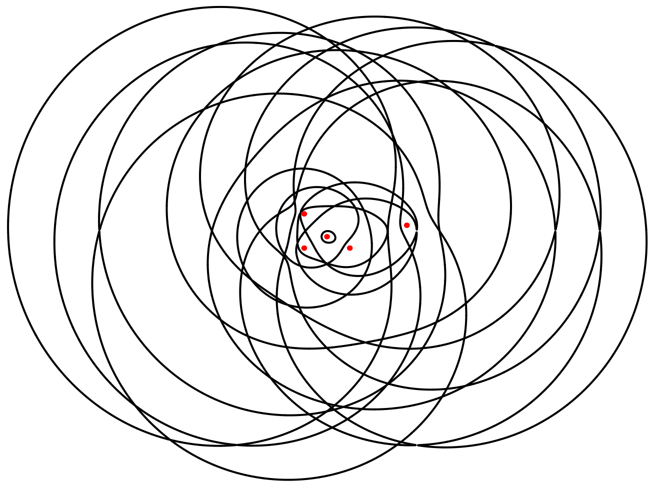

FIG. 2. *The Zariski closure of the 5-ellipse is an algebraic curve of degree 32. The tiny curve in the center is the 5-ellipse.*

$$
\begin{matrix}
y-v_3 & 0 & 0 & 0 \\
0 & y-v_3 & 0 & 0 \\
0 & 0 & y-v_3 & 0 \\
0 & 0 & 0 & y-v_3 \\
d+x-u_1-u_2+u_3 & y-v_1 & y-v_2 & 0 \\
y-v_1 & d-x+u_1-u_2+u_3 & 0 & y-v_2 \\
y-v_2 & 0 & d-x-u_1+u_2+u_3 & y-v_1 \\
0 & y-v_2 & y-v_1 & d-3x+u_1+u_2+u_3
\end{matrix}
\Bigg] .
$$

The full expansion of this 8×8-determinant has $2,355$ terms. Next, the 4-ellipse is a curve of degree ten which is represented by a symmetric 16×16-matrix, etc....

This paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2. Section 3 is devoted to geometric aspects and connections to semidefinite programming. While the k-ellipse itself is a convex curve, its Zariski closure $\{ \, p_k(x,y) = 0 \, \}$ has many extra branches outside the convex set $\mathcal{E}_k$. They are arranged in a beautiful pattern known as a *Helton-Vinnikov curve* [5]. This pattern is shown in Figure 2 for $k = 5$ points. In Section 4 we generalize our results to the weighted case and to higher dimensions, and we discuss the computation of the *Fermat-Weber point* of the given points (u_i, v_i). A list of open problems and future directions is presented in Section 5.

2. Derivation of the matrix representation. We begin with a discussion of the degree of the k-ellipse.

LEMMA 2.1. *The defining polynomial of the k-ellipse has degree at most 2^k in the variables (x, y) and it is monic of degree 2^k in the radius parameter d.*

Proof. We claim that the defining polynomial of the k-ellipse can be written as follows:

$$p_k(x, y) \quad = \prod_{\sigma \in \{-1,+1\}^k} \left(d - \sum_{i=1}^{k} \sigma_i \cdot \sqrt{(x - u_i)^2 + (y - v_i)^2} \right). \quad (2.1)$$

Obviously, the right hand side vanishes on the k-ellipse. The following Galois theory argument shows that this expression is a polynomial and that it is irreducible. Consider the polynomial ring $R = \mathbb{Q}[x, y, d, u_1, v_1, \ldots, u_k, v_k]$. The field of fractions of R is the field $K = \mathbb{Q}(x, y, d, u_1, v_1, \ldots, u_k, v_k)$ of rational functions in all unknowns. Adjoining the square roots in (1.3) to K gives an algebraic field extension L of degree 2^k over K. The Galois group of the extension L/K is $(\mathbb{Z}/2\mathbb{Z})^k$, and the product in (2.1) is over the orbit of the element $d - \sum_{i=1}^{k} \sqrt{(x - u_i)^2 + (y - v_i)^2}$ of L under the action of the Galois group. Thus this product in (2.1) lies in the ground field K. Moreover, each factor in the product is integral over R, and therefore the product lies in the polynomial ring R. To see that this polynomial is irreducible, it suffices to observe that no proper subproduct of the right hand side in (2.1) lies in the ground field K. □

The statement *degree at most* 2^k is the crux in Lemma 2.1. Indeed, the degree in (x, y) can be strictly smaller than 2^k as the case of the classical ellipse ($k = 2$) demonstrates. When evaluating the product (2.1) some unexpected cancellations may occur. This phenomenon happens for all even k, as we shall see later in this section.

We now turn to the matrix representation promised by Theorem 1.1. We recall the following standard definition from matrix theory (e.g., [6]). Let A be a real $m \times m$-matrix and B a real $n \times n$-matrix. The *tensor sum* of A and B is the $mn \times mn$ matrix $A \oplus B := A \otimes I_n + I_m \otimes B$. The tensor sum of square matrices is an associative operation which is not commutative. For instance, for three matrices A, B, C we have

$$A \oplus B \oplus C \quad = \quad A \otimes I \otimes I \,+\, I \otimes B \otimes I \,+\, I \otimes I \otimes C.$$

Here $\otimes$ denotes the *tensor product*, which is also associative but not commutative. Tensor products and tensor sums of matrices are also known as *Kronecker products* and *Kronecker sums* [2, 6]. Tensor sums of symmetric matrices can be diagonalized by treating the summands separately:

LEMMA 2.2. *Let $M_1, \ldots, M_k$ be symmetric matrices, let $U_1, \ldots, U_k$ be orthogonal matrices, and let $\Lambda_1, \ldots, \Lambda_k$ be diagonal matrices such that $M_i = U_i \cdot \Lambda_i \cdot U_i^T$ for $i = 1, \ldots, k$. Then*

$$(U_1 \otimes \cdots \otimes U_k)^T \cdot (M_1 \oplus \cdots \oplus M_k) \cdot (U_1 \otimes \cdots \otimes U_k) \;=\; \Lambda_1 \oplus \cdots \oplus \Lambda_k.$$

In particular, the eigenvalues of the tensor sum $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ *are the sums* $\lambda_1 + \lambda_2 + \cdots + \lambda_k$ *where* λ_1 *is any eigenvalue of* M_1*,* λ_2 *is any eigenvalue of* M_2*, etc.*

The proof of this lemma is an exercise in (multi)-linear algebra. We are now prepared to state our formula for the explicit determinantal representation of the k-ellipse.

THEOREM 2.1. *Given points* $(u_1, v_1), \ldots, (u_k, v_k)$ *in* $\mathbb{R}^2$*, we define the* $2^k \times 2^k$ *matrix*

$$L_k(x,y) := d \cdot I_{2^k} + \begin{bmatrix} x-u_1 & y-v_1 \\ y-v_1 & -x+u_1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} x-u_k & y-v_k \\ y-v_k & -x+u_k \end{bmatrix} \quad (2.2)$$

which is affine in x, y *and* d*. Then the* k*-ellipse has the determinantal representation*

$$p_k(x,y) \;=\; \det L_k(x,y). \quad (2.3)$$

The convex region bounded by the k*-ellipse is defined by the following matrix inequality:*

$$\mathcal{E}_k \;=\; \left\{ (x,y) \in \mathbb{R}^2 \,:\, L_k(x,y) \;\succeq\; 0 \right\}. \quad (2.4)$$

Proof. Consider the 2×2 matrix that appears as a tensor summand in (2.2):

$$\begin{bmatrix} x-u_i & y-v_i \\ y-v_i & -x+u_i \end{bmatrix}.$$

A computation shows that this matrix is orthogonally similar to

$$\begin{bmatrix} \sqrt{(x-u_i)^2+(y-v_i)^2} & 0 \\ 0 & -\sqrt{(x-u_i)^2+(y-v_i)^2} \end{bmatrix}.$$

These computations take place in the field L which was considered in the proof of Lemma 2.1 above. Lemma 2.2 is valid over any field, and it implies that the matrix $L_k(x,y)$ is orthogonally similar to a $2^k \times 2^k$ diagonal matrix with diagonal entries

$$d \,+\, \sum_{i=1}^{k} \sigma_i \cdot \sqrt{(x-u_i)^2+(y-v_i)^2}, \qquad \sigma_i \in \{-1,+1\}. \quad (2.5)$$

The desired identity (2.3) now follows directly from (2.1) and the fact that the determinant of a matrix is the product of its eigenvalues. For the characterization of the convex set $\mathcal{E}_k$, notice that positive semidefiniteness of $L_k(x,y)$ is equivalent to nonnegativity of all the eigenvalues (2.5). It suffices to consider the smallest eigenvalue, which equals

$$d \,-\, \sum_{i=1}^{k} \sqrt{(x-u_i)^2+(y-v_i)^2}.$$

Indeed, this quantity is nonnegative if and only if the point (x, y) lies in $\mathcal{E}_k$. ∎

Proof. [Proof of Theorem 1.1] The assertions in the second and third sentence have just been proved in Theorem 2.1. What remains to be shown is the first assertion concerning the degree of $p_k(x, y)$ as a polynomial in (x, y). To this end, we consider the univariate polynomial $g(t) := p_k(t\cos\theta, t\sin\theta)$ where θ is a generic angle. We must prove that

$$\deg_t\big(g(t)\big) \quad = \quad \begin{cases} 2^k & \text{if } k \text{ is odd,} \\ 2^k - \binom{k}{k/2} & \text{if } k \text{ is even.} \end{cases}$$

The polynomial $g(t)$ is the determinant of the symmetric $2^k \times 2^k$-matrix

$$L_k(t\cos\theta, t\sin\theta) = t \cdot \left(\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \right) + C_k. \quad (2.6)$$

The matrix C_k does not depend on t. We now define the $2^k \times 2^k$ orthogonal matrix

$$U \;:=\; \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}} \qquad \text{where} \quad V \;:=\; \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{bmatrix},$$

and we note the matrix identity

$$V^T \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \cdot V \quad = \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Pre- and post-multiplying (2.6) by U^T and U, we find that

$$U^T \cdot L_k(t\cos\theta, t\sin\theta) \cdot U = t \cdot \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)}_{E_k} + U^T \cdot C_k \cdot U.$$

The determinant of this matrix is our univariate polynomial $g(t)$. The matrix E_k is a diagonal matrix of dimension $2^k \times 2^k$. Its diagonal entries are obtained by summing k copies of -1 or $+1$ in all 2^k possible ways. None of these sums are zero when k is odd, and precisely $\binom{k}{k/2}$ of these sums are zero when k is even. This shows that the rank of E_k is 2^k when k is odd, and it is $2^k - \binom{k}{k/2}$ when k is even. We conclude that the univariate polynomial $g(t) = \det\big(t \cdot E_k + U^T C_k U\big)$ has the desired degree. ∎

3. More pictures and some semidefinite aspects. In this section we examine the geometry of the k-ellipse, we look at some pictures, and we discuss aspects relevant to the theory of semidefinite programming. In Figure 1 several k-ellipses are shown, for $k = 3, 4, 5$. One immediately observes that, in contrast to the classical circle and ellipse, a k-ellipse does

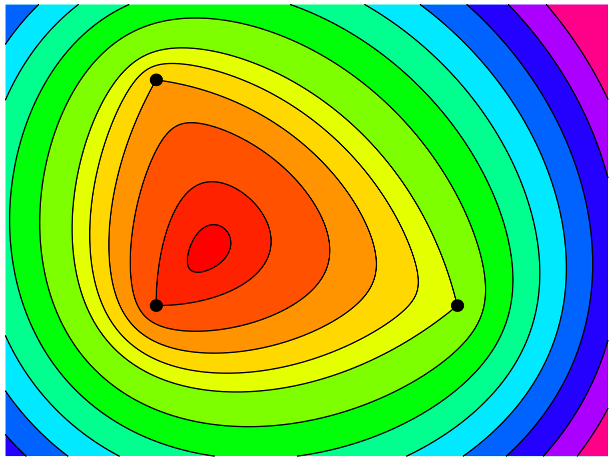

FIG. 3. *A pencil of 3-ellipses with fixed foci (the three bold dots) and different radii. These 3-elliptical curves are always smooth unless they contain one of the foci.*

not necessarily contain the foci in its interior. The interior $\mathcal{E}_k$ of the k-ellipse is a sublevel set of the convex function

$$(x, y) \quad \mapsto \quad \sum_{i=1}^{k} \sqrt{(x - u_i)^2 + (y - v_i)^2}. \qquad (3.1)$$

This function is strictly convex, provided the foci $\{(u_i, v_i)\}_{i=1}^k$ are not collinear [14]. This explains why the k-ellipse is a convex curve. In order for $\mathcal{E}_k$ to be nonempty, it is necessary and sufficient that the radius d be greater than or equal to the global minimum d_* of the convex function (3.1).

The point (x_*, y_*) at which the global minimum d_* is achieved is called the *Fermat-Weber point* of the foci. This point minimizes the sum of the distances to the k given points (u_i, v_i), and it is of importance in the facility location problem. See [1, 3, 7], and [15] for a historical reference. For a given set of foci, we can vary the radius d, and this results in a pencil of confocal k-ellipses, as in Figure 3. The sum of distances function (3.1) is differentiable everywhere except at the (u_i, v_i), where the square root function has a singularity. As a consequence, the k-ellipse is a smooth convex curve, except when that curve contains one of the foci.

An algebraic geometer would argue that there is more to the k-ellipse than meets the eye in Figures 1 and 3. We define the *algebraic k-ellipse* to be the Zariski closure of the k-ellipse, or, equivalently, the zero set of the polynomial $p_k(x, y)$. The algebraic k-ellipse is an algebraic curve, and it can be considered in either the real plane $\mathbb{R}^2$, in the complex plane $\mathbb{C}^2$, or (even better) in the complex projective plane $\mathbb{CP}^2$.

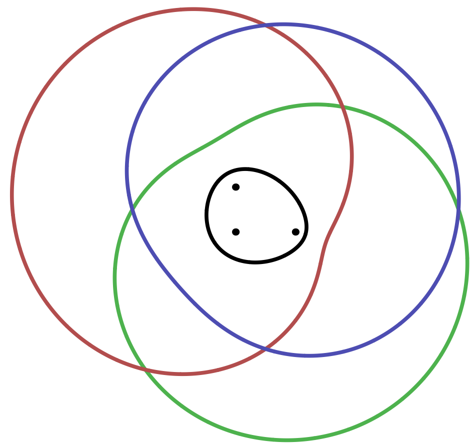

FIG. 4. *The Zariski closure of the 3-ellipse is an algebraic curve of degree eight.*

Figure 2 shows an algebraic 5-ellipse. In that picture, the actual 5-ellipse is the tiny convex curve in the center. It surrounds only one of the five foci.

For a less dizzying illustration see Figure 4. That picture shows an algebraic 3-ellipse. The curve has degree eight, and it is given algebraically by the 8×8-determinant displayed in the Introduction. We see that the set of real points on the algebraic 3-ellipse consists of four ovals, corresponding to the equations

$$\pm\sqrt{(x-u_1)^2+(y-v_1)^2}\pm\sqrt{(x-u_2)^2+(y-v_2)^2}\pm\sqrt{(x-u_3)^2+(y-v_3)^2}=d.$$

Thus Figure 4 visualizes the Galois theory argument in the proof of Lemma 2.1.

If we regard the radius d as an unknown, in addition to the two unknowns x and y, then the determinant in Theorem 1.1 specifies an irreducible surface $\{p_k(x,y,d)=0\}$ in three-dimensional space. That surface has degree 2^k. For an algebraic geometer, this surface would live in complex projective 3-space $\mathbb{CP}^3$, but we are interested in its points in real affine 3-space $\mathbb{R}^3$. Figure 5 shows this surface for $k=3$. The bowl-shaped convex branch near the top is the graph of the sum of distances function (3.1), while each of the other three branches is associated with a different combination of signs in the product (2.1). The surface has a total of $2^k=8$ branches, but only the four in the half-space $d\geq 0$ are shown, as it is symmetric with respect to the plane $d=0$. Note that the Fermat-Weber point (x_*,y_*,d_*) is a highly singular point of the surface.

The time has now come for us to explain the adjective "semidefinite" in the title of this paper. *Semidefinite programming* (SDP) is a widely used method in convex optimization. Introductory references include [17, 19].

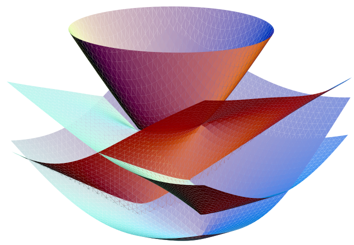

FIG. 5. *The irreducible surface* $\{ p_3(x,y,d) = 0\}$. *Taking horizontal slices gives the pencil of algebraic 3-ellipses for fixed foci and different radii d, as shown in Figure 3.*

An algebraic perspective was recently given in [12]. The problem of SDP is to minimize a linear functional over the solution set of a linear matrix inequality (LMI). An example of an LMI is

$$x \cdot A_k + y \cdot B_k + d \cdot I_{2^k} + \tilde{C}_k \ \succeq\ 0. \tag{3.2}$$

Here $\tilde{C}_k$ is the matrix gotten from C_k by setting $d = 0$, so that $C_k = d \cdot I_{2^k} + \tilde{C}_k$. If d is a fixed positive real number then the solution set to the LMI (3.2) is the convex region $\mathcal{E}_k$ bounded by the k-ellipse. If d is an unknown then the solution set to the LMI (3.2) is the epigraph of (3.1), or, geometrically, the unbounded 3-dimensional convex region interior to the bowl-shaped surface in Figure 5. The bottom point of that convex region is the Fermat-Weber point (x_*, y_*, d_*), and it can be computed by solving the SDP

$$\text{Minimize } d \text{ subject to (3.2).} \tag{3.3}$$

Similarly, for fixed radius d, the k-ellipse is the set of all solutions to

$$\text{Minimize } \ \alpha x + \beta y \ \text{ subject to (3.2)} \tag{3.4}$$

where α, β run over $\mathbb{R}$. This explains the term *semidefinite representation* in our title.

While the Fermat-Weber SDP (3.3) has only three unknowns, it has the serious disadvantage that the matrices are exponentially large (size 2^k). For computing (x_*, y_*, d_*) in practice, it is better to introduce slack variables $d_1, d_2, \ldots, d_k$, and to solve

$$\text{Minimize} \sum_{i=1}^{k} d_i \ \text{ subject to } \begin{bmatrix} d_i{+}x{-}a_i & y{-}b_i \\ y{-}b_i & d_i{-}x{+}a_i \end{bmatrix} \succeq 0 \ \ (i{=}1,\ldots,k). \tag{3.5}$$

This system can be written as a single LMI by stacking the 2×2-matrices to form a block matrix of size $2k \times 2k$. The size of the resulting LMI is linear in k while the size of the LMI (3.3) is exponential in k. If we take the LMI

(3.5) and add the linear constraint $d_1 + d_2 + \cdots + d_k = d$, for some fixed $d > 0$, then this furnishes a natural and concise *lifted* semidefinite representation of our k-ellipse[1]. Geometrically, the representation expresses $\mathcal{E}_k$ as the *projection* of a convex set defined by linear matrix inequalities in a higher-dimensional space. Theorem 1.1 solves the algebraic *LMI elimination problem* corresponding to this projection, but at the expense of an exponential increase in size, which is due to the exponential growing of degrees of k-ellipses.

Our last topic in this section is the relationship of the k-ellipse to the celebrated work of Helton and Vinnikov [5] on LMI representations of planar convex sets, which led to the resolution of the Lax conjecture in [8]. A semialgebraic set in the plane is called *rigidly convex* if its boundary has the property that every line passing through its interior intersects the Zariski closure of the boundary only in real points. Helton and Vinnikov [5, Thm. 2.2] proved that a plane curve of degree d has an LMI representation by symmetric $d \times d$ matrices if and only if the region bounded by this curve is rigidly convex. In arbitrary dimensions, rigid convexity holds for every region bounded by a hypersurface that is given by an LMI representation, but the strong form of the converse, where the degree of the hypersurface precisely matches the matrix size of the LMI, is only valid in two dimensions.

It follows from the LMI representation in Theorem 1.1 that the region bounded by a k-ellipse is rigidly convex. Rigid convexity can be seen in Figures 4 and 2. Every line that passes through the interior of the 3-ellipse intersects the algebraic 3-ellipse in eight real points, and lines through the 5-ellipse meet its Zariski closure in 32 real points. Combining our Theorem 1.1 with the Helton-Vinnikov Theorem, we conclude:

Corollary 3.1. *The k-ellipse is rigidly convex. If k is odd, it can be represented by an LMI of size 2^k, and if k is even, it can be represented as by LMI of size $2^k - \binom{k}{k/2}$.*

We have not found yet an *explicit* representation of size $2^k - \binom{k}{k/2}$ when k is even and $k \geq 4$. For the classical ellipse ($k = 2$), the determinantal representation (1.2) presented in the Introduction has size 4×4, while Corollary 3.1 guarantees the existence of a 2×2 representation. One such representation of the ellipse with foci (u_1, v_1) and (u_2, v_2) is:

$$\big(d^2 + (u_1 - u_2)(2x - u_1 - u_2) + (v_1 - v_2)(2y - v_1 - v_2)\big) \cdot I_2 + 2d \cdot \begin{bmatrix} x - u_2 & y - v_2 \\ y - v_2 & -x + u_2 \end{bmatrix} \succeq 0.$$

Notice that in this LMI representation of the ellipse, the matrix entries are linear in x and y, as required, but they are quadratic in the radius

[1] The formulation (3.5) actually provides a *second-order cone* representation of the k-ellipse. Second-order cone programming (SOCP) is another important class of convex optimization problems, of complexity roughly intermediate between that of linear and semidefinite programming; see [9] for a survey of the basic theory and its many applications.

parameter d and the foci u_i, v_i. What is the nicest generalization of this representation to the k-ellipse for k even?

4. Generalizations. The semidefinite representation of the k-ellipse we have found in Theorem 1.1 can be generalized in several directions. Our first generalization corresponds to the inclusion of arbitrary positive weights for the distances, while the second one extends the results from plane curves to higher dimensions. The resulting geometric shapes are known as *Tschirnhaus'sche Eiflächen* (or "egg surfaces") in the classical literature [11].

4.1. Weighted k-ellipse. Consider k points $(u_1, v_1), \ldots, (u_k, v_k)$ in the real plane $\mathbb{R}^2$, a positive radius d, and positive *weights* $w_1, \ldots, w_k$. The *weighted k-ellipse* is the plane curve defined as

$$\left\{ (x,y) \in \mathbb{R}^2 \;:\; \sum_{i=1}^{k} w_i \cdot \sqrt{(x-u_i)^2 + (y-v_i)^2} \;=\; d \right\},$$

where w_i indicates the relative weight of the distance from (x, y) to the i-th focus (u_i, v_i). The *algebraic weighted k-ellipse* is the Zariski closure of this curve. It is the zero set of an irreducible polynomial $p_k^w(x, y)$ that can be constructed as in equation (2.1). The interior of the weighted k-ellipse is the bounded convex region

$$\mathcal{E}_k(w) \;:=\; \left\{ (x,y) \in \mathbb{R}^2 \;:\; \sum_{i=1}^{k} w_i \cdot \sqrt{(x-u_i)^2 + (y-v_i)^2} \;\leq\; d \right\}.$$

The matrix construction from the unweighted case in (2.2) generalizes as follows:

$$L_k^w(x,y) := d \cdot I_{2^k} + w_1 \cdot \begin{bmatrix} x-u_1 & y-v_1 \\ y-v_1 & -x+u_1 \end{bmatrix} \oplus \cdots \oplus w_k \cdot \begin{bmatrix} x-u_k & y-v_k \\ y-v_k & -x+u_k \end{bmatrix}. \quad (4.1)$$

Each tensor summand is simply multiplied by the corresponding weight. The following representation theorem and degree formula are a direct generalization of Theorem 2.1:

THEOREM 4.1. *The algebraic weighted k-ellipse has the semidefinite representation*

$$p_k^w(x,y) \;=\; \det L_k^w(x,y),$$

and the convex region in its interior satisfies

$$\mathcal{E}_k(w) \;=\; \left\{ (x,y) \in \mathbb{R}^2 \;:\; L_k^w(x,y) \succeq 0 \right\}.$$

The degree of the weighted k-ellipse is given by

$$\deg p_k^w(x,y) \;=\; 2^k - |\mathcal{P}(w)|,$$

where $\mathcal{P}(w) = \{\delta \in \{-1,1\}^k : \sum_{i=1}^k \delta_i w_i = 0\}$.

Proof. The proof is entirely analogous to that of Theorem 2.1. ☐

A consequence of the characterization above is the following cute complexity result.

COROLLARY 4.1. *The decision problem "Given a weighted k-ellipse with fixed foci and positive integer weights, is its algebraic degree smaller than* 2^k*?" is NP-complete. The function problem "What is the algebraic degree?" is #P-hard.*

Proof. Since the number partitioning problem is NP-hard [4], the fact that our decision problem is NP-hard follows from the degree formula in Theorem 4.1. But it is also in NP because if the degree is less than 2^k, we can certify this by exhibiting a choice of δ_i for which $\sum_{i=1}^k \delta_i w_i = 0$. Computing the algebraic degree is equivalent to counting the number of solutions of a number partitioning problem, thus proving its #P-hardness. ☐

4.2. k-Ellipsoids. The definition of a k-ellipse in the plane can be naturally extended to a higher-dimensional space to obtain k-ellipsoids. Consider k fixed points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathbb{R}^n$, with $\mathbf{u}_i = (u_{i1}, u_{i2}, \ldots, u_{in})$. The *$k$-ellipsoid* in $\mathbb{R}^n$ with these foci is the hypersurface

$$\left\{\mathbf{x}\in\mathbb{R}^n : \sum_{i=1}^{k} \|\mathbf{u}_i - \mathbf{x}\| = d\right\} = \left\{\mathbf{x}\in\mathbb{R}^n : \sum_{i=1}^{k} \sqrt{\sum_{j=1}^{n}(u_{ij}-x_j)^2} = d\right\}. \quad (4.2)$$

This hypersurface encloses the convex region

$$\mathcal{E}_k^n \quad = \quad \left\{\mathbf{x} \in \mathbb{R}^n \; : \; \sum_{i=1}^{k} \|\mathbf{u}_i - \mathbf{x}\| \; \leq \; d\right\}.$$

The Zariski closure of the k-ellipsoid is the hypersurface defined by an irreducible polynomial $p_k^n(\mathbf{x}) = p_k^n(x_1, x_2, \ldots, x_n)$. By the same reasoning as in Section 2, we can prove the following:

THEOREM 4.2. *The defining irreducible polynomial* $p_n^k(\mathbf{x})$ *of the k-ellipsoid is monic of degree* 2^k *in the parameter d, it has degree* 2^k *in* $\mathbf{x}$ *if k is odd, and it has degree* $2^k - \binom{k}{k/2}$ *if k is even.*

We shall represent the polynomial $p_n^k(\mathbf{x})$ as a factor of the determinant of a symmetric matrix of affine-linear forms. To construct this semidefinite representation of the k-ellipsoid, we proceed as follows. Fix an integer $m \geq 2$. Let $\mathbf{U}_i(\mathbf{x})$ be any symmetric $m\times m$-matrix of rank 2 whose entries are affine-linear forms in $\mathbf{x}$, and whose two non-zero eigenvalues are $\pm\|\mathbf{u}_i - \mathbf{x}\|$. Forming the tensor sum of these matrices, as in the proof of Theorem 2.1, we find that $p_k^n(\mathbf{x})$ is a factor of

$$\det\big(\, d \cdot I_{m^k} \;+\; \mathbf{U}_1(\mathbf{x}) \oplus \mathbf{U}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{U}_k(\mathbf{x})\,\big). \quad (4.3)$$

However, there are many extraneous factors. They are powers of the irreducible polynomials that define the k'-ellipsoids whose foci are subsets of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$.

There is a standard choice for the matrices $\mathbf{U}_i(\mathbf{x})$ that is symmetric with respect to permutations of the n coordinates. Namely, we can take $m = n + 1$ and

$$\mathbf{U}_i(\mathbf{x}) \quad = \quad \begin{bmatrix} 0 & x_1 - u_{i1} & x_2 - u_{i2} & \cdots & x_n - u_{in} \\ x_1 - u_{i1} & 0 & 0 & \cdots & 0 \\ x_2 - u_{i2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n - u_{in} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

However, in view of the extraneous factors in (4.3), it is desirable to replace these by matrices of smaller size m, possibly at the expense of having additional nonzero eigenvalues. It is not hard to see that $m = n$ is always possible, and the following result shows that, without extra factors, this is indeed the smallest possible matrix size.

LEMMA 4.1. *Let $A(x) := A_0 + A_1 x_1 + \cdots + A_n x_n$, where $A_0, A_1, \ldots, A_n$ are real symmetric $m \times m$ matrices. If $\det A(x) = d^2 - \sum_{j=1}^n x_j^2$, then $m \geq n$.*

Proof. Assume, on the contrary, that $m < n$. For any fixed vector ξ with $\|\xi\| = d$ the matrix $A(\xi)$ must be singular. Therefore, there exists a nonzero vector $\eta \in \mathbb{R}^m$ such that $A(\xi)\eta = 0$. The set $\{x \in \mathbb{R}^n : A(x)\eta = 0\}$ is thus a nonempty affine subspace of positive dimension (at least $n - m$). The polynomial $d^2 - \sum_{j=1}^n x_j^2$ should then also vanish on this subspace, but this is not possible since $\{x \in \mathbb{R}^n : d^2 - \sum_{j=1}^n x_j^2 = 0\}$ is compact. ∎

However, if we allow extraneous factors or complex matrices, then the smallest possible value of m might drop. These questions are closely related to finding the *determinant complexity* of a given polynomial, as discussed in [10]. Note that in the applications to complexity theory considered there, the matrices of linear forms need not be symmetric.

5. Open questions and further research. The k-ellipse is an appealing example of an object from algebraic geometry. Its definition is elementary and intuitive, and yet it serves well in illustrating the intriguing interplay between algebraic concepts and convex optimization, in particular semidefinite programming. The developments presented in this paper motivate many natural questions. For most of these, to the best of our knowledge, we currently lack definite answers. Here is a short list of open problems and possible topics of future research.

Singularities and genus. Both the circle and the ellipse are rational curves, i.e., have genus zero. What is the genus of the (projective) algebraic k-ellipse? The first values, from $k = 1$ to $k = 4$, are 0,0,3,6. What is the formula for the genus in general? The genus is related to the class of the curve, i.e. the degree of the dual curve, and this number is the algebraic degree [12] of the problem (3.4). Moreover, is there a nice geometric characterization of all (complex) singular points of the algebraic k-ellipse?

Algebraic degree of the Fermat-Weber point. The Fermat-Weber point (x_*, y_*) is the unique solution of an algebraic optimization problem, formulated in (3.3) or (3.5), and hence it has a well-defined algebraic degree over $\mathbb{Q}(u_1, v_1, \ldots, u_k, v_k)$. However, that algebraic degree will depend on the combinatorial configuration of the foci. For instance, in the case $k = 4$ and foci forming a convex quadrilateral, the Fermat-Weber point lies in the intersection of the two diagonals [1], and therefore its algebraic degree is equal to one. What are the possible values for this degree? Perhaps a possible approach to this question would be to combine the results and formulas in [12] with the semidefinite characterizations obtained in this paper.

Reduced SDP representations of rigidly convex curves. A natural question motivated by our discussion in Section 3 is how to systematically produce minimal determinantal representations for a rigidly convex curve, when a non-minimal one is available. This is likely an easier task than finding a representation directly from the defining polynomial, since in this case we have a certificate of its rigid convexity.

Concretely, given real symmetric $n \times n$ matrices A and B such that

$$p(x, y) = \det(A \cdot x + B \cdot y + I_n)$$

is a polynomial of degree $r < n$, we want to produce $r \times r$ matrices $\tilde{A}$ and $\tilde{B}$ such that

$$p(x, y) = \det(\tilde{A} \cdot x + \tilde{B} \cdot y + I_r).$$

The existence of such matrices is guaranteed by the results in [5, 8]. In fact, explicit formulas in terms of theta functions of a Jacobian variety are presented in [5]. But isn't there a simpler algebraic construction in this special case?

Elimination in semidefinite programming. The projection of an algebraic variety is (up to Zariski closure, and over an algebraically closed field) again an algebraic variety. That projection can be computed using elimination theory or Gröbner bases. The projection of a polyhedron into a lower-dimensional subspace is a polyhedron. That projection can be computed using Fourier-Motzkin elimination. In contrast to these examples, the class of feasible sets of semidefinite programs is not closed under projections. As a simple concrete example, consider the convex set

$$\left\{ (x, y, t) \in \mathbb{R}^3 \;:\; \begin{bmatrix} 1 & x - t \\ x - t & y \end{bmatrix} \succeq 0, \quad t \geq 0 \right\}.$$

Its projection onto the (x, y)-plane is a convex set that is not rigidly convex, and hence cannot be expressed as $\{(x, y) \in \mathbb{R}^2 : Ax + By + C \succeq 0\}$.

In fact, that projection is not even basic semialgebraic. In some cases, however, this closure property nevertheless does hold. We saw this for the projection that transforms the representation (3.5) of the k-ellipse to the representation (3.3). Are there general conditions that ensure the semidefinite representability of the projections? Are there situations where the projection does not lead to an exponential blowup in the size of the representation?

Hypersurfaces defined by eigenvalue sums. Our construction of the (weighted) k-ellipsoid as the determinant of a tensor sum has the following natural generalization. Let $\mathbf{U}_1(\mathbf{x}), \mathbf{U}_2(\mathbf{x}), \ldots, \mathbf{U}_k(\mathbf{x})$ be any symmetric $m \times m$-matrices whose entries are affine-linear forms in $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then we consider the polynomial

$$p(\mathbf{x}) \quad = \quad \det\bigl(d \cdot I_{m^k} + \mathbf{U}_1(\mathbf{x}) \oplus \mathbf{U}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{U}_k(\mathbf{x})\bigr). \tag{5.1}$$

We also consider the corresponding rigidly convex set

$$\bigl\{\, \mathbf{x} \in \mathbb{R}^n \ : \ d \cdot I_{m^k} + \mathbf{U}_1(\mathbf{x}) \oplus \mathbf{U}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{U}_k(\mathbf{x}) \succeq 0 \,\bigr\}.$$

The boundary of this convex set is a hypersurface whose Zariski closure is the set of zeroes of the polynomial $p(\mathbf{x})$. It would be worthwhile to study the hypersurfaces of the special form (5.1) from the point of view of computational algebraic geometry.

These hypersurfaces specified by eigenvalue sums of symmetric matrices of linear forms have a natural generalization in terms of *resultant sums* of hyperbolic polynomials. For concreteness, let us take $k = 2$. If $p(\mathbf{x})$ and $q(\mathbf{x})$ are hyperbolic polynomials in n unknowns, with respect to a common direction $\mathbf{e}$ in $\mathbb{R}^n$, then the polynomial

$$(p \oplus q)(\mathbf{x}) \quad := \quad \mathrm{Res}_t\bigl(p(\mathbf{x} - t\mathbf{e}), q(\mathbf{x} + t\mathbf{e})\bigr)$$

is also hyperbolic with respect to $\mathbf{e}$. This construction mirrors the operation of taking Minkowski sums in the context of convex polyhedra, and we believe that it is fundamental for future studies of hyperbolicity in polynomial optimization [8, 13].

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